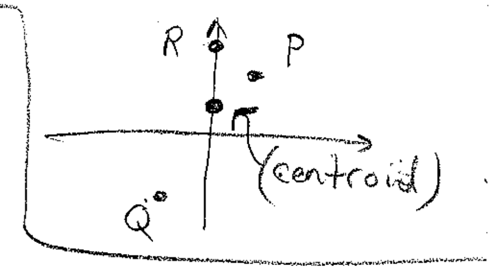
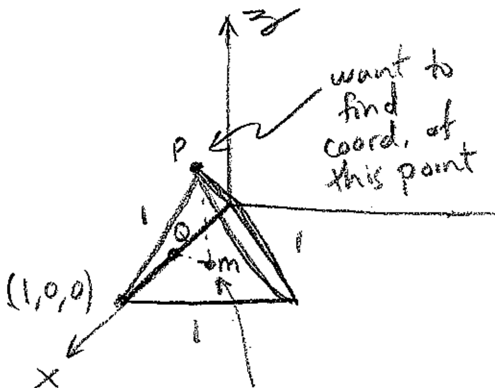
PROBLEM 1 If $(x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{Z}$ then $\boxed{x \in [0,1],\ y \in \mathbb{R},\ z \in \mathbb{Z}}$

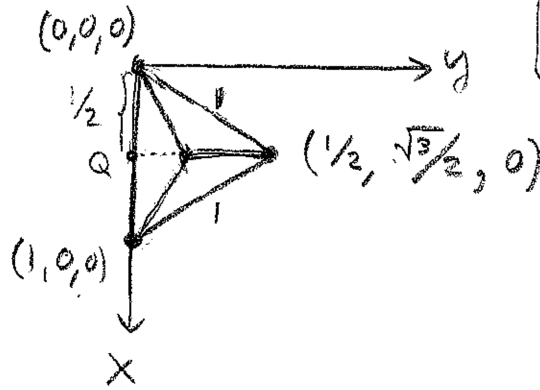
PROBLEM 2 To find midpoint calculate vector average,

$$\frac{1}{3}(P+Q+R) = \frac{1}{3}\big[(1,2)+(-1,-2)+(0,3)\big] = \boxed{(0,1)}$$

PROBLEM 3 Find the height of unit tetrahedron.

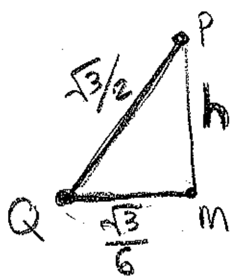


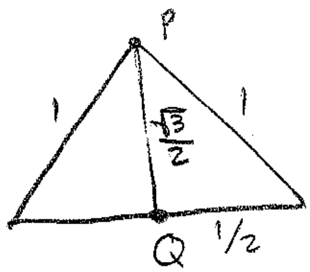




$$m = \frac{1}{3}\Big[(0,0,0)+(1,0,0)+\big(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\big)\Big] = \frac{1}{3}\Big(\frac{3}{2}, \frac{\sqrt{3}}{2}, 0\Big)$$

$$\Rightarrow Q = \frac{1}{3}\Big(\frac{3}{2}, 0, 0\Big)$$

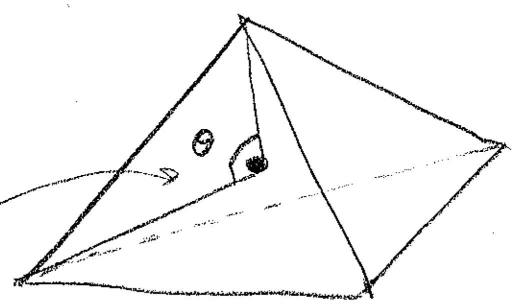




$$h = \sqrt{\Big(\frac{\sqrt{3}}{2}\Big)^2 - \Big(\frac{\sqrt{3}}{6}\Big)^2} = \sqrt{\frac{3}{4} - \frac{3}{36}} = \sqrt{\frac{27-3}{36}} = \sqrt{\frac{2}{3}}$$

Thus $\vec{P} = \Big\langle \frac{1}{2}, \frac{\sqrt{3}}{6}, \sqrt{\frac{2}{3}} \Big\rangle$.

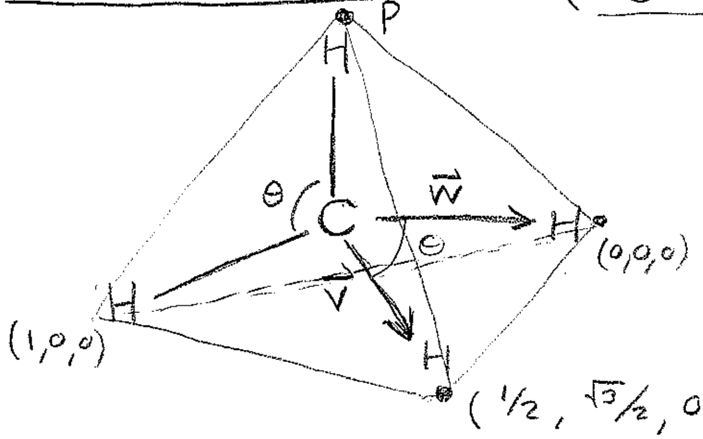
$$\Rightarrow \boxed{\text{Height} = \sqrt{2/3}}$$



this is the angle I had in mind, the angle between the lines connecting the center and two vertices.

