

Homework 6, CALCULUS III

(1)

§17.6 #19) Find parametric representation of the surface: plane containing $(1, 2, -3)$ and the vectors $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, 1 \rangle$. This has normal

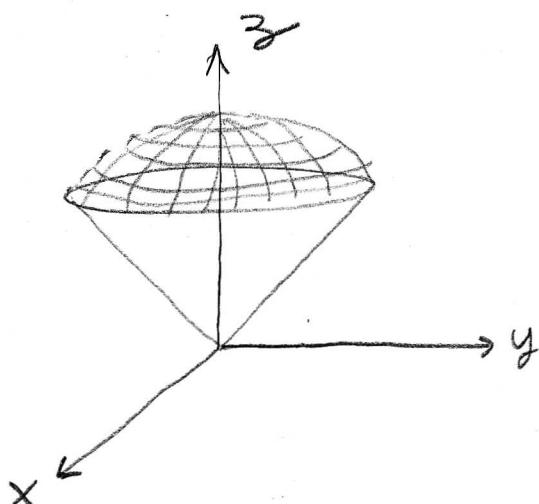
$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 2, -2, -2 \rangle$$

I'll use normal $\langle 1, -1, -1 \rangle$ thus the Cartesian Representation of the plane is $x - 1 - (y - 2) - (z + 3) = 0$ Which gives $z = x - y - 2$. Choose parameters to be x & y then

$$\vec{\Gamma}(x, y) = \vec{\Sigma}(x, y) = \langle x, y, x - y - 2 \rangle$$

for $(x, y) \in \mathbb{R}^2$ is a parametrization
for the plane

§17.6 #23) The parametrization of sphere $x^2 + y^2 + z^2 = 4$ that lies above cone $z = \sqrt{x^2 + y^2}$.



Notice that $z = \sqrt{x^2 + y^2}$ is cone making angle

$$\phi = \frac{\pi}{4} = 45^\circ$$

Look at $y = 0$, $z = \sqrt{x^2} = \pm x$ or $x = 0$, $z = \sqrt{y^2} = \pm y$ these are lines with slope 1 on the zx and yz coordinate planes hence $\phi = 45^\circ$ or $\pi/4$.

Thus using spherical coordinates the parametrization is naturally found using $\rho = 2$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/4$

$$\vec{\Sigma}(\theta, \phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi \rangle \quad \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/4 \end{matrix}$$

②

§15.7 #11 Find and classify the local max/min of

$$f(x, y) = x^3 - 12xy + 8y^3$$

Critical points occur where either $\nabla f = 0$ or ∇f don't exist. Observe,

$$\begin{aligned}\nabla f &= \langle f_x, f_y \rangle \\ &= \langle 3x^2 - 12y, -12x + 24y^2 \rangle\end{aligned}$$

Thus ∇f exists everywhere. Critical pts. come from

$$\begin{aligned}f_x &= 3x^2 - 12y = 0 \rightarrow y = \frac{3}{12}x^2 = \frac{x^2}{4} \\ f_y &= -12x + 24y^2 = 0\end{aligned} \quad (I)$$

Substitute into $f_y = 0$,

$$-12x + 24\left(\frac{x^2}{4}\right)^2 = 0$$

$$24x - 48\left(\frac{x^4}{16}\right) = 0$$

$$24x - 3x^4 = 0$$

$$x(24 - 3x^3) = 0$$

$$\Rightarrow x = 0 \text{ or } 24 - 3x^3 = 0 \Rightarrow x^3 = 8$$

Sols occur at $x_1 = 0$ or $x_2 = 2$

corresponding $y = \frac{x^2}{4}$ so $y_1 = 0$ & $y_2 = 1$

The critical pts. are $(0,0)$ and $(2,1)$

The Hessian $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix} = 288xy - 144$

will tell us if $(0,0)$ or $(2,1)$ is min/max etc...

$$1.) D(0,0) = 288(0) - 144 = -144 < 0 \Rightarrow (0,0) \text{ not at local max/min}$$

$$2.) D(2,1) = 288(2)(1) - 144 = 432 > 0$$

$$\text{and } f_{xx}(2,1) = 6x \Big|_{(2,1)} = 12 > 0 \Rightarrow (2,1) \text{ yields } f(2,1) = -8 \text{ which is a local minimum}$$

Both of these conclusions follow from

Thm (3) the 2nd Derivative Test on pg. 960.

