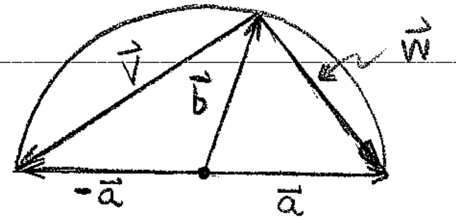
PROBLEM 26 Show a triangle inscribed in a semi-circle must be a right triangle.



By vector addition & picture,

$$\vec{V} = -\vec{b} - \vec{a}$$
$$\vec{W} = -\vec{b} + \vec{a}$$

Consider that
$$\begin{aligned}\vec{V}\cdot\vec{W} &= -(\vec{b}+\vec{a})\cdot(\vec{a}-\vec{b}) \\ &= -[\cancel{\vec{b}\cdot\vec{a}} - \vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a} - \cancel{\vec{a}\cdot\vec{b}}] \\ &= b^2 - a^2 \quad : \text{ using } \|\vec{a}\| = a \ \& \ \|\vec{b}\| = b \\ &= R^2 - R^2 \quad : \text{ since } \vec{a}, \vec{b} \text{ are radial vectors they both are of the radius length } R. \\ &= 0.\end{aligned}$$

Thus $\vec{V} \perp \vec{W}$ and we find the right triangle in the picture.

PROBLEM 27 $\vec{r}(t) = \langle 1+t, 2-3t, 3+4t \rangle$ parametrizes a line. Find where this line intersects $x+y+z=8$

The line has $x = 1+t$, $y = 2-3t$ and $z = 3+4t$ thus substituting into $x+y+z=8$ yields

$$1+t+2-3t+3+4t = 8 \Rightarrow 2t = 2 \Rightarrow \underline{t=1}$$

Thus, $\boxed{\vec{r}(1) = \langle 2, -1, 7 \rangle}$ is where the line intercepts the plane.

PROBLEM 29) $(\vec{A}+\vec{B}) \cdot (\vec{A}\times\vec{B}) = \vec{A}\cdot(\vec{A}\times\vec{B}) + \vec{B}\cdot(\vec{A}\times\vec{B})$

$= 0 + 0$ ← by the construction of $\vec{A}\times\vec{B}$. Recall we sought out a vector $\vec{A}\times\vec{B}$ $\perp$ to both $\vec{A}$ & $\vec{B}$.

$= 0.$

PROBLEM 30) $(\vec{A}-\vec{B})\cdot(\vec{A}+\vec{B}) = \vec{A}\cdot(\vec{A}+\vec{B}) - \vec{B}\cdot(\vec{A}+\vec{B})$ ← dist. prop. of dot products.

$= \vec{A}\cdot\vec{A} + \cancel{\vec{A}\cdot\vec{B}} - \cancel{\vec{B}\cdot\vec{A}} - \vec{B}\cdot\vec{B}$ ∴ commutative $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$= A^2 - B^2$

Thus $(\vec{A}-\vec{B}) \perp (\vec{A}+\vec{B})$ iff $A = B$ (need same magnitude for both $\vec{A}$ & $\vec{B}$)

PROBLEM 31) Use a computer

$\boxed{\text{PROBLEM 32}}$ Identify and plot the surfaces below. (Plots given on next page.)

(a.) $\underbrace{z = x^2+y^2}$

the set of all (x,y,z) satisfying this equation form a paraboloid (a.k.a. elliptical paraboloid) with symmetry axis along z.

(b.) $\underbrace{x^2+y^2-3z^2 = 1}$

Hyperboloid of one sheet, axis of symmetry is z.

Note $\underbrace{x^2+y^2 = 1+3z^2}$ hence $z = \text{const.} \Rightarrow$ circle of radius $\sqrt{1+3z^2}$.

think about it, no matter what z is we find sol^{n}'s for x,y.

(c.) $(x-1)^2+(y+2)^2+(z-3)^2 = 1$

this is a unit-sphere centered at $(1,-2,3)$.

From a vector view, we have the equation for $\vec{r} = \langle x, y, z\rangle$ and $\vec{c} = \langle 1,-2,3\rangle$ that

$\|\vec{r}-\vec{c}\|^2 = 1$. (distance between $\vec{r}$ & $\vec{c}$ is one for all choices of $\vec{r}$... this is a sphere.)