PROBLEM 3 Continued

(in Chemistry this angle is of interest
C = CARBON H = HYDROGEN)



C in center of tetrahedron. We can find the position vector by averaging

$$C = \frac{1}{4}\left[\left\langle \frac{1}{2}, \frac{\sqrt{3}}{6}, \sqrt{\frac{2}{3}} \right\rangle + \langle 1,0,0 \rangle + \langle 1/2, \sqrt{3}/2, 0 \rangle + \langle 0,0,0 \rangle\right]$$

$$= \frac{1}{4}\left[\left\langle 2, \sqrt{3}\left(\frac{1}{6}+\frac{1}{2}\right), \sqrt{\frac{2}{3}} \right\rangle\right]$$

$$= \underbrace{\left\langle 1/2, \left(\frac{\sqrt{3}}{4}\right)\left(\frac{4}{6}\right), \frac{1}{4}\sqrt{\frac{2}{3}} \right\rangle}_{\text{center of tetrahedron.}}$$

Let $\vec{v} = (1/2, \sqrt{3}/2, 0) - \left(1/2, \sqrt{3}/6, \frac{1}{4}\sqrt{\frac{2}{3}}\right)$

$\hookrightarrow \vec{v} = \left\langle 0, \frac{2\sqrt{3}}{6}, -\frac{1}{4}\sqrt{\frac{2}{3}} \right\rangle$

$\vec{w} = O - C = \left\langle -1/2, -\sqrt{3}/6, -\frac{1}{4}\sqrt{\frac{2}{3}} \right\rangle$

Note, $\vec{v} \cdot \vec{w} = \frac{-2(\sqrt{3})^2}{6^2} + \frac{1}{16}\left(\frac{2}{3}\right) = \frac{-6}{6^2} + \frac{1}{24} = \frac{-1}{6} + \frac{1}{24} = \underline{\frac{-3}{24}}$.

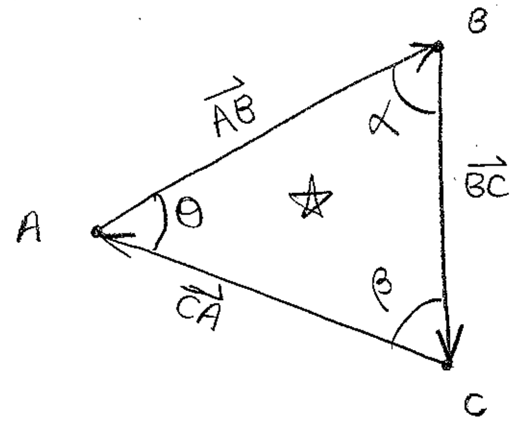
$$\|\vec{v}\| = \sqrt{\frac{4(3)}{36} + \frac{2}{16(3)}} = \sqrt{\frac{12}{36} + \frac{1}{24}} = \sqrt{\frac{1}{3} + \frac{1}{24}} = \sqrt{\frac{9}{24}} = \frac{3}{\sqrt{24}}$$

$$\|\vec{w}\| = \sqrt{\frac{1}{4} + \frac{3}{36} + \frac{2}{48}} = \sqrt{\frac{1}{4} + \frac{1}{12} + \frac{1}{24}} = \sqrt{\frac{9}{24}} = \frac{3}{\sqrt{24}} \quad \text{(duh.)}$$

Thus $\Theta = \cos^{-1}\left[\frac{\vec{v} \cdot \vec{w}}{vw}\right] = \cos^{-1}\left[\frac{-3/24}{9/24}\right] = \cos^{-1}\left[\frac{-1}{3}\right] = \boxed{109.47^\circ}$

PROBLEM 4) Let $A = (1,2,3)$, $B = (1,1,-2)$, $C = (4,4,4)$

(a.)



$$\boxed{\begin{aligned} \overrightarrow{AB} &= B - A = \langle 0,-1,-5\rangle \\ \overrightarrow{BC} &= C - B = \langle 3,3,6\rangle \\ \overrightarrow{CA} &= A - C = \langle -3,-2,-1\rangle \end{aligned}}$$

(b.) $\overrightarrow{AB} + \overrightarrow{BC} = \langle 0,-1,-5\rangle + \langle 3,3,6\rangle = \boxed{\langle 3,2,1\rangle}$

(c.) $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = \langle 3,2,1\rangle + \langle -3,-2,-1\rangle = \underbrace{\boxed{\langle 0,0,0\rangle}}_{\text{makes sense see picture ☆}}$

(d.) $\overrightarrow{AB} \cdot \overrightarrow{AC} = \langle 0,-1,-5\rangle \cdot \langle 3,2,1\rangle$

$= -2 - 5$

$= -7 = \sqrt{26}\sqrt{14}\cos\theta$

$\theta = \cos^{-1}\left(\dfrac{-7}{\sqrt{26(14)}}\right) = \boxed{111.52° = \theta}$

(e.) $\overrightarrow{CA} \cdot \overrightarrow{CB} = \langle -3,-2,-1\rangle \cdot \langle -3,-3,-6\rangle$

$= 9 + 6 + 6$

$= 21 = \sqrt{14}\sqrt{54}\cos\beta$

$\beta = \cos^{-1}\left(\dfrac{21}{\sqrt{14(54)}}\right) = \boxed{40.20° = \beta}$

(f.) $\overrightarrow{BC} \cdot \overrightarrow{BA} = \langle 3,3,6\rangle \cdot \langle 0,1,5\rangle$

$= 3 + 30$

$= 33 = \sqrt{54}\sqrt{26}\cos\alpha$

$\alpha = \cos^{-1}\left(\dfrac{33}{\sqrt{54(26)}}\right) = \boxed{28.27° = \alpha}$

(g.) $\theta + \beta + \alpha = 111.52° + 40.20° + 28.27° = \underline{179.99° \approx 180°}$

YES, the interior angles of a triangle sum to 180°.

PROBLEM 5 Let $\vec{v} = \langle 1, 0, 4 \rangle$ and $\vec{w} = \langle 0, 2, 0 \rangle$.

