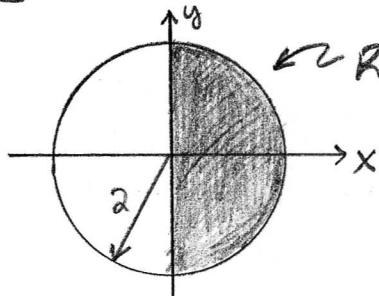


Homework 9 CALCULUS III

(1)

§16.4 #10 Let $R = \{(x, y) / x^2 + y^2 \leq 4, x \geq 0\}$



In polar coordinates we can see R is simply

$$0 \leq r \leq 2$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Caution:

Now some folks might insist that $0 \leq \theta < 2\pi$ but I'll not be so picky in this course. I allow use of $\theta \in \mathbb{R}$ with the understanding θ and $\theta + 2\pi$ measure the same standard angle. If we were to be picky here then I would have to break up the θ description and say

$$0 \leq \theta \leq \pi/2 \quad \text{OR} \quad 3\pi/2 \leq \theta \leq 2\pi$$

then the integration would split into two pieces as well but in the end we'd get the same answer.

$$\begin{aligned} \iint_R \sqrt{4-x^2-y^2} dA &= \int_0^2 \int_{-\pi/2}^{\pi/2} \sqrt{4-r^2} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| d\theta dr \\ &= \int_0^2 \int_{-\pi/2}^{\pi/2} r \sqrt{4-r^2} d\theta dr \\ &= \int_0^2 r \sqrt{4-r^2} (\theta) \Big|_{-\pi/2}^{\pi/2} dr \\ &= \int_0^2 \pi r \sqrt{4-r^2} dr \\ &= \int_4^0 \frac{-\pi}{2} \sqrt{u} du \\ &= -\frac{\pi}{2} \frac{2}{3} u^{3/2} \Big|_4^0 \\ &= \frac{\pi}{3} (4^{3/2}) \\ &= \boxed{\frac{8\pi}{3}} \end{aligned}$$

using remark (*)

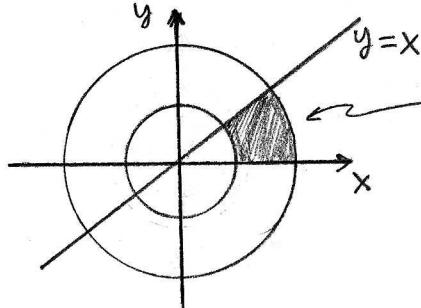
Let $u = 4-r^2$
 $u(2) = 0, u(0) = 4$
 $du = -2rdr$
 $dr = \frac{-du}{2r}$

Remark(*) Polar coordinates are (r, θ) where $x = r\cos\theta$ & $y = r\sin\theta$. The change of variables Thm states we need to put the Jacobian factor $\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \left| \det \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} \right| = \left| \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$

this essentially means $dA = r dr d\theta$ in polar. Now sometimes I'll write $dA = r d\theta dr$, the bounds must reflect the ordering. You don't have to say all this in each polar coordinate integral, but you should understand these comments at some time before the test.

§16.4 #13] Let $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$

(2)



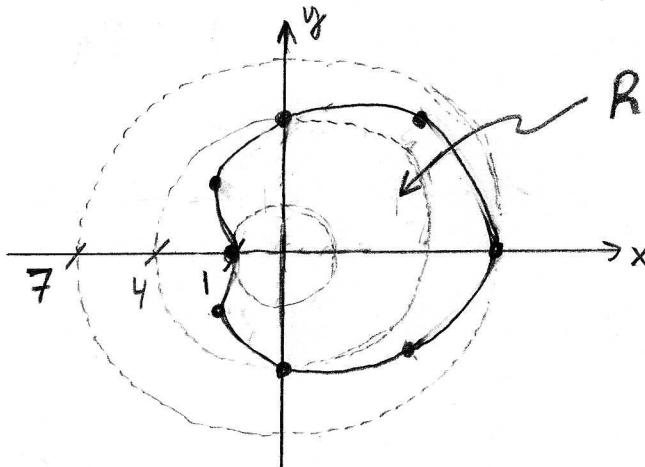
$$R: \quad 1 \leq r \leq 2 \\ 0 \leq \theta \leq \frac{\pi}{4}$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ \frac{y}{x} &= \frac{r \sin \theta}{r \cos \theta} = \tan \theta \\ \Rightarrow \theta &= \tan^{-1}(y/x). \end{aligned}$$