(d.) $x^2+2y^2 = 1$ describes an elliptical cylinder along the z-axis. At each slice $z = \text{const}$ we get an ellipse.

(e.) $x^2+y^2+z^2+2xy+2xz = 1$: oh noes, cross-terms

$\Rightarrow$ non-standard example

$\Rightarrow$ use CAS to plot is only 231-technique to grapple with this.

Forbidden Mathematical Jitzu

$$\det\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2\right] - 1(1-\lambda) + 1(-(1-\lambda))$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 2\right]$$

$$= (1-\lambda)(1-\lambda+\sqrt{2})(1-\lambda+\sqrt{2}) = 0$$

$\lambda_1 = 1 > 0$

$\lambda_2 = 1+\sqrt{2} > 0$

$\lambda_3 = 1-\sqrt{2} < 0$

I deduce this is a hyperboloid of one sheet.

PROBLEM 33 continued

(c.) $(x-1)^2 + (y+2)^2 + (z-3)^2 = 1$

(i.) $z = 3 + \sqrt{1-(x-1)^2-(y+2)^2}$ (only gets top-half)

$$\rightarrow \boxed{\vec{r}(x,y) = \left\langle x,\ y,\ 3+\sqrt{1-(x-1)^2-(y+2)^2}\right\rangle}$$

for $(x,y) \in \mathbb{R}^2$ such that $1-(x-1)^2-(y+2)^2 \geq 0$ that is $(x-1)^2+(y+2)^2 \leq 1$. The $\text{dom}(\vec{r})$ = disk of radius 1 centered at $(1,-2)$.

OR

(ii.)
$$\left.\begin{aligned} x &= 1+\cos\theta\sin\varphi \\ y &= -2+\sin\theta\sin\varphi \\ z &= 3+\cos\varphi \end{aligned}\right\}$$
by analogy to parametrization for sphere given in lecture.

Covers whole sphere. $\rightarrow$
$$\boxed{\begin{gathered}\vec{r}(\varphi,\theta) = \langle 1+\cos\theta\sin\varphi,\ -2+\sin\theta\sin\varphi,\ 3+\cos\varphi\rangle \\ \text{for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq \pi\end{gathered}}$$

<u>Remark</u>: the use of θ & φ above breaks from the standard reserved usage. For example, $z \neq \rho\cos\varphi$ for the φ used above.

(d.) $x^2 + 2y^2 = 1$

I'll choose z as a free parameter and use β to parametrize the ellipse $x^2+2y^2=1$ by setting $x = \cos\beta$ and $y = \frac{1}{\sqrt{2}}\sin\beta$

covers the whole cylinder $\rightarrow$
$$\boxed{\begin{gathered}\vec{r}(z,\beta) = \left\langle \cos\beta,\ \tfrac{1}{\sqrt{2}}\sin\beta,\ z\right\rangle \\ \text{for } z \in \mathbb{R} \text{ and } 0 \leq \beta \leq 2\pi\end{gathered}}$$

PROBLEM 33 Parametrize 32a, b, c, d

- For a given surface there are ∞ly many possible parametrizations! (sorry to the grader this is a "pain")

(a.) $z = x^2 + y^2$ (all of these in (a.) cover the whole paraboloid.)

(i.) $\boxed{\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle}$ for $(x,y) \in \mathbb{R}^2$.

OR

(ii.) $\boxed{\vec{r}(\theta, z) = \langle \sqrt{z}\cos\theta, \sqrt{z}\sin\theta, z \rangle}$ for $z \geq 0$ and $\theta \in [0, 2\pi]$.

OR

(iii.) $\boxed{\vec{r}(\theta, R) = \langle R\cos\theta, R\sin\theta, R^2 \rangle}$ for $R \geq 0$, $\theta \in [0, 2\pi]$.

(b.) $x^2 + y^2 - 3z^2 = 1 \rightsquigarrow z^2 = \dfrac{x^2+y^2-1}{3}$

(i.) $\boxed{\vec{r}(x,y) = \left\langle x, y, \sqrt{\dfrac{x^2+y^2-1}{3}} \right\rangle}$ ← only gets top-half of hyperboloid.

for $(x,y) \in \mathbb{R}^2$ such that $\dfrac{x^2+y^2-1}{3} \geq 0$ which is better written $x^2+y^2 \geq 1$, the domain is the exterior of the unit circle for this choice of $\vec{r}$.