Note $\vec{v} \cdot \vec{w} = 1(0) + 0(2) + 4(0) = 0$ $\therefore$ $\boxed{\vec{v} \,\&\, \vec{w} \text{ are orthogonal}}$

PROBLEM 6 Let $\vec{v} = \langle 1, 1, 1 \rangle$ and $\vec{w} = 2\hat{y} - \hat{z}$.

$$\begin{aligned}
\text{Proj}_{\vec{w}}(\vec{v}) &= (\vec{v} \cdot \hat{w})\hat{w} \\
&= \left(\frac{\vec{v} \cdot \vec{w}}{w^2}\right)\vec{w} \qquad \text{(happy ?)} \\
&= \left(\frac{\langle 1,1,1 \rangle \cdot \langle 0,2,-1 \rangle}{5}\right)\langle 0, 2, -1 \rangle \\
&= \underline{\tfrac{1}{5}\langle 0, 2, -1 \rangle} \qquad \text{(colinear with } \vec{w}\text{)}
\end{aligned}$$

$$\begin{aligned}
\text{Orth}_{\vec{w}}(\vec{v}) &= \vec{v} - \text{Proj}_{\vec{w}}(\vec{v}) \\
&= \langle 1,1,1 \rangle - \langle 0, 2/5, -1/5 \rangle \\
&= \tfrac{1}{5}\langle 5, 5-2, 5+1 \rangle \\
&= \underline{\langle 1, 3/5, 6/5 \rangle} \quad \text{orthogonal to } \vec{w}
\end{aligned}$$

(Believe It! $\vec{w} \cdot \text{Orth}_w(\vec{v}) = \langle 0,2,-1 \rangle \cdot \langle 1, 3/5, 6/5 \rangle$
$= 0 + 6/5 - 6/5$
$= 0.$

Note $\vec{v} = \underbrace{\langle 0, 2/5, -1/5 \rangle}_{\text{colinear to } \vec{w}} + \underbrace{\langle 1, 3/5, 6/5 \rangle}_{\text{orthogonal to } \vec{w}}.$

PROBLEM 7] Suppose $\vec{A} = \hat{x} + \hat{y}$, $\vec{B} = \hat{z}$, $\vec{C} = \hat{y}$

(a.) $$\begin{aligned}\vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{A} \cdot (\hat{z} \times \hat{y}) \\ &= (\hat{x} + \hat{y}) \cdot (-\hat{x}) \\ &= -\hat{x} \cdot \hat{x} - \cancelto{0}{\hat{y} \cdot \hat{x}} \\ &= -1.\end{aligned}$$

(b.) $$\begin{aligned}\vec{B} \cdot (\vec{A} \times \vec{C}) &= \hat{z} \cdot [(\hat{x} + \hat{y}) \times \hat{y}] \\ &= \hat{z} \cdot [\hat{x} \times \hat{y} + \cancelto{0}{\hat{y} \times \hat{y}}] \\ &= \hat{z} \cdot \hat{z} \\ &= 1.\end{aligned}$$

(c.) Since volume is positive and we discussed in lecture that $Vol = |\vec{A} \cdot (\vec{B} \times \vec{C})|$ it follows calculation (b.) gave $\underline{Vol = 1}$.

PROBLEM 8] Well, put together the items I gave in lecture,

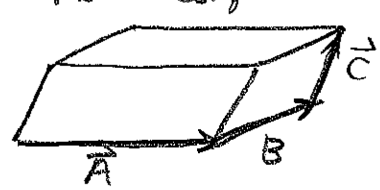
1.) $\det(\vec{A}|\vec{B}|\vec{C}) > 0$ iff $\{\vec{A}, \vec{B}, \vec{C}\}$ is RIGHT-HANDED

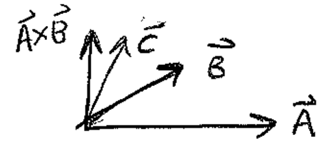
~~2.) $\det(\vec{A}|\vec{B}|\vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})$~~

3.) $Vol = |\vec{A} \cdot (\vec{B} \times \vec{C})|$

It follows $\vec{A} \cdot (\vec{B} \times \vec{C}) = Vol$ provided the vectors $\{\vec{A}, \vec{B}, \vec{C}\}$ are right-handed.

(the diagram in the notes assumes $\vec{A}, \vec{B}, \vec{C}$ are right-handed, this makes $\vec{C}$ on same side as $\vec{A} \times \vec{B}$)

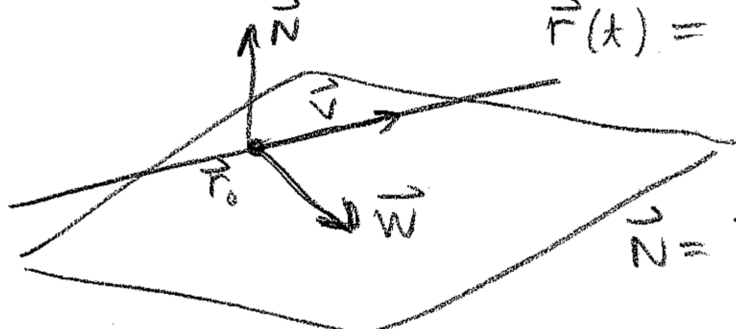




PROBLEM 8) Not that I req^{d} this of you but, perhaps it is of interest,

$$\det\left(\vec{A}\,|\,\vec{B}\,|\,\vec{C}\right) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i B_j C_k$$
$$= \sum_{k=1}^{3} C_k \sum_{i,j=1}^{3} \epsilon_{ijk} A_i B_j$$
$$= \sum_{k=1}^{3} C_k \left(\vec{A}\times\vec{B}\right)_k$$
$$= \underline{\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)}.$$

PROBLEM 9 | Find eq^{n} of plane which contains line parametrized by $\vec{r}(t) = \langle 1+t,\ 2-t,\ 3\rangle$ and the vector $\vec{w} = \langle 1, 2, 3\rangle$



$$\vec{r}(t) = \underbrace{\langle 1, 2, 3\rangle}_{\vec{r}_o} + t\underbrace{\langle 1, -1, 0\rangle}_{\vec{v}}$$

$$\vec{N} = \vec{v}\times\vec{w}$$
$$= \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$
$$= \hat{x}(-3) - \hat{y}(3) + \hat{z}(2+1)$$
$$= \langle -3, -3, 3\rangle$$

$$\therefore\quad \boxed{-3(x-1) - 3(y-2) + 3(z-3) = 0}$$

(used base point $\vec{r}_o$ and normal $\vec{N}$ from above)

