

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

**Problem 1** Your signature below indicates you have:

- (a.) I have read §1.1 – 1.4 of Cook: \_\_\_\_\_.
- (b.) I have attempted homeworks from Salas and Hille as listed below: \_\_\_\_\_.

The following homeworks from the text are good rudimentary skill problems. These are not collected or graded. However, they are all odd problems thus there are answers given within Salas, Hille and Etgen's text:

- § 12.1 #'s 3, 9, 15, 17
- § 12.3 #'s 1, 5, 11, 21, 25, 27, 31, 35, 39
- § 12.4 #'s 3, 5, 9, 11, 13, 21, 29
- § 12.5 #'s 1, 7, 9, 17, 21, 25, 27
- § 12.6 #'s 7, 13, 19, 35
- § 12.7 #'s 3, 5, 9, 21

**Problem 2** Find the midpoint of  $P = (1, 2, 3)$  and  $Q = (5, 5, 0)$ . Then, find the equation of the sphere centered on the midpoint which has  $\overrightarrow{PQ}$  as a diameter.

**Problem 3** Let  $P = (-3, -2)$  and  $Q = (1, 4)$ . Find the Cartesian components of  $\vec{A} = \overrightarrow{PQ}$ . In addition, find the length of  $\vec{A}$ , its direction vector, and its standard angle.

**Problem 4** Suppose a tetrahedron is formed by joining four equilateral triangles of side length 1. If the base face of the tetrahedron is on the  $xy$ -plane and one vertex is at the origin and another at  $(1, 0, 0)$  then find the coordinates of the remaining vertices.

**Problem 5** Prove Proposition 1.1.12 part (4.) then use that result to show  $\|c\vec{A}\| = |c|\|\vec{A}\|$  where  $|c|$  denotes absolute value of  $c$ .

**Problem 6** Consider the triangle formed by the points  $P = (2, 0, 1)$ ,  $Q = (3, 1, 1)$  and  $R = (-3, -3, -3)$ . Find the interior angle at each vertex of the triangle. Hint: use vectors!

**Problem 7** Suppose  $\vec{A}$  is a vector for which  $\vec{A} \cdot \hat{x} = 2$  and  $\vec{A} \cdot \hat{y} = 1$ . If  $\vec{A}$  is perpendicular to  $\vec{B} = \langle 1, 2, 3 \rangle$  then specify the form of  $\vec{A}$  as much as possible. If there are infinitely many cases or no solutions then explain.

**Problem 8** Suppose a vector has magnitude<sup>1</sup>  $v = 10$  and a standard angle of  $\theta = 130^\circ$ . Find the Cartesian components of  $\vec{v}$ . Also, find  $\hat{v}$  for which  $\vec{v} = v\hat{v}$ .

**Problem 9** Let  $\vec{A} = \langle 1, 0, 1 \rangle$  and  $\vec{B} = \langle 2, 2, 2 \rangle$  and  $\vec{C} = \langle 0, 0, 1 \rangle$ . Find the volume of the parallel-piped with edges  $\vec{A}, \vec{B}, \vec{C}$ .

**Problem 10** Let  $\vec{A} = \langle 1, 0, 1 \rangle$  and  $\vec{B} = \langle 2, 2, 2 \rangle$ . Find the area of the parallelogram with sides  $\vec{A}$  and  $\vec{B}$ . Also, find the angle between the given vectors.

**Problem 11** Find the parametric equations of a line through  $P = (2, 3, 4)$  and  $Q = (5, 0, -4)$ . Also, find the symmetric equations of the line.

**Problem 12** Find the parametric equations of a plane containing points  $P = (2, 2, 2)$  and  $Q = (3, 3, 3)$  and  $R = (0, 4, 0)$ . Also, find the Cartesian equations of the plane.

**Problem 13** Find the point on  $2x + 3y - 6z = 10$  which is closest the point  $(7, 7, 7)$  by finding the intersection of the normal line to the plane which goes through  $(7, 7, 7)$ . (it is geometrically clear this gives the closest point, although, later we prove this by other methods)

**Problem 14** Suppose  $A, B, C$  are not all zero and  $\alpha \neq \beta$ . Find the distance between the parallel planes  $Ax + By + Cz + \alpha = 0$  and  $Ax + By + Cz + \beta = 0$ .

**Problem 15** Consider  $\vec{r}(t) = \langle 3 + t, 2 + 5t, -4 + 6t \rangle$ . Does this line intersect the plane  $x + y + z = 10$ . If so, where, and for what value of  $t$ ?

**Problem 16** If  $\vec{A} = \frac{1}{5}\langle 3, 4, 0 \rangle$  and  $\vec{B} = \frac{1}{5}\langle 4, -3, 0 \rangle$ . If  $\{\vec{A}, \vec{B}, \vec{C}\}$  forms a right-handed frame of vectors then find  $\vec{C}$ .

**Problem 17** Show that if  $\vec{A}$  is a vector such that  $\vec{A} \times \hat{x} = 0$  and  $\vec{A} \times \hat{y} = 0$  then  $\vec{A} = 0$ .