OR

Remark: the formulas for ii. cover the whole sheet

(ii.) Let $x = \cos\theta\cosh\gamma$, $y = \sin\theta\cosh\gamma$ and $z = \frac{1}{\sqrt{3}}\sinh\gamma$ and note,

$$\begin{aligned} x^2+y^2-3z^2 &= \cos^2\theta\cosh^2\gamma + \sin^2\theta\cosh^2\gamma - \sinh^2\gamma \\ &= \cosh^2\gamma - \sinh^2\gamma \\ &= 1. \end{aligned}$$

(which shows my parameters θ, γ are right on target)

$$\boxed{\vec{r}(\theta,\gamma) = \left\langle \cos\theta\cosh\gamma, \sin\theta\cosh\gamma, \tfrac{1}{\sqrt{3}}\sinh\gamma \right\rangle}$$

for $0 \leq \theta \leq 2\pi$ and $\gamma \in \mathbb{R}$

PROBLEM 34 Parametrize $x+3y-z=10$ for which $1 \leq x \leq 3$ and $2 \leq y \leq 4$

Natural to use x, y as parameters. Solve for $z = x+3y-10$ and write

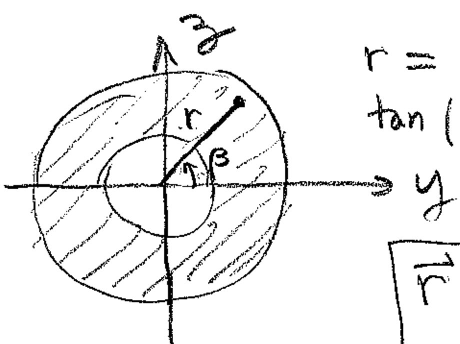
$$\boxed{\begin{array}{l}\vec{r}(x,y) = \langle x,\, y,\, x+3y-10\rangle \\ \text{for } \text{dom}(\vec{r}) = [1,3]\times[2,4]\end{array}}$$

PROBLEM 35 Parametrize $x+3y-z=10$ for which $1 \leq y^2+z^2 \leq 4$

Now we should solve for $x = 10+z-3y$ since y, z parameters.

$$\boxed{\vec{r}(y,z) = \langle 10+z-3y,\, y,\, z\rangle \text{ for } \text{dom}(\vec{r}) = \{(y,z) \mid 1 \leq y^2+z^2 \leq 4\}}$$

However, the domain is a bit ugly and later when we find ourselves integrating you might better appreciate the approach I take below



$$r = \sqrt{y^2+z^2} \qquad y = r\cos\beta$$
$$\tan(\beta) = z/y \qquad z = r\sin\beta$$

$$\underbrace{\boxed{\begin{array}{l}\vec{r}(r,\beta) = \langle 10 + r\sin\beta - 3r\cos\beta,\, r\cos\beta,\, r\sin\beta\rangle \\ \text{for } 1 \leq r \leq 2 \text{ and } 0 \leq \beta \leq 2\pi\end{array}}}$$

later this means we integrate w/o variable bounds. Trust me, a good thing.

$\boxed{\text{PROBLEM 36}}$ Convert to spherical or cylindrical and comment on result

(a.) $\underbrace{1 \le x^2+y^2+z^2 \le 3}$ $\Rightarrow$ $1 \le \rho^2 \le 3$ $\Rightarrow$ $\boxed{1 \le \rho \le \sqrt{3}}$

Spherical shell from $\rho = 1$ to $\rho = \sqrt{3}$.

(b.) $\underbrace{0 \le x^2+y^2 \le 4}$ $\Rightarrow$ $0 \le r^2 \le 4$ $\Rightarrow$ $\boxed{0 \le r \le 2}$

solid cylindrical region of radius 2.

Remark: the following were not what I wanted, but the wording allows for

(a.) $1 \le r^2 + z^2 \le 3$

(b.) $0 \le x^2+y^2+z^2-z^2 = \rho^2 - \rho^2\cos^2\phi \le 4$

You can convert to cylindrical / spherical for a/b but those are not good choices here.