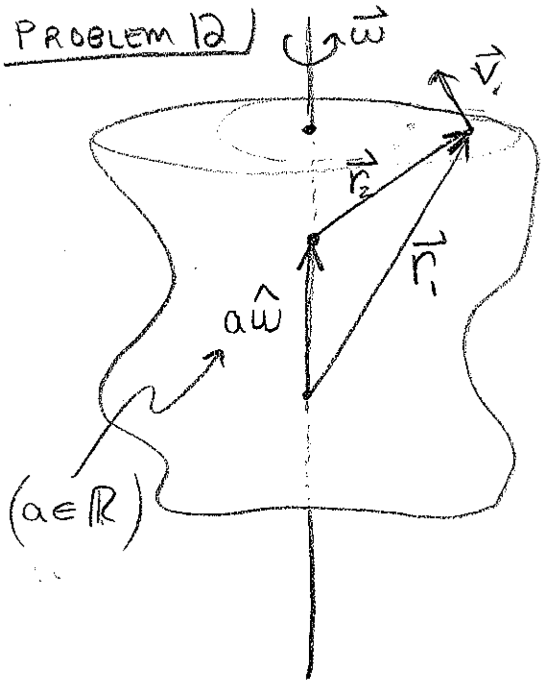
PROBLEM 10) ask me if interested

PROBLEM 11) Suppose $\vec{F} = 100\hat{x}$ and calculate the work done by $\vec{F}$ as a particle moves from $(1,2,3)$ to $(4,4,4)$.

$$W = \vec{F} \cdot \vec{\Delta x} = (100\hat{x}) \cdot \langle 4-1, 4-2, 4-3 \rangle$$
$$= (100\hat{x}) \cdot (3\hat{x} + \underbrace{2\hat{y}}_{0} + \underbrace{\hat{z}}_{0})$$
$$= \boxed{300}$$

[I need to supply direction for $\vec{F}$ otherwise this problem is not as interesting notice when $\vec{F} \perp \vec{\Delta x}$ we find $\vec{F}$ does zero work.]

PROBLEM 12)



Key Observation is
$$\vec{r}_1 = \vec{r}_2 + a\hat{\omega}$$

Note,
$$\vec{\omega} \times \vec{r}_1 = \omega\hat{\omega} \times (\vec{r}_2 + a\hat{\omega})$$
$$= \omega\hat{\omega} \times \vec{r}_2 + \omega a \underbrace{\hat{\omega} \times \hat{\omega}}_{0}$$
$$= \vec{\omega} \times \vec{r}_2.$$

Thus $\vec{v} = \vec{\omega} \times \vec{r}$ is independent of where we base $\vec{r}$ on the $\vec{\omega}$-axis

PROBLEM 13 Direction of line of intersection given by cross product of the normals (I prove in notes)

$$x+y+z=3 \hookrightarrow \vec{n}_1 = \langle 1,1,1\rangle$$
$$2x-3y-4z=7 \hookrightarrow \vec{n}_2 = \langle 2,-3,-4\rangle$$

$$\vec{n}_1\times\vec{n}_2 = \langle -4+3,\ 2+4,\ -3-2\rangle = \underbrace{\boxed{\langle -1,6,-5\rangle}}_{\perp \text{ to } \vec{n}_1 \,\&\, \vec{n}_2 \text{ is good check}}$$

Alternatively

$$x+y+z=3 \qquad 2x-3y-4z=7 \rightarrow \begin{pmatrix} 2x+2y+2z=6 \\ 2x-3y-4z=7 \end{pmatrix}$$

$$-5y-6z=1 \Rightarrow -5y = 1+6z$$

$$5x+5z = 15-5y = 15+1+6z$$

$$\therefore\ 5x+5z = 16+6z$$

$$\underline{5x = z+16}\ . \text{ or } \underbrace{z=5x-16}_{*}$$

Subst. * into $x+y+z=3$

$$x+y+5x-16 = 3 \longrightarrow \underbrace{6x=-y+19}_{**}$$

$$\left.\begin{array}{ll} 6* : & 6z = 30x-96 \\ 5** : & 30x = -5y+95 \end{array}\right\} \quad 6z+\cancel{30x} = \cancel{30x}-5y+1$$

$$6z = -5y-1 \longrightarrow z = \frac{-5}{6}y - \frac{1}{6}$$

Thus, $z = 5x-16 = \frac{-5}{6}y - \frac{1}{6}$

$$\Rightarrow \frac{z}{5} = \frac{5x-16}{5} = \frac{-y-1/5}{6}$$

$$\Rightarrow \frac{z}{5} = \frac{x-16/5}{1} = \frac{y+1/5}{-6}$$

$$\Rightarrow \boxed{\langle 1,-6,5\rangle \text{ direction vector}}$$

PROBLEM 14

$$\left.\begin{array}{ll} \text{I} & x = u+v \\ \text{II} & y = u-v \\ \text{III} & z = 1+u \end{array}\right\} \quad \begin{aligned} \vec{r}(u,v) &= \langle u+v,\, u-v,\, 1+u \rangle \\ &= \underbrace{\langle 0,0,1\rangle + u\langle 1,1,1\rangle + v\langle 1,-1,0\rangle} \end{aligned}$$

plane with point $(0,0,1)$ and containing vectors $\langle 1,1,1\rangle$ and $\langle 1,-1,0\rangle$.