$$\begin{aligned} \iint_R \tan^{-1}(y/x) dA &= \int_1^2 \int_0^{\pi/4} \theta r d\theta dr \\ &= \int_1^2 \left(\frac{\theta^2}{2} \Big|_0^{\pi/4} \right) r dr \\ &= \int_1^2 \frac{\pi^2}{32} r dr \\ &= \frac{\pi^2}{32} \left(\frac{r^2}{2} \Big|_1^2 \right) \\ &= \boxed{\frac{3\pi^2}{64}} \end{aligned}$$

§16.4 #16] Find area enclosed by the curve $r = 4 + 3\cos \theta$. We must determine the bounds need for the integration. We need to graph this thing.

θ	$r = 4 + 3\cos \theta$
0	7
$\pi/2$	4
π	1
$3\pi/2$	4
2π	7
$\pi/4$	$4 + 3/\sqrt{2}$
$3\pi/4$	$4 - 3/\sqrt{2}$
$5\pi/4$	$4 - 3/\sqrt{2}$
$7\pi/4$	$4 + 3/\sqrt{2}$



My picture isn't pretty, but we can clearly see that $0 \leq r \leq 4 + \cos \theta$ and $0 \leq \theta \leq 2\pi$ will capture the entire region.

- Sometimes we can figure out what the region looks like w/o plotting points, but if other methods fail its not a bad idea to make a table of values and see where it takes you. A compass would help here.

§16.4 #16 (Continued)

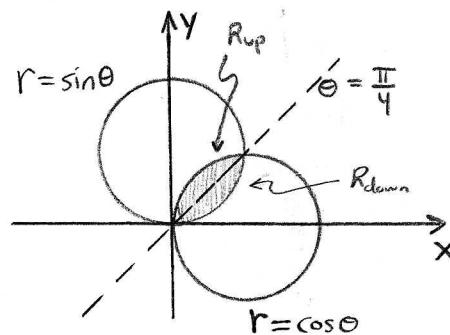
(3)

$$\begin{aligned}
 A &= \iint_R dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r dr d\theta \\
 &= \int_0^{2\pi} \left(\frac{1}{2} r^2 \Big|_0^{4+3\cos\theta} \right) d\theta \\
 &= \int_0^{2\pi} \frac{1}{2} (4 + 3\cos\theta)^2 d\theta \\
 &= \int_0^{2\pi} \frac{1}{2} (16 + 24\cos\theta + 9\cos^2\theta) d\theta \\
 &= \int_0^{2\pi} \left(8 + 12\cos\theta + \frac{9}{4} + \frac{9}{4}\cos(2\theta) \right) d\theta \\
 &= \left(\frac{41}{4}\theta + 12\sin\theta + \frac{9}{8}\sin(2\theta) \right) \Big|_0^{2\pi} \\
 &= \boxed{\frac{41\pi}{2}}
 \end{aligned}$$

§16.4 #17 Calculate the area between the circles $r = \cos\theta$ and $r = \sin\theta$

Notice $r = \cos\theta \rightarrow r^2 = r\cos\theta \rightarrow x^2 + y^2 = x \rightarrow (x-1)^2 + y^2 = 1$

$r = \sin\theta \rightarrow r^2 = r\sin\theta \rightarrow x^2 + y^2 = y \rightarrow x^2 + (y-1)^2 = 1$