§ | S. 7 # 35 Find absolute max/min values of f on the set D . Let (3)

$$f(x, y) = 2x^3 + y^4 \quad \text{with} \quad D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

Notice $\nabla f = \langle 6x^2, 4y^3 \rangle$ thus all critical points come from $\nabla f = 0$ since ∇f exists for all x, y .

$$\begin{aligned} 6x^2 &= 0 \rightarrow x = 0 \\ 4y^3 &= 0 \rightarrow y = 0 \end{aligned} \quad \Rightarrow \quad \begin{array}{l} (0, 0) \text{ only critical pt.} \\ \text{Note } f(0, 0) = 0. \end{array}$$

Calculate Hessian

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x & 0 \\ 0 & 12y^2 \end{vmatrix} = 144xy^2 \quad \text{thus } D(0, 0) = 0$$

So we note $f(0, 0) = 2(0) + 0 = 0$ falls outside the scope of the 2nd Derivative Test.

On the boundary ∂D of D we have $x^2 + y^2 = 1$ lets eliminate y via $y^2 = 1 - x^2$. We have

$$\begin{aligned} f \Big|_{\partial D}(x) &= f(x, y) \Big|_{y^2=1-x^2} \\ &= 2x^3 + (1-x^2)^2 \\ &= x^4 + 2x^3 - 2x^2 + 1 = \underline{g(x)} \end{aligned}$$

The restriction of f to ∂D we have introduced g for convenience denoted g . We now have an ordinary of notation. calc. I extreme values problem for $g(x)$ on the interval $[-1, 1]$. (notice $x^2 + y^2 = 1$ has $-1 \leq x \leq 1$)

$$\begin{aligned} g'(x) &= 4x^3 + 6x^2 - 4x \\ &= 2x(2x^2 + 3x - 2) \\ &= 2x(2x - 1)(x + 2) \end{aligned}$$

$$x_1 = 0, x_2 = \frac{1}{2}, x_3 = -2 \quad \text{critical pts. for } g$$

Since $x^2 + y^2 = 1$ on ∂D we have $y_1 = 1, y_2 = \sqrt{3}/2$. We need not consider $x_3 = -2$ since $x_3 \notin [-1, 1]$, its not in D .

§15.7 #35 (Continued)

(4)

We found the critical points for g were at $x_1 = 0$ and $x_2 = \frac{1}{2}$. This gives us the points $(0, 1)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ on ∂D . We need to find $f(0, 1) \neq f(\frac{1}{2}, \frac{\sqrt{3}}{2})$ so we can compare against $f(0, 0) = 0$ and see which is the biggest / most negative,

$$f(0, 1) = 2(0)^3 + (1)^4 = 1 = \frac{16}{16}$$

$$f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{\sqrt{3}}{2}\right)^4 = \frac{2}{8} + \frac{9}{16} = \frac{11}{16}$$

Thus, using the Extreme Value Th^m for two variables we find $f(0, 0) = 0$ is the absolute minimum and $f(0, 1) = 1$ is the absolute maximum of $f(x, y) = 2x^3 + y^4$ on $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Remark: Some things are known for $f(x, y, z)$ or $f(w, x, y, z)$ etc... but that is beyond the scope of this course so far as max/min analysis is concerned. We can do directional derivatives $D_{\hat{U}} f(p) = \nabla f(p) \cdot \hat{U}$ for $n=3, 4, 5, \dots$ BUT those just study the change in f at a single given point $p \in \mathbb{R}^n$. In contrast, §15.7 allows us to make global statements about $f(x, y)$.