To find Cartesian Eq^{n}'s we can either calc. $\langle 1,1,1\rangle \times \langle 1,-1,0\rangle$ to find normal etc... or simply eliminate u, v from the given triple of eq^{n}'s

$$\text{I} + \text{II} \quad \underbrace{x + y = 2u}_{\text{IV}} \Rightarrow \quad 2\,\text{III} - \text{IV} : 2z - x - y = 2 + \cancel{2u} - \cancel{2u}$$

$$\therefore \boxed{-x - y + 2z = 2} \quad ☆$$

Alternatively: find normal to find eq^{n} from that,

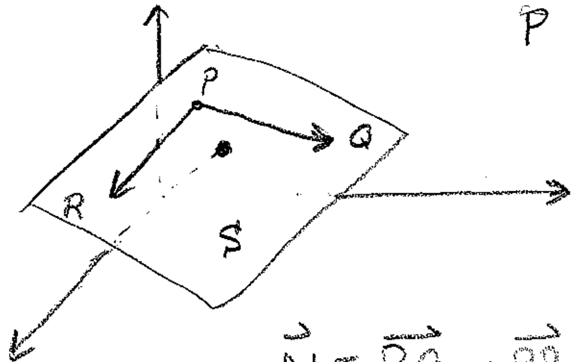
$$\begin{aligned} \langle 1,1,1\rangle \times \langle 1,-1,0\rangle &= (\hat{x} + \hat{y} + \hat{z}) \times (\hat{x} - \hat{y}) \\ &= -\hat{x}\times\hat{y} + \hat{y}\times\hat{x} + \hat{z}\times\hat{x} - \hat{z}\times\hat{y} \\ &= -2\hat{z} + \hat{y} + \hat{x} \end{aligned}$$

$$\hookrightarrow \vec{N} = \langle 1, 1, -2\rangle$$

BASE POINT $(0,0,1)$ & $\vec{N} = \langle 1,1,-2\rangle$

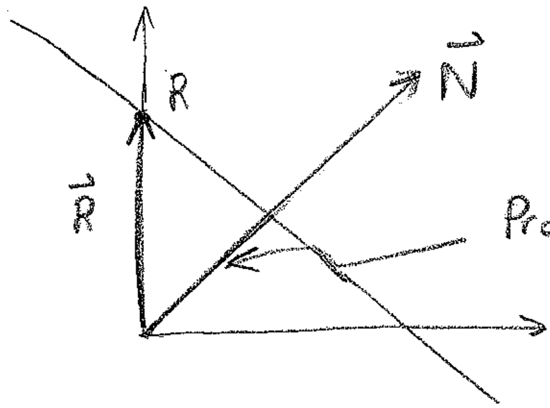
$$\therefore \boxed{x + y - 2(z-1) = 0} \quad (\text{same as } ☆)$$

PROBLEM 15 Suppose $\underset{P}{(1,0,2)}, \underset{Q}{(3,4,1)}, \underset{R}{(0,0,1)} \in S$



$$\overrightarrow{PQ} = Q - P = \langle 2, 4, -1 \rangle$$
$$\overrightarrow{PR} = R - P = \langle -1, 0, -1 \rangle$$

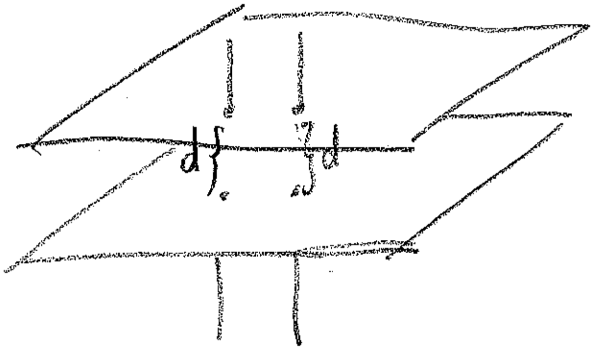
$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -4,\ 1+2,\ 4 \rangle = \langle -4, 3, 4 \rangle$$



$$\operatorname{Proj}_{\vec{N}}(\vec{R}) = (\vec{R} \cdot \hat{N})\hat{N}$$
$$= \left(\frac{\vec{R} \cdot \vec{N}}{N^2}\right)\vec{N}$$
$$= \left(\frac{4}{16+9+16}\right)\langle -4, 3, 4 \rangle$$
$$= \frac{4}{41}\langle -4, 3, 4 \rangle$$
$$= \boxed{\left\langle -16/41,\ 12/41,\ 16/41 \right\rangle}$$

PROBLEM 16 (Problem statement bad)

All points on a pair of parallel planes are equidistant from the point on ∥-plane as connected by normal line.



PROBLEM 17 $\mathfrak{P}$ $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ for some $\vec{a}$. Does it follow $\vec{b} = \vec{c}$?

No. Notice $\vec{0}\cdot\vec{b} = \vec{0}\cdot\vec{c} = 0$ and $\vec{b}$ & $\vec{c}$ could be anything. Other counter examples exist!
Let $\vec{a} = \hat{x}$ then $\hat{x}\cdot\vec{b} = b_1$ and $\hat{x}\cdot\vec{c} = c_1$
note for $\vec{b} = \langle 1, 0, 0\rangle$ and $\vec{c} = \langle 1, 1, 1\rangle$ we have $b_1 = c_1 = 1$ yet $\vec{b} \neq \vec{c}$. There are many correct answers here, they all should expose that a single-dot product will not fix a vector generally.

PROBLEM 18 $\mathfrak{P}$ $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ for all vectors $\vec{a}$. Is $\vec{b} = \vec{c}$?

YES. Let's examine the three-dimensional case.
We're given $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ for all $\vec{a}$ in $\mathbb{R}^3$.

Choose $\vec{a} = \hat{x}$ then $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \hat{x}\cdot\vec{b} = \hat{x}\cdot\vec{c} \Rightarrow \underline{b_1 = c_1}$

Choose $\vec{a} = \hat{y}$ then $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \hat{y}\cdot\vec{b} = \hat{y}\cdot\vec{c} \Rightarrow \underline{b_2 = c_2}$

Choose $\vec{a} = \hat{z}$ then $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \hat{z}\cdot\vec{b} = \hat{z}\cdot\vec{c} \Rightarrow \underline{b_3 = c_3}$

Therefore, $\langle b_1, b_2, b_3\rangle = \langle c_1, c_2, c_3\rangle$ which is $\vec{b} = \vec{c}$.

(the proof for $\mathbb{R}^n$ is similar,
choose $\vec{a} = \hat{x}_j$ and note $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \hat{x}_j\cdot\vec{b} = \hat{x}_j\cdot\vec{c}$

$$\Rightarrow b_j = c_j$$

But, j was arbitrary hence $\vec{b} = \vec{c}$.