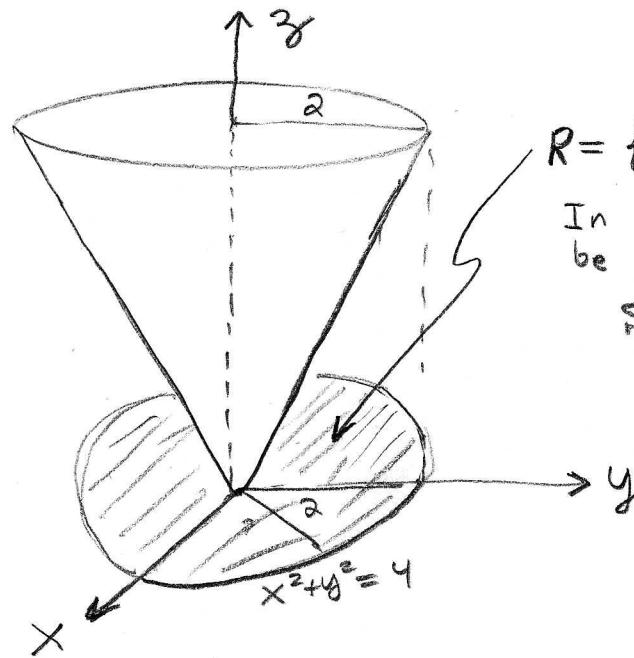


from the picture it is clear that for
 $0 \leq \theta \leq \pi/4 \Rightarrow 0 \leq r \leq \sin\theta$
whereas when
 $\pi/4 \leq \theta \leq \pi/2 \Rightarrow 0 \leq r \leq \cos\theta$

$$\begin{aligned}
 A &= \iint_{R_{down}} dA + \iint_{R_{up}} dA = \int_0^{\pi/4} \int_0^{\sin\theta} r dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\cos\theta} r dr d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \sin^2\theta d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos^2\theta d\theta \\
 &= \frac{1}{4} \int_0^{\pi/4} (1 - \cos 2\theta) d\theta + \frac{1}{4} \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= \frac{1}{4} \left(\theta - \frac{1}{2} \sin(2\theta) \right) \Big|_0^{\pi/4} + \frac{1}{4} \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Big|_{\pi/4}^{\pi/2} \\
 &= \frac{1}{4} \left[\frac{\pi}{4} - \frac{1}{2} \sin\left(\frac{\pi}{2}\right) - 0 + \frac{1}{2} \sin(0) \right] + \frac{1}{4} \left[\frac{\pi}{2} + \frac{1}{2} \sin(\pi) - \frac{\pi}{4} - \frac{1}{2} \sin\left(\frac{\pi}{2}\right) \right] \\
 &= \boxed{\frac{\pi}{8} - \frac{1}{4} = \frac{1}{8}(\pi - 2)}
 \end{aligned}$$

could also just calculate
 R_{up} or R_{down} 's area
then multiply by two
using symmetry

§6.4 #19 Find volume under the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \leq 4$ (4)



$$R = \{(x, y) \mid 0 \leq x^2 + y^2 \leq 4\}$$

In polar coordinates R can be restated as

$$S = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$V = \iint_R \sqrt{x^2 + y^2} dA$$

$$= \iint_S \sqrt{r^2} r dr d\theta$$

$$= \int_0^2 \int_0^{2\pi} r^2 dr d\theta$$

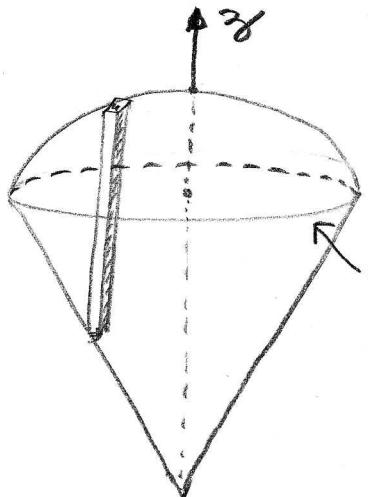
$$= \int_0^2 (\theta \Big|_0^{2\pi}) r^2 dr$$