**Problem 18** Find  $k$  for which  $(a\vec{A} + b\vec{B}) \times (c\vec{A} + d\vec{B}) = k\vec{A} \times \vec{B}$ .

**Problem 19** Proposition 1.2.11 part (1.). Show that  $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ . Note: if you use the idea of the proof for the other difficult identity there it is much easier than brute force.

**Problem 20** Let  $\hat{u}$  be a unit-vector. Let  $\vec{A}$  be an arbitrary vector. Show that:

$$\vec{A} = [\vec{A} \cdot \hat{u}] \hat{u} + [\hat{u} \times \vec{A}] \times \hat{u}.$$

Then, identify the given formulas with proj and orth operations as discussed in my notes (Definition 1.1.24)

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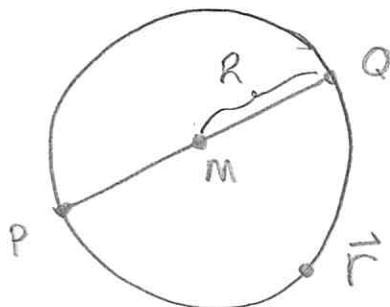
<sup>1</sup>pop pop

# Mission 1 Solution, MATH 231

**PROBLEM 2**  $P = (1, 2, 3)$  and  $Q = (5, 5, 0)$

midpoint is at  $M = \frac{1}{2}(P+Q) = \frac{1}{2}((1, 2, 3) + (5, 5, 0))$

Thus  $M = \frac{1}{2}(6, 7, 3) = \underbrace{(3, 7/2, 3/2)}_{\text{midpt.}}$



center of sphere  
is at midpt. of  
diameter

Equation of sphere:  $\|\vec{r} - M\| = R$  where  $R$  is distance from  $M$  to  $Q$  which is the same as  $\frac{1}{2} \|\overrightarrow{PQ}\| = \frac{1}{2} \|Q - P\| = \frac{1}{2} \|\langle 4, 3, -3 \rangle\| = \frac{\sqrt{34}}{2}$

Therefore,  $\|\vec{r} - (3, 7/2, 3/2)\| = \frac{\sqrt{34}}{2}$  which

for  $\vec{r} = \langle x, y, z \rangle$  given,

$$\sqrt{(x-3)^2 + (y-7/2)^2 + (z-3/2)^2} = \frac{\sqrt{34}}{2}$$

or,  $(x-3)^2 + (y-7/2)^2 + (z-3/2)^2 = \frac{17}{2}$

(I didn't specify how to present the eq<sup>2</sup>  
so there are a few reasonable answers  
here ... )

### PROBLEM 3

$$P = (-3, -2) \quad \& \quad Q = (1, 4)$$

$$\vec{A} = \overrightarrow{PQ} = Q - P = (1, 4) - (-3, -2) = (1, 4) + (3, 2) = (4, 6).$$

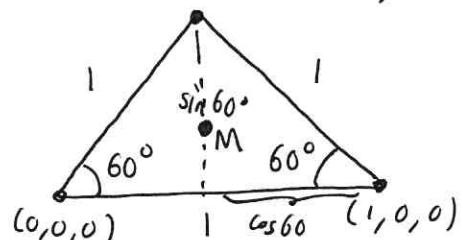
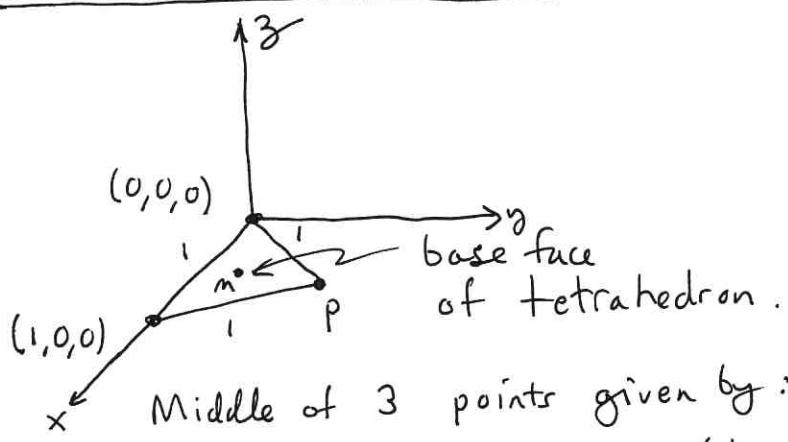
Hence  $\boxed{\vec{A} = \langle 4, 6 \rangle} \Rightarrow \underbrace{A_x = 4}_{\text{Cartesian components.}} \quad \text{and} \quad A_y = 6$

$$\|\vec{A}\| = A = \sqrt{4^2 + 6^2} = \boxed{\sqrt{52}}. \quad (\text{length of } \vec{A} \text{ is } A)$$

$$\hat{A} = \frac{\vec{A}}{A} = \boxed{\frac{1}{\sqrt{52}} \langle 4, 6 \rangle}. \quad (\hat{A} \text{ is direction of } \vec{A})$$

$$\theta = \tan^{-1}(6/4) = \boxed{56.31^\circ}. \quad (\vec{A} \text{ is clearly in quadrant I hence } \tan^{-1} \text{ yields correct } \theta.)$$