$\boxed{\text{PROBLEM 37}}$ Let P be the point $(\sqrt{3}, 1, 2)$.

(a.) find cylindrical coordinates for P,

$$r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6.$$

Thus $\boxed{r = 2,\ \theta = \pi/6,\ z = 2}$

(b.) find spherical coordinates for P,

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3+1+4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \pi/6$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left[\frac{2}{2\sqrt{2}}\right] = \pi/4$$

Hence, $\underbrace{\boxed{\rho = 2\sqrt{2},\ \phi = \pi/4,\ \theta = \pi/6}}$

Spherical coord. for P.

PROBLEM 38 Let $\vec{V} = \langle 1, 2, 3 \rangle$. Find a, b, c in terms of ρ, ϕ, θ such that $\vec{V} = a\hat{\rho} + b\hat{\varphi} + c\hat{\theta}$

Recall that $\hat{\rho}\cdot\hat{\theta} = \hat{\rho}\cdot\hat{\varphi} = \hat{\theta}\cdot\hat{\varphi} = 0$, $\hat{\rho}\cdot\hat{\rho} = \hat{\theta}\cdot\hat{\theta} = \hat{\varphi}\cdot\hat{\varphi} = 1$. Hence,

$$\begin{aligned}
\vec{V}\cdot\hat{\rho} &= (a\hat{\rho} + b\hat{\varphi} + c\hat{\theta})\cdot\hat{\rho} \\
&= a\hat{\rho}\cdot\hat{\rho} + b\hat{\varphi}\cdot\hat{\rho} + c\hat{\theta}\cdot\hat{\rho} \\
&= a.
\end{aligned}$$

Likewise, $\vec{V}\cdot\hat{\varphi} = b$ and $\vec{V}\cdot\hat{\theta} = c$. We need only recall the f-la's for $\hat{\rho}, \hat{\varphi}, \hat{\theta}$ and we'll find the desired a, b, c from a short calculation,

$$\begin{aligned}
\vec{V}\cdot\hat{\rho} &= \langle 1, 2, 3 \rangle \cdot \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi \rangle \\
&= \boxed{\cos\theta\sin\varphi + 2\sin\theta\sin\varphi + 3\cos\varphi = a}
\end{aligned}$$

Next,

$$\begin{aligned}
\vec{V}\cdot\hat{\varphi} &= \langle 1, 2, 3 \rangle \cdot \langle -\cos\varphi\cos\theta, -\cos\varphi\sin\theta, \sin\phi \rangle \\
&= \boxed{-\cos\phi\cos\theta - 2\cos\phi\sin\theta + 3\sin\phi = b}
\end{aligned}$$

Next,

$$\begin{aligned}
\vec{V}\cdot\hat{\theta} &= \langle 1, 2, 3 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \\
&= \boxed{-\sin\theta + 2\cos\theta = c}
\end{aligned}$$