$$= 2\pi \int_0^2 r^2 dr$$

$$= \frac{2\pi}{3} r^3 \Big|_0^2$$

$$= \boxed{\frac{16\pi}{3}}$$

§16.4 #25 / Find volume above $z = \sqrt{x^2+y^2}$ and below $x^2+y^2+z^2=1$ (5)



curve of intersection will determine the appropriate integration region to calculate the volume.

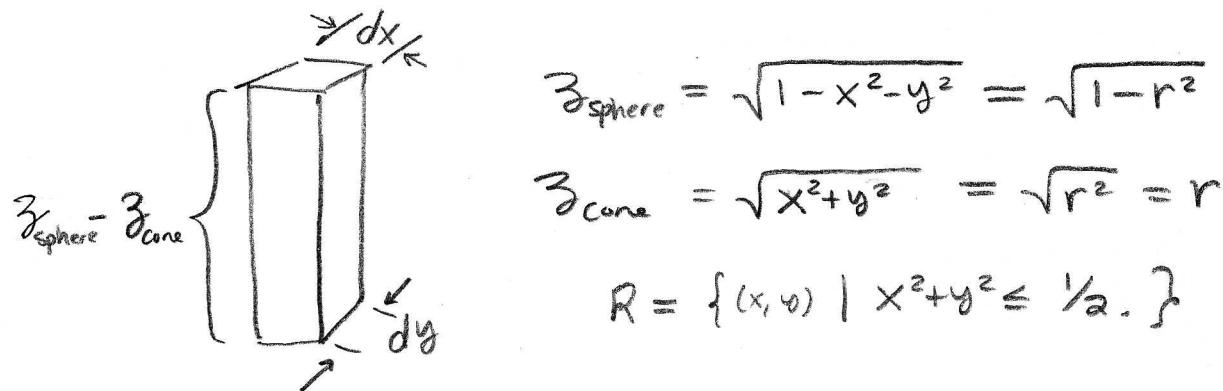
$$z^2 = x^2 + y^2 = 1 - x^2 - y^2$$

$$2(x^2 + y^2) = 1$$

$$x^2 + y^2 = \frac{1}{2} \text{ & } z^2 = 1 - \frac{1}{2}$$

The intersection is a circle $x^2 + y^2 = \frac{1}{2}$ in the $z = 1/\sqrt{2}$ plane.

A typical approximating rectangular box looks like



Thus

$$V = \iint_R (z_{\text{sphere}} - z_{\text{cone}}) dA$$

$$= \int_0^{2\pi} \int_0^{1/\sqrt{2}} (\sqrt{1-r^2} - r) r dr d\theta \quad : \text{ note } \int_0^{2\pi} d\theta = 2\pi,$$

$$= 2\pi \left(\int_0^{1/\sqrt{2}} r \sqrt{1-r^2} dr - \int_0^{1/\sqrt{2}} r^2 dr \right)$$

$$= 2\pi \left(\int_1^{1/\sqrt{2}} \frac{1}{2} \sqrt{u} du - \frac{1}{3} \left(\frac{1}{\sqrt{2}}\right)^3 \right)$$

$$= 2\pi \left(-\frac{1}{3} (u)^{3/2} \Big|_1^{1/\sqrt{2}} - \frac{1}{6\sqrt{2}} \right)$$

$$= 2\pi \left(-\frac{1}{6\sqrt{2}} + \frac{1}{3} - \frac{1}{6\sqrt{2}} \right) = \boxed{\frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right)}$$

$u = 1 - r^2$ $u(0) = 1, u(\frac{1}{\sqrt{2}}) = \frac{1}{2}$ $du = -2rdr$
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