### PROBLEM 4 (I to use combination of midpoint idea's generalization to 3 pts. and vector length)

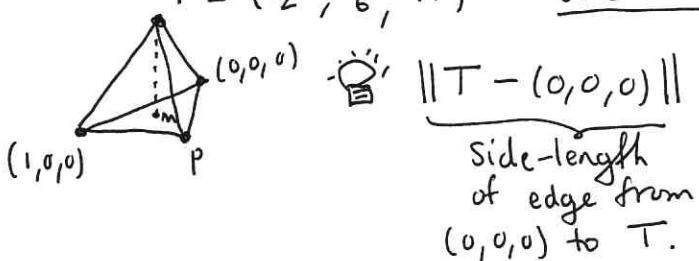


Middle of 3 points given by:

$$M = \frac{1}{3} ((0,0,0) + (1,0,0) + (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, 0\right)$$

xy-coord. of the  
top point T

$$T = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, h\right) \quad \text{we need to find } h.$$



$$\|T - (0,0,0)\| = 1 \rightarrow \frac{1}{4} + \frac{3}{36} + h^2 = 1$$

$$h^2 = 1 - \frac{1}{4} - \frac{1}{12} = \frac{12-3-1}{12}$$

$$h^2 = \frac{8}{12} = \frac{2}{3}$$

$$\therefore h = \sqrt{2/3}$$

$$(0,0,0), (1,0,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{2/3}}{3}\right)$$

### PROBLEM 5

Prove Prop. 1.1.12 part 4 and use that result to show  $\|c\vec{A}\| = |c|\|\vec{A}\|$  where  $|c|$  denotes absolute value of  $c$ .

Show:  $\vec{A} \cdot (c\vec{B}) = (c\vec{A}) \cdot \vec{B} = c(\vec{A} \cdot \vec{B})$  for  $c \in \mathbb{R}$ ,  $\vec{A}, \vec{B} \in \mathbb{R}^n$ .

Proof: note  $\vec{A} \cdot \vec{B} = \sum_{i=1}^n A_i B_i$  is most efficient notation here.

$$\begin{aligned}\vec{A} \cdot (c\vec{B}) &= \sum_{i=1}^n A_i (c\vec{B})_i && : \text{def}^{\text{a}} \text{ of dot-product.} \\ &= \sum_{i=1}^n A_i c B_i && : \text{def}^{\text{a}} \text{ of scalar mult. } c\vec{B} \\ &= c \left( \sum_{i=1}^n A_i B_i \right) && : \text{property of finite sum.} \\ &= c (\vec{A} \cdot \vec{B}) && : \text{again by def}^{\text{a}} \text{ of dot-product.}\end{aligned}$$

Hence  $\vec{A} \cdot (c\vec{B}) = c(\vec{A} \cdot \vec{B})$ . Notice that it then follows

$$(\vec{A}) \cdot \vec{B} = \vec{B} \cdot (\vec{A}) = \underbrace{c(\vec{B} \cdot \vec{A})}_{\substack{\text{using part 1} \\ \text{of Prop. 1.1.12}}} = c(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (c\vec{B}).$$

Hence (4) holds true. Alternatively, could argue

$$\begin{aligned}\vec{A} \cdot (c\vec{B}) &= A_1(c\vec{B})_1 + A_2(c\vec{B})_2 + \dots + A_n(c\vec{B})_n \\ &= A_1 c B_1 + A_2 c B_2 + \dots + A_n c B_n \\ &= c(A_1 B_1 + A_2 B_2 + \dots + A_n B_n) \\ &= c(\vec{A} \cdot \vec{B}) \text{ etc...}\end{aligned}$$

I prefer  $\sum$  to the  $+\dots$  notation for questions like this. Anyway, recall  $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$  thus,

$$\begin{aligned}\|c\vec{A}\|^2 &= (c\vec{A}) \cdot (c\vec{A}) \rightarrow \text{apply (4.) twice} \\ &= c^2 (\vec{A} \cdot \vec{A}) \\ &= c^2 \|\vec{A}\|^2\end{aligned}$$

$$\begin{aligned}\text{However, } c^2, \|c\vec{A}\|^2, \|\vec{A}\|^2 \geq 0 \Rightarrow \|c\vec{A}\| &= \sqrt{c^2} \|\vec{A}\| \\ \Rightarrow \|c\vec{A}\| &= |c| \|\vec{A}\|.\end{aligned}$$