(I told you to do it for $\mathbb{R}^3$ since most of you are not comfortable with index notation at this time.)

PROBLEM 19 Suppose $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ for some vector $\vec{a}$, does $\vec{b}=\vec{c}$?

No. Again, one easy counterexample is found from $\vec{0}$ since $\vec{0}\times\vec{b}=\vec{0}$ and $\vec{0}\times\vec{c}=\vec{0}$ but, $\vec{b},\vec{c}$ are arbitrary. For example,

$\vec{0}\times\hat{x}=\vec{0}\times\vec{y}$ yet $\hat{x}\neq\hat{y}$.

(many other counterexamples possible, for example $\langle 1,0,0\rangle\times\langle 2,0,0\rangle=\langle 1,0,0\rangle\times\langle 3,0,0\rangle=\langle 0,0,0\rangle$, etc... there are also examples where $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}\neq\vec{0}$ and yet $\vec{b}\neq\vec{c}$, my examples use $\vec{0}$ since it's easy and we just need one counterexample here.

PROBLEM 20 Suppose $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ for all vectors $\vec{a}$. We'll show $\vec{b}=\vec{c}$ given this data.

Let $\vec{b}=\langle b_1,b_2,b_3\rangle$ and $\vec{c}=\langle c_1,c_2,c_3\rangle$.

Let $\vec{a}=\hat{x}$ and note that

$$\hat{x}\times\vec{b}=\hat{x}\times(b_1\hat{x}+b_2\hat{y}+b_3\hat{z})=b_2\hat{z}-b_3\hat{y}$$

Likewise $\hat{x}\times\vec{c}=c_2\hat{z}-c_3\hat{y}$. Thus $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ in the case $\vec{a}=\hat{x}$ yields $\langle 0,-b_3,b_2\rangle=\langle 0,-c_3,c_2\rangle$ hence, $\boxed{b_3=c_3}$ and $\boxed{b_2=c_2}$.

Next, let $\vec{a}=\hat{y}$ and note that

$$\hat{y}\times\vec{b}=\hat{y}\times(b_1\hat{x}+b_2\hat{y}+b_3\hat{z})=-b_1\hat{z}+b_3\hat{x}$$

Likewise, $\hat{y}\times\vec{c}=-c_1\hat{z}+c_3\hat{x}$ thus we find from $\hat{y}\times\vec{b}=\hat{y}\times\vec{c}$ that $\langle b_3,0,-b_1\rangle=\langle c_3,0,-c_1\rangle$ thus $\boxed{b_1=c_1}$. Therefore, $b_1=c_1$, $b_2=c_2$, $b_3=c_3$ which shows $\vec{b}=\vec{c}$.

<u>Remark</u>: looks like we can weaken the $\forall$ considerably and still obtain $\vec{b}=\vec{c}$.

PROBLEM 21 Suppose $\exists$ nonzero vectors $\vec{v}, \vec{b}$ and a constant c such that $\vec{v} \cdot \vec{x} = c$ and $\vec{v} \times \vec{x} = \vec{b}$. Solve for $\vec{x}$ in terms of $\vec{c}$ and $\vec{b}$

There's doubtless a better sol$^{\underline{n}}$, but I'll use brute-force this time.

$$V_1 X_1 + V_2 X_2 + V_3 X_3 = c$$

$$\langle V_2 X_3 - V_3 X_2,\ V_3 X_1 - V_1 X_3,\ V_1 X_2 - V_2 X_1 \rangle = \langle b_1, b_2, b_3 \rangle$$

The unknowns here are x_1, x_2, x_3 and we have 4 eq$^{\underline{n}}$'s with these 3 unknowns,

$$\begin{bmatrix} V_1 & V_2 & V_3 \\ 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} c \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Since $\det \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix} = 0$ I am certain the last three eq$^{\underline{n}}$'s are dependent on one another. We must use the 1$^{\underline{st}}$ row and two of the last three for best results. Assume $V_3 \neq 0$,

$$\begin{bmatrix} V_1 & V_2 & V_3 \\ 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} c \\ b_1 \\ b_2 \end{bmatrix}$$

We can solve via Kramer's Rule from highschool algebra,

$$X_1 = \frac{\det \begin{bmatrix} c & V_2 & V_3 \\ b_1 & -V_3 & V_2 \\ b_2 & 0 & -V_1 \end{bmatrix}}{\det \begin{bmatrix} V_1 & V_2 & V_3 \\ 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \end{bmatrix}} = \frac{c(V_3 V_1) - V_2(-b_1 V_1 - b_2 V_2) + V_3(b_2 V_3)}{V_1(V_3 V_1) - V_2(-V_2 V_3) + V_3(V_3 V_3)}$$

$$\boxed{X_1 = \frac{c V_1 V_3 + b_1 (V_1 V_2) + b_2 (V_2^2 + V_3^2)}{V_3 (V_1^2 + V_2^2 + V_3^2)}}$$

PROBLEM 21 continued

$$X_2 = \frac{\det\begin{bmatrix} V_1 & c & V_3 \\ 0 & b_1 & V_2 \\ V_3 & b_2 & -V_1 \end{bmatrix}}{V_3(V^2)} = \frac{V_1(-b_1V_1 - b_2V_2) - c(-V_2V_3) + V_3(-b_1V_3)}{V_3V^2}$$

$$\boxed{X_2 = \frac{cV_2V_3 - b_1(V_1^2 + V_1V_3) - b_2V_1V_2}{V_3V^2}}$$

and we can solve for X_3

$$X_3 = \frac{\det\begin{bmatrix} V_1 & V_2 & c \\ 0 & -V_3 & b_1 \\ V_3 & 0 & b_2 \end{bmatrix}}{V_3(V_1^2 + V_2^2 + V_3^2)} = \frac{V_1(-V_3b_2) - V_2(-b_1V_3) + c(V_3^2)}{V_3(V^2)}$$

