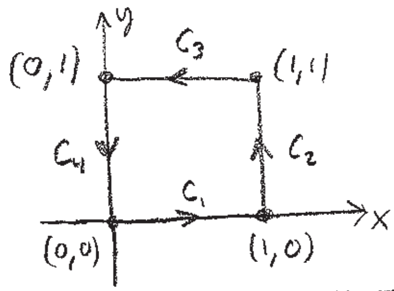
PROBLEM 198 Calculate $\int_C (y^2 dx + x^2 dy)$ for C pictured below in the following 2 ways



a.) direct computation of line-integrals.

b.) via Green's Th$^{\underline{m}}$

(a.) C_1: $x = t$, $y = 0$
$dx = dt$, $dy = 0$
$0 \le t \le 1$

$$\int_{C_1} y^2 dx + x^2 dy = \int_0^1 (0+0)\,dt = \underline{0.}$$

C_2: $x = 1$, $y = t$
$dx = 0$, $dy = dt$
$0 \le t \le 1$

$$\int_{C_2} y^2 dx + x^2 dy = \int_0^1 t^2(0)dt + 1\,dt = \underline{1.}$$

C_3: $x = 1-t$, $y = 1$
$dx = -dt$, $dy = 0$
$0 \le t \le 1$

$$\int_{C_3} y^2 dx + x^2 dy = \int_0^1 1(-dt) + 0 = \underline{-1.}$$

C_4: $x = 0$, $y = 1-t$
$dx = 0$, $dy = -dt$
$0 \le t \le 1$

$$\int_{C_4} y^2 dx + x^2 dy = \int (0+0)\,dt = \underline{0.}$$

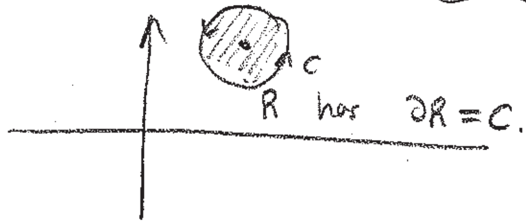
Thus $\int_C y^2 dx + x^2 dy = 0 + 1 - 1 + 0 = \boxed{0.}$

(b.)
$$\int_{C_1} y^2 dx + x^2 dy = \int_0^1 \int_0^1 \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dx\,dy$$
$$= \int_0^1 \int_0^1 (2x - 2y)\,dx\,dy$$
$$= \int_0^1 (x^2 - 2xy\,\big|_0^1\,dy$$
$$= \int_0^1 (1-2y)\,dy$$
$$= (y - y^2\,\big|_0^1 = 1 - 1 = \boxed{0.}$$

PROBLEM 199

$\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \Big\langle \underbrace{y - \ln(x^2+y^2)}_{P},\ \underbrace{2\tan^{-1}(y/x)}_{Q} \Big\rangle$

where



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R \left[\cancel{\frac{2}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)} - \cancel{\frac{-2y}{x^2+y^2}} - 1 \right] dA$$

$$= -\iint dA$$

$$= \boxed{-\pi}$$

why? well, let

$$x - 2 = \tilde{r}\cos\tilde{\theta}$$
$$y - 3 = \tilde{r}\sin\tilde{\theta}$$

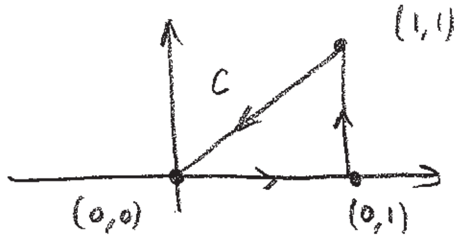
$$x = 2 + \tilde{r}\cos\tilde{\theta}$$
$$y = 3 + \tilde{r}\sin\tilde{\theta}$$

$$\iint_{(x-2)^2+(y-3)^2\leq 1} dx\,dy = \iint_{\tilde{r}^2\leq 1} \frac{\partial(x,y)}{\partial(\tilde{r},\tilde{\theta})}\, d\tilde{r}\, d\tilde{\theta} = \int_0^{2\pi}\int_0^1 \tilde{r}\, d\tilde{r}\, d\theta$$

$$= \pi.$$

$\boxed{\text{PROBLEM 200}}$ Left to the reader.

PROBLEM 201



Calculate $\oint_C xy\,dx + x^2 dy$

Apply Green's Th$^{\underline{m}}$. Note that $R = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$ has $\partial R = C$ thus,

$$\oint_C xy\,dx + x^2 dy = \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy) \right) dA$$

$$= \int_0^1 \int_0^x [2x - x]\, dy\, dx$$

$$= \int_0^1 \left(xy \Big|_0^x \right) dx$$

$$= \int_0^1 x^2 dx$$

$$= \boxed{1/3.}$$

PROBLEM 202

Let C be the CCW circle $x^2 + y^2 = 1$. Calculate $\oint_C 3y\,dx + 5x\,dy$

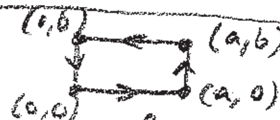
Apply Green's Th$^{\underline{m}}$,

$$\oint_C 3y\,dx + 5x\,dy = \iint_{x^2+y^2 \leq 1} \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3y) \right) dA$$

$$= 2 \iint_{x^2+y^2 \leq 1} dA$$ ← area of unit circle is π.

$$= \boxed{2\pi.}$$

PROBLEM 203 Let R be the CCW oriented rectangle with vertices $(0,0)$, $(a,0)$, (a,b), $(0,b)$. Calculate $\oint_C x^2\,dy$



$$\oint_C x^2\,dy = \int_0^a\int_0^b \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(0)\right) dy\,dx$$

$$= \int_0^a\int_0^b 2x\,dy\,dx$$

$$= \int_0^a (2xb)\,dx$$

$$= \frac{2x^2b}{2}\Big|_0^a$$

$$= \boxed{a^2 b} \quad \text{nice pattern.}$$

PROBLEM 204 Suppose f, g are continuously differentiable on an simply connected open set R. Show that if C is a piecewise simple closed curve in R then

$$\oint_C (f\,\nabla g + g\,\nabla f)\cdot d\vec{r} = 0$$

Consider $h = fg$ note that FTC for line-integrals implies $\oint_C \nabla h \cdot d\vec{r} = 0$. However, the product-rule for gradients is easily proved $\longrightarrow$

$$\nabla(fg) = \sum_{i=1}^n \hat{x}_i \partial_i (fg) = \sum_{i=1}^n \hat{x}_i \left[(\partial_i f)g + f(\partial_i g)\right]$$

$$= g\left(\sum_{i=1}^n \hat{x}_i\,\partial_i f\right) + f\sum_{i=1}^n \hat{x}_i\,\partial_i g$$

$$= g\,\nabla f + f\,\nabla g.$$

hence,

$$\oint_C (\nabla h)\cdot d\vec{r} = \underline{\oint_C (f\,\nabla g + g\nabla f)\cdot d\vec{r} = 0}.$$