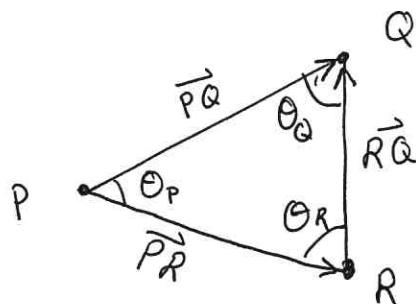
## PROBLEM 6

$$P = (2, 0, 1)$$

$$Q = (3, 1, 1)$$

$$R = (-3, -3, -3)$$

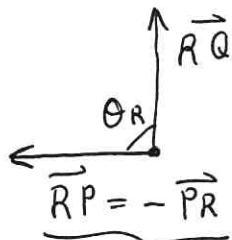
$$\begin{aligned}\overrightarrow{PQ} &= Q - P = \langle 3, 1, 1 \rangle - \langle 2, 0, 1 \rangle = \langle 1, 1, 0 \rangle \\ \overrightarrow{PR} &= R - P = \langle -3, -3, -3 \rangle - \langle 2, 0, 1 \rangle = \langle -5, -3, -4 \rangle \\ \overrightarrow{RQ} &= Q - R = \langle 3, 1, 1 \rangle - \langle -3, -3, -3 \rangle = \langle 6, 4, 4 \rangle\end{aligned}$$



Note:  $\overrightarrow{PQ} \neq \overrightarrow{PR} + \overrightarrow{RQ}$  (checks my work)  
think tip-to-tail vector addition this must hold.

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \|\overrightarrow{PQ}\| \|\overrightarrow{PR}\| \cos \theta_p$$

$$\theta_p = \cos^{-1} \left[ \frac{\langle 1, 1, 0 \rangle \cdot \langle -5, -3, -4 \rangle}{\sqrt{2} \sqrt{25+9+16}} \right] = \cos^{-1} \left[ \frac{-8}{\sqrt{2(50)}} \right] = \boxed{143.13^\circ}$$



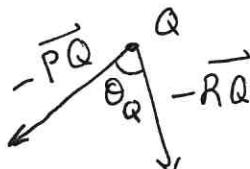
common mistake  
is to forget  
this detail.

$$\overrightarrow{RQ} \cdot (-\overrightarrow{PR}) = \|\overrightarrow{RQ}\| \|\overrightarrow{PR}\| \cos \theta_R$$

$$-\langle 6, 4, 4 \rangle \cdot \langle -5, -3, -4 \rangle = \sqrt{36+16+16} \sqrt{25+9+16} \cos \theta_R$$

$$-(-30 - 12 - 16) = \sqrt{68} \sqrt{50} \cos \theta_R$$

$$\cos \theta_R = \frac{58}{\sqrt{68(50)}} \quad \therefore \quad \theta_R = \cos^{-1} \left[ \frac{58}{\sqrt{68(50)}} \right] = \boxed{5.91^\circ}$$



cancel, so you might be unaware of issue and still correct

$$(-\overrightarrow{PQ}) \cdot (-\overrightarrow{RQ}) = \|\overrightarrow{PQ}\| \|\overrightarrow{RQ}\| \cos \theta_Q$$

$$\underbrace{\langle 1, 1, 0 \rangle \cdot \langle 6, 4, 4 \rangle}_{10} = \sqrt{2} \sqrt{68} \cos \theta_Q$$

careful  
round-off big issue  
here with being  
so close to 1...

$$\theta_Q = \cos^{-1} \left[ \frac{10}{\sqrt{2(68)}} \right] = \boxed{30.96^\circ}$$

Notice:  $\theta_p + \theta_Q + \theta_R = 143.13^\circ + 30.96^\circ + 5.91^\circ = 180^\circ$ .

### PROBLEM 7:

$\vec{A} \cdot \hat{x} = 2$ ,  $\vec{A} \cdot \hat{y} = 1$  and  $\vec{A} \perp \vec{B}$  where  $\vec{B} = \langle 1, 2, 3 \rangle$ .

$$\vec{A} = (\vec{A} \cdot \hat{x})\hat{x} + (\vec{A} \cdot \hat{y})\hat{y} + (\vec{A} \cdot \hat{z})\hat{z} = 2\hat{x} + \hat{y} + c\hat{z} = \langle 2, 1, c \rangle$$

we don't know  $c$ . However,  $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$

$$\text{thus } \langle 2, 1, c \rangle \cdot \langle 1, 2, 3 \rangle = 2 + 2 + 3c = 0$$

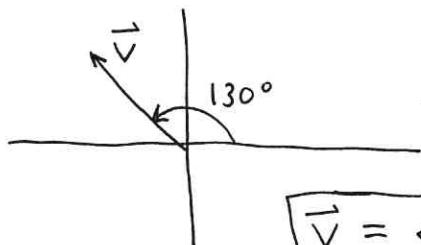
$$\therefore 3c = -4 \text{ and we find } c = -\frac{4}{3} \text{ hence}$$

$$\boxed{\vec{A} = \langle 2, 1, -\frac{4}{3} \rangle}$$

(just one answer)  
just one soln possible  
to the given constraints.