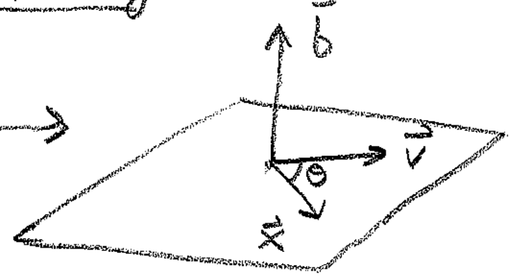
$$\boxed{X_3 = \frac{cV_3^2 + b_1V_2V_3 - b_2V_1V_3}{V_3V^2}}$$

Well, that's a sol^{n} for $V_3 \neq 0$. I'm pretty sure similar sol^{n}'s exist for $V_1 \neq 0$ or $V_2 \neq 0$. But, is there a better sol^{n}? Geometrically:

① $\vec{V} \times \vec{X} = \vec{b}$ $\longrightarrow$

② $\underbrace{\vec{V} \cdot \vec{X} = c}$

$\hookrightarrow \theta = \cos^{-1}\left(\frac{\vec{V}\cdot\vec{X}}{\|\vec{V}\|\|\vec{X}\|}\right)$



In words, $\vec{V} \times \vec{X} = \vec{b}$ fixes a plane where $\vec{x}, \vec{v}$ are vectors in the plane then $\vec{v} \cdot \vec{x} = c$ shows the angle between $\vec{x}$ and $\vec{v}$.

Problem 21 continued

Let us choose coordinates such that $\vec{v} = v\hat{x}$ then the equations $\vec{v} \times \vec{x} = \vec{b}$ and $\vec{v} \cdot \vec{x} = c$ simplify considerably,

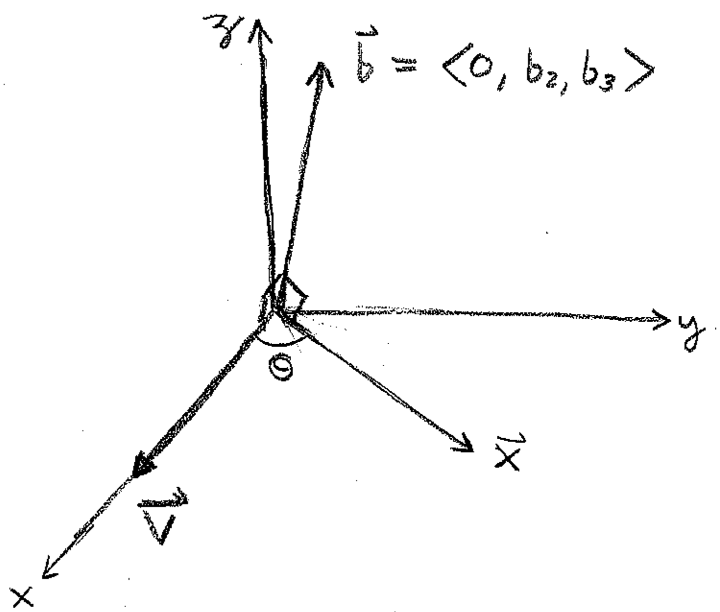
$$\underbrace{(v\hat{x}) \times \langle x_1, x_2, x_3 \rangle}_{v x_2 \hat{z} - v x_3 \hat{y}} = \langle b_1, b_2, b_3 \rangle$$

$$v x_2 \hat{z} - v x_3 \hat{y} = \langle 0, -v x_3, v x_2 \rangle = \langle b_1, b_2, b_3 \rangle$$

It follows that $-v x_3 = b_2$ and $v x_2 = b_3$ $\therefore$ $\underline{x_2 = \frac{b_3}{v},\ x_3 = \frac{-b_2}{v}}$.

Likewise $\vec{v} \cdot \vec{x} = c \Rightarrow (v\hat{x}) \cdot \langle x_1, x_2, x_3 \rangle = v x_1 = c$ $\therefore$ $\underline{x_1 = \frac{c}{v}}$.

We find $\boxed{\vec{x} = \langle c/v,\ b_3/v,\ -b_2/v \rangle}$. Also, note in our coordinates we find $b_1 = 0$ so the picture of what goes on is



$$\Theta = \cos^{-1}\left[\frac{\vec{v} \cdot \vec{x}}{vx}\right] = \cos^{-1}\left[\frac{c}{v\sqrt{c^2/v^2 + b_3^2/v^2 + b_2^2/v^2}}\right]$$

$$= \cos^{-1}\left[\frac{c}{\sqrt{c^2 + b_2^2 + b_3^2 + b_1^2}}\right]$$

$$\Theta = \cos^{-1}\left[\frac{c}{\sqrt{c^2 + b^2}}\right]$$

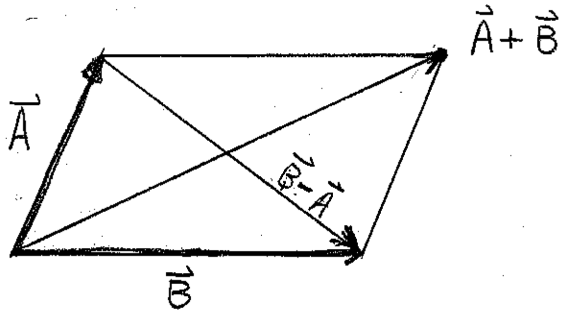
PROBLEM 22 Given $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are <u>coplanar</u> <u>vectors</u> in $\mathbb{R}^3$. Show that $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = 0$

Sol^{n} If $\vec{A}, \vec{B}$ lie in plane S then $\vec{A}\times\vec{B} = k_1\hat{n}$ where $k_1 \in \mathbb{R}$ and $\hat{n}$ is a unit-normal to the plane. Likewise, If $\vec{C}, \vec{D}$ lie in S then $\vec{C}\times\vec{D} = k_2\hat{n}$ for some $k_2 \in \mathbb{R}$.
Thus $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (k_1\hat{n})\times(k_2\hat{n}) = k_1k_2\,\hat{n}\times\hat{n} = 0$.