### PROBLEM 8:

Given  $V = 10$  and  $\Theta = 130^\circ$ . Find cartesian components of  $\vec{V}$  and find  $\hat{v}$  for which  $\vec{V} = V\hat{v}$



$$V_x = V \cos 130^\circ = -6.428$$

$$V_y = V \sin 130^\circ = 7.660$$

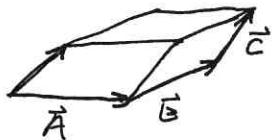
$$\boxed{\vec{V} = \langle -6.428, 7.660 \rangle = 10 \underbrace{\langle -0.6428, 0.7660 \rangle}_{\hat{v}}}$$

### PROBLEM 9:

Let  $\vec{A} = \langle 1, 0, 1 \rangle$ ,  $\vec{B} = \langle 2, 2, 2 \rangle$ ,  $\vec{C} = \langle 0, 0, 1 \rangle$

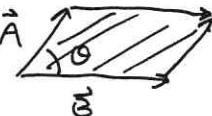
$$\text{Vol}(\vec{A}, \vec{B}, \vec{C}) = \left| \det \begin{bmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right| = \left| 1(2-0) + 1(2(0)-2(0)) \right| = \boxed{2}$$

Same as  $\vec{A} \cdot (\vec{B} \times \vec{C})$   
gives  $\pm \text{Vol}(\vec{A}, \vec{B}, \vec{C})$



PROBLEM 10:

$$\begin{array}{l} \vec{A} = \langle 1, 0, 1 \rangle \\ \vec{B} = \langle 2, 2, 2 \rangle \end{array} \left. \begin{array}{l} \text{find area} \\ \text{and } \theta \end{array} \right\}$$



$$\vec{A} \times \vec{B} = \langle -2, 0, 2 \rangle \quad (\text{as a check, } (\vec{A} \times \vec{B}) \cdot \vec{A} = 0) \\ (\vec{A} \times \vec{B}) \cdot \vec{B} = 0)$$

$$\|\vec{A} \times \vec{B}\| = \sqrt{4+4} = AB \sin \theta$$

$$\sin \theta = \frac{\sqrt{8}}{AB} = \frac{\sqrt{8}}{\sqrt{2}\sqrt{12}} = \sqrt{\frac{8}{24}} = \frac{1}{\sqrt{3}} \Rightarrow \underbrace{\theta = 35.26^\circ}_{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

$$\text{Area} = \|\vec{A} \times \vec{B}\| = \boxed{\sqrt{8}}$$

PROBLEM 11: find line through  $P = (2, 3, 4)$  and  $Q = (5, 0, -4)$   
both in terms of parametrization & as a s.t. set

Standard technique,

symmetric eq's.

$$\vec{r}(t) = P + t(Q-P) \quad \text{has } \vec{r}(0) = P \text{ & } \vec{r}(1) = Q$$

$$= (2, 3, 4) + t \langle 3, -3, -8 \rangle$$

$$= \boxed{\langle 2+3t, 3-3t, 4-8t \rangle = \vec{r}(t)} \quad (\text{for } t \in \mathbb{R})$$

To find symmetric eq's we want two eq's in just  $x, y, z$ . We can solve

$$x = 2+3t, \quad y = 3-3t, \quad z = 4-8t$$

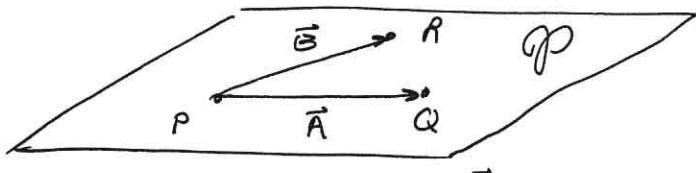
for  $t$ ,

$$t = \frac{x-2}{3} = \frac{y-3}{-3} = \frac{z-4}{-8}$$

Symmetric eq's for line.

Remark:  $Q-P = \langle 3, -3, -8 \rangle$  is ~~a~~ direction vector for the line.

PROBLEM 12:  $P = (2, 2, 2)$ ,  $Q = (3, 3, 3)$ ,  $R = (0, 4, 0)$ . Find parametric & cartesian description of plane



$$\vec{A} = \langle 1, 1, 1 \rangle$$

$$\vec{B} = \langle -2, 2, -2 \rangle$$

$$\vec{r}(u, v) = P + u \underbrace{(Q - P)}_{\vec{A}} + v \underbrace{(R - P)}_{\vec{B}}$$

Note  $\vec{r}(0, 0) = P$ ,  $\vec{r}(1, 0) = Q$ ,  $\vec{r}(0, 1) = R$

$$\boxed{\vec{r}(u, v) = (2, 2, 2) + u \langle 1, 1, 1 \rangle + v \langle -2, 2, -2 \rangle}$$

This gives scalar parametric eq<sup>n</sup>s,

$$x(u, v) = 2 + u - 2v$$

$$y(u, v) = 2 + u + 2v$$

$$z(u, v) = 2 + u - 2v$$

The cartesian eq<sup>n</sup> can be found from eliminating  $u, v$ , but, I recommend a geometric approach.

$\vec{A} \times \vec{B}$  is normal to  $OP$  so it gives the  $x, y, z$  eq<sup>n</sup> with ease.

$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ -2 & 2 & -2 \end{bmatrix} = \langle -4, 0, 4 \rangle$$

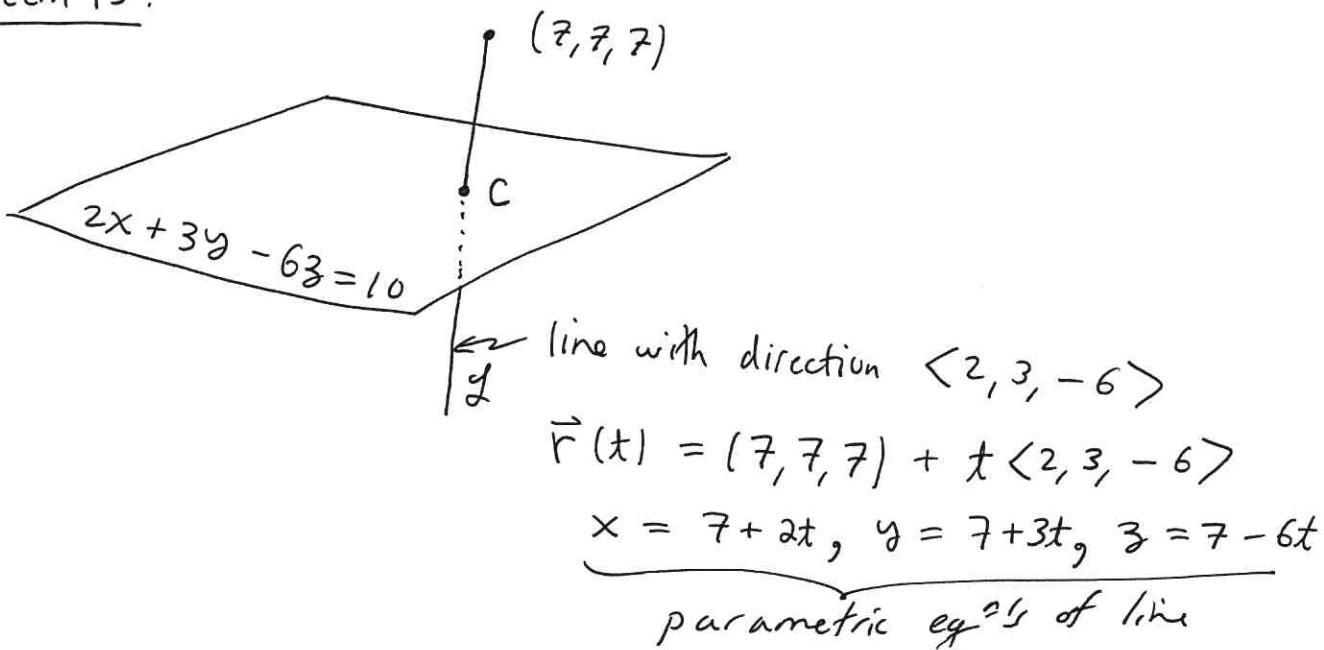
Thus, by the standard Cartesian Eq<sup>n</sup> for plane with normal  $\langle a, b, c \rangle$  and basepoint  $(x_0, y_0, z_0)$

$$\frac{(x_0, y_0, z_0)}{(2, 2, 2)}$$

$$\boxed{-4(x-2) + 4(z-2) = 0}$$

$$\Rightarrow \boxed{x = z}$$

PROBLEM 13:



Substitute into eq<sup>2</sup> for  $S'$  to find intersection of  $S'$  and  $L$

$$2(7+2t) + 3(7+3t) - 6(7-6t) = 10$$

$$14 + 4t + 21 + 9t - 42 + 36t = 10$$

$$49t = 17 \quad \therefore t = 17/49$$

$$\begin{aligned} C = \vec{r}\left(\frac{17}{49}\right) &= \left(7 + 2\left(\frac{17}{49}\right), 7 + 3\left(\frac{17}{49}\right), 7 - 6\left(\frac{17}{49}\right)\right) \\ &= \left(\frac{377}{49}, \frac{394}{49}, \frac{241}{49}\right) \approx (7.694, 8.041, 4.918) \end{aligned}$$

PROBLEM 14:

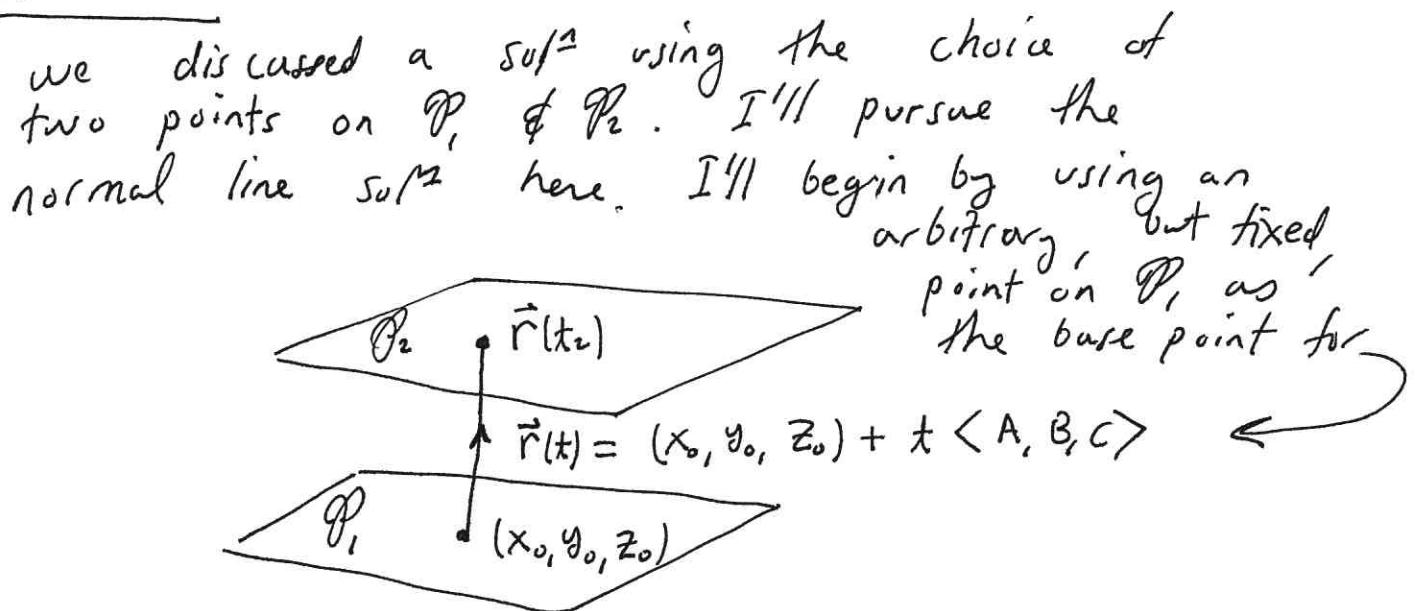
Suppose  $A, B, C$  are constants, not all zero, and  $\alpha, \beta$  are constants such that  $\alpha \neq \beta$ . Find the distance between the parallel planes

$$P_1: Ax + By + Cz + \alpha = 0$$

$$P_2: Ax + By + Cz + \beta = 0$$

sol<sup>1/2</sup> ↗

PROBLEM 14:



$$\vec{r}(t) = (x_0 + tA, y_0 + tB, z_0 + tC)$$

$\vec{r}(t)$  intersects  $P_2$  where

$$A(x_0 + tA) + B(y_0 + tB) + C(z_0 + tC) - \beta = 0$$

$$\Rightarrow \underbrace{Ax_0 + By_0 + Cz_0}_{-\alpha} + t_2(A^2 + B^2 + C^2) = -\beta$$

Thus, as  $(x_0, y_0, z_0) \in P_1$  we obtain,

$$-\alpha + t_2(A^2 + B^2 + C^2) = -\beta$$

$$\text{Thus, } t_2 = \frac{\alpha - \beta}{A^2 + B^2 + C^2} \quad \left( \begin{array}{l} \text{division by } A^2 + B^2 + C^2 \neq 0 \text{ ok} \\ \text{as one (at least) of } A, B, C \text{ is non zero} \end{array} \right)$$

The distance is

simply  $\|\vec{r}(t_2) - (x_0, y_0, z_0)\| \Rightarrow$

$$= \|(x_0, y_0, z_0) + t_2 \langle A, B, C \rangle - (x_0, y_0, z_0)\|$$

$$= |t_2| \|\langle A, B, C \rangle\|$$

$$= \frac{|\alpha - \beta|}{A^2 + B^2 + C^2} \sqrt{A^2 + B^2 + C^2} = \boxed{\frac{|\alpha - \beta|}{\sqrt{A^2 + B^2 + C^2}}}$$

To grader: I allowed them to assume  $C \neq 0$  for their sol.

PROBLEM 15:  $\vec{r}(t) = \langle 3+t, 2+5t, -4+6t \rangle$

Does this line intersect  $x+y+z=10$ ? Where? When?

$$(3+t) + (2+5t) + (-4+6t) = 10 \quad \text{at pt. of intersection}$$

This eq<sup>2</sup> has sol<sup>1</sup> as  $10t+1=10 \Rightarrow t = \frac{9}{10}$  when.

where?  $\boxed{\vec{r}(\frac{9}{10}) = (3.9, 6.5, 1.4)} \quad [\text{set } t=0.9 \text{ in } \vec{r}(t)]$

PROBLEM 16:

$\vec{A} = \frac{1}{5} \langle 3, 4, 0 \rangle, \vec{B} = \frac{1}{5} \langle 4, -3, 0 \rangle$ . If  $\{\vec{A}, \vec{B}, \vec{C}\}$  forms right-handed frame of vectors find  $\vec{C}$

$$\text{By def} \quad \vec{C} = \vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \end{bmatrix} = \langle 0, 0, -\frac{9}{25} - \frac{16}{25} \rangle$$

thus  $\boxed{\vec{C} = \langle 0, 0, -1 \rangle}$

(note,  $\|\vec{A}\| = \|\vec{B}\| = 1$  and  $\vec{A} \cdot \vec{B} = 0$  so we have  
 $\{\vec{A}, \vec{B}, \vec{C}\}$  forms an orthonormal right-handed frame)  
 $\|\vec{C}\|=1$  also.