PROBLEM 23 Show diagonals of a parallelogram are orthogonal iff the parallelogram is a rhombus

all 4 sides equal length. ↑ Def^{n}

Parallelograms have $\parallel$-sides.



From picture & vector addition

diagonal 1 $= \vec{d}_1 = \vec{A}+\vec{B}$

diagonal 2 $= \vec{d}_2 = \vec{B}-\vec{A}$

$\Rightarrow$ Suppose $\vec{d}_1 \perp \vec{d}_2$ then $\vec{d}_1\cdot\vec{d}_2 = 0$

Observe $(\vec{A}+\vec{B})\cdot(\vec{B}-\vec{A}) = \cancel{\vec{A}\cdot\vec{B}} - \vec{A}\cdot\vec{A} + \vec{B}\cdot\vec{B} - \cancel{\vec{B}\cdot\vec{A}} = B^2 - A^2$.

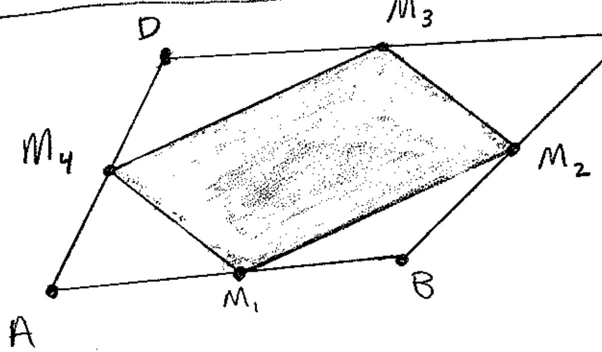
Therefore, $\vec{d}_1\cdot\vec{d}_2 = 0 \Rightarrow B^2 - A^2 = 0 \Rightarrow A = \pm B \Rightarrow \boxed{A = B}$
(since $A, B > 0$)

$\Leftarrow$ Suppose $A = B$ then by the identity
$(\vec{A}+\vec{B})\cdot(\vec{B}-\vec{A}) = B^2 - A^2$ we find $\vec{d}_1\cdot\vec{d}_2 = B^2 - A^2 = A^2 - A^2 = 0$

Therefore, $\vec{d}_1 \perp \vec{d}_2$.

(you could certainly present this as a $\Longleftrightarrow$ argument) but, I thought this might be useful to some of you.

PROBLEM 24 | Let A, B, C, D be points where no 3 lie on a single line. Show that $M_1M_2M_3M_4$ is a ||-gram where M_1, M_2, M_3, M_4 are the midpoints of the quadrilateral ABCD



picture it

GOAL: show $M_1M_2M_3M_4$ is a parallelogram. We should show that $M_4M_3 \parallel M_1M_2$ and $M_1M_4 \parallel M_2M_3$ since that will show $M_1M_2M_3M_4$ is a ||-gram. Midpoint of P and Q is found at $\frac{1}{2}(P+Q)$

$$M_1 = \tfrac{1}{2}(A+B)$$
$$M_2 = \tfrac{1}{2}(B+C)$$
$$M_3 = \tfrac{1}{2}(C+D)$$
$$M_4 = \tfrac{1}{2}(D+A)$$

Thus, the vectors $M_iM_j = M_j - M_i$ are found,

$$M_4M_3 = M_3 - M_4 = \tfrac{1}{2}(C+D) - \tfrac{1}{2}(D+A) = \tfrac{1}{2}(C-A).$$
$$M_1M_2 = M_2 - M_1 = \tfrac{1}{2}(B+C) - \tfrac{1}{2}(A+B) = \tfrac{1}{2}(C-A).$$
$$M_1M_4 = M_4 - M_1 = \tfrac{1}{2}(D+A) - \tfrac{1}{2}(A+B) = \tfrac{1}{2}(D-B).$$
$$M_2M_3 = M_3 - M_2 = \tfrac{1}{2}(C+D) - \tfrac{1}{2}(B+C) = \tfrac{1}{2}(D-B).$$

Therefore $M_4M_3 \parallel M_1M_2$ and $M_1M_4 \parallel M_2M_3$.

QUESTION: why do we need the assumption the triples of points (A, B, C), (A, C, D), (B, C, D), (A, B, D) must all define planes? (another way of saying 3 pts. do not lie on line is to say they define a plane uniquely.

PROBLEM 25 Let $\vec{v} = \langle 2, 4, \sqrt{5} \rangle$

(a.) find angle α relative to $x>0$ axis,
(b.) find angle β relative to $y>0$ axis,
(c.) find angle γ relative to $z>0$ axis,
(d.) what does this say about $\vec{v}$?

(a.) $\vec{v} \cdot \hat{x} = v\cos\alpha = 2$

$$\cos\alpha = \frac{2}{v} = \frac{2}{\sqrt{4+16+5}} = \frac{2}{\sqrt{25}} = \frac{2}{5}.$$

$$\boxed{\cos\alpha = 2/5}$$

(b.) $\vec{v} \cdot \hat{y} = v\cos\beta = 4$

$$\cos\beta = \frac{4}{v} = \frac{4}{5} \quad \therefore \quad \boxed{\cos\beta = \frac{4}{5}}$$

(c.) $\vec{v} \cdot \hat{z} = v\cos\gamma = \sqrt{5}$

$$\cos\gamma = \frac{\sqrt{5}}{5} \quad \therefore \quad \boxed{\cos\gamma = \frac{1}{\sqrt{5}}}$$

(d.) Recall

$$\vec{v} = (\vec{v}\cdot\hat{x})\hat{x} + (\vec{v}\cdot\hat{y})\hat{y} + (\vec{v}\cdot\hat{z})\hat{z}$$
$$= v(\cos\alpha)\hat{x} + v(\cos\beta)\hat{y} + v(\cos\gamma)\hat{z}$$
$$= v\underbrace{\langle \cos\alpha, \cos\beta, \cos\gamma \rangle}_{\hat{v}}$$

$\hat{v}$ = unit-vector for $\vec{v}$
(the direction of $\vec{v}$)