PROBLEM 17:

Show: if  $\vec{A}$  is vector with  $\vec{A} \times \hat{x} = 0$  and  $\vec{A} \times \hat{y} = 0$   
then  $\vec{A} = 0$

Let  $\vec{A} = \langle a, b, c \rangle$ .

$$\vec{A} \times \hat{x} = (a\hat{x} + b\hat{y} + c\hat{z}) \times \hat{x} = b(\hat{y} \times \hat{x}) + c(\hat{z} \times \hat{x}) = -b\hat{z} + c\hat{y}$$

$$\vec{A} \times \hat{y} = (a\hat{x} + b\hat{y} + c\hat{z}) \times \hat{y} = a(\hat{x} \times \hat{y}) + b(\hat{x} \times \hat{y}) + c(\hat{z} \times \hat{y}) = a\hat{z} - c\hat{x}$$

0 due.

$$\text{Thus } \vec{A} \times \hat{x} = \langle 0, c, -b \rangle = \langle 0, 0, 0 \rangle \Rightarrow \underline{c=0}, \underline{b=0}.$$

$$\text{and } \vec{A} \times \hat{y} = \langle -c, 0, a \rangle = \langle 0, 0, 0 \rangle \Rightarrow \underline{c=0}, \underline{a=0}.$$

Thus  $\vec{A} = \langle 0, 0, 0 \rangle$  as claimed. //

PROBLEM 18: Find  $k$  such that

$$(a \vec{A} + b \vec{B}) \times (c \vec{A} + d \vec{B}) = k \vec{A} \times \vec{B}$$

$$(a \vec{A} + b \vec{B}) \times (c \vec{A} + d \vec{B}) = a \vec{A} \times (c \vec{A} + d \vec{B}) + b \vec{B} \times (c \vec{A} + d \vec{B})$$

$$= ac \cancel{\vec{A} \times \vec{A}} + ad \vec{A} \times \vec{B} + bc \cancel{\vec{B} \times \vec{A}} + bd \vec{B} \times \vec{B}^0$$

$$= ad \vec{A} \times \vec{B} + bc (-\vec{A} \times \vec{B})$$

$$= (ad - bc) \vec{A} \times \vec{B} \quad \therefore [k = ad - bc]$$

PROBLEM 19: Show  $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

We know  $(\vec{B} \times \vec{C})_k = \sum_{i,j} \epsilon_{ijk} B_i C_j$  hence

$$\begin{aligned} [\vec{A} \times (\vec{B} \times \vec{C})]_l &= \sum_{m,n,k} \epsilon_{mnk} A_m (\vec{B} \times \vec{C})_k \\ &= \sum_{m,n} \epsilon_{mnk} A_m \sum_{i,j} \epsilon_{ijk} B_i C_j \\ &= \sum_{m,n,i,j} \epsilon_{mnk} \epsilon_{ijk} A_m B_i C_j \quad \epsilon_{mnk} = -\epsilon_{mlk} \\ &= - \sum_{m,i,j} \left( \sum_k \epsilon_{mlk} \epsilon_{ijk} A_m B_i C_j \right) \quad \text{see of page 36 notes} \\ &= - \sum_{m,i,j} (\delta_{mi} \delta_{lj} - \delta_{mj} \delta_{li}) A_m B_i C_j \\ &= \sum_{m,i,j} \delta_{mj} \delta_{li} A_m B_i C_j - \sum_{m,i,j} \delta_{mi} \delta_{lj} A_m B_i C_j \\ &= \sum_{i,j} \delta_{li} A_j B_i C_j - \sum_{i,j} \delta_{lj} A_i B_i C_j \\ &= \left( \sum_j A_j C_j \right) B_l - \left( \sum_i A_i B_i \right) C_l \\ &= [(\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}]_l \quad \text{and as this holds for } l=1, 2, 3 \text{ we're done.} \end{aligned}$$

Problem 20: Let  $\hat{u}$  be a unit vector.  
 Let  $\vec{A}$  be an arbitrary vector. Show that

$$\vec{A} = [\vec{A} \cdot \hat{u}] \hat{u} + [\hat{u} \times \vec{A}] \times \hat{u}$$

Note, from Prob. 19 we have

$$[\hat{u} \times \vec{A}] \times \hat{u} = -\hat{u} \times [\hat{u} \times \vec{A}] \\ = -(\hat{u} \cdot \vec{A}) \hat{u} + (\hat{u} \cdot \hat{u}) \vec{A}$$

$$\text{Thus, } \vec{A} = \underbrace{(\hat{u} \cdot \vec{A}) \hat{u}}_{\text{Proj}_{\hat{u}}(\vec{A})} + \underbrace{[\hat{u} \times \vec{A}] \times \hat{u}}_{\text{Orth}_{\hat{u}}(\vec{A})}.$$

Remark: without 19, this might be harder...