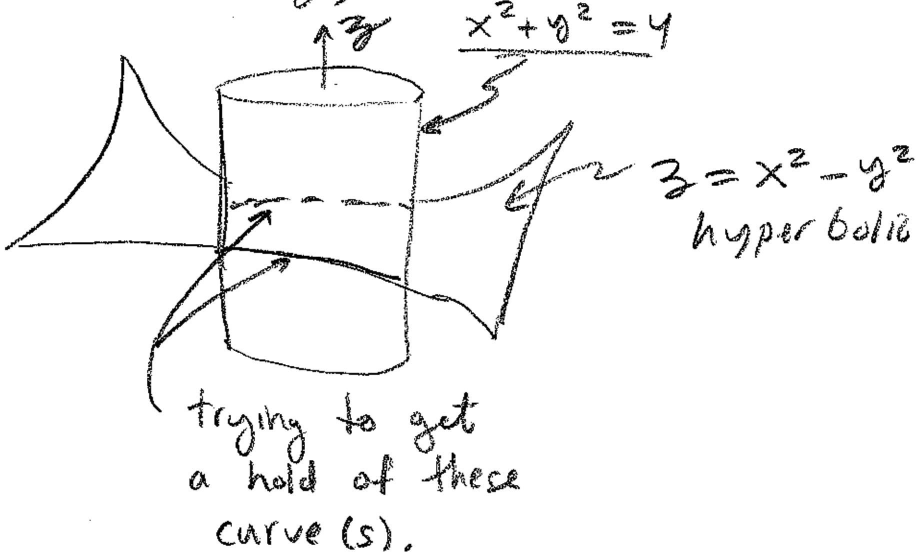
Assemble this all in one piece,

$$\boxed{\begin{aligned}
\vec{V} = &\ (\cos\theta\sin\varphi + 2\sin\theta\sin\varphi + 3\cos\varphi)\,\hat{\rho} \\
&+ (3\sin\phi - \cos\phi\cos\theta - 2\cos\phi\sin\theta)\,\hat{\phi} \\
&+ (2\cos\theta - \sin\theta)\,\hat{\theta}
\end{aligned}}$$

PROBLEM 39 Parametrize the curve of intersection for $x^2+y^2=4$ and $z=x^2-y^2$

Two Eq^{n}s and Three Unknowns $\Rightarrow$ 1 free variable (usually.)

Graphically,



Note, $x^2+y^2=4 \Rightarrow \underbrace{x^2=4-y^2}$ subst. into $z=x^2-y^2$ to obtain $z=4-y^2-y^2$

Thus $\vec{r}_+(y) = \langle \sqrt{4-y^2},\ y,\ 4-2y^2 \rangle$ covers half whereas to get the $x<0$ piece use $\vec{r}_-(y) = \langle -\sqrt{4-y^2},\ y,\ 4-2y^2 \rangle$ both $\text{dom}(\vec{r}_\pm) = [-2,2]$

(this is a sol^{n}, but I think what follows is better)

Idea 1: the curve is on a cylinder so we can trace out by β where $x = 2\cos\beta$ and $y = 2\sin\beta$

Idea 2: to get z just stick $x=2\cos\beta$, $y=2\sin\beta$ into the hyperboloid eq^{n}, $z = x^2-y^2 = 4(\cos^2\beta - \sin^2\beta)$.

trigonometry (that you "know")

$$\boxed{\vec{r}(\beta) = \langle 2\cos\beta,\ 2\sin\beta,\ 4\cos 2\beta \rangle \quad \text{for } 0 \le \beta \le 2\pi}$$

$\boxed{\text{PROBLEM 40}}$ Parametrize intersection of $x+y+z=10$ and $z = x^2+y^2$

<u>Cartesian naive approach:</u> (no offense, but, this creates ugly eq^{n}'s which hurt us later.)

$$x = 10 - y - z$$
$$x = \pm\sqrt{z-y^2}$$

Hence, $10 - y - z = \sqrt{z-y^2}$

$$(10-y-z)^2 = z - y^2$$

$$100 - 20(y+z) + (y+z)^2 = z - y^2$$

$$100 - 20y - 20z + y^2 + 2yz + z^2 = z - y^2$$

$$z^2 + [2y - 21]z + 100 - 20y + y^2 = 0$$

Quadratic in z can solve for $z = z(y)$,

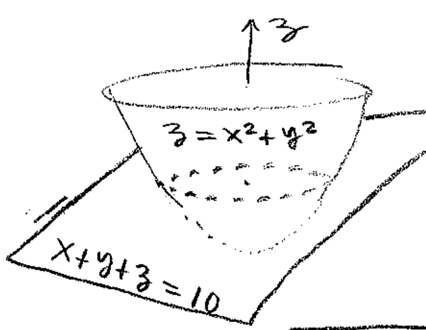
$$z = \frac{21 - 2y \pm \sqrt{(21-2y)^2 - 4(100-20y+y^2)}}{2}$$

Thus,

$\vec{r}(y) = \langle \sqrt{z-y^2}, y, z \rangle$ where z is given as above (choose (+))

parametrizes curve of intersection by y.

OR:



$$x + y + x^2 + y^2 = 10$$

$$\left(x + \tfrac{1}{2}\right)^2 + \left(y + \tfrac{1}{2}\right)^2 = 10 + \tfrac{1}{2} = \tfrac{21}{2}$$

Let $x = \frac{-1}{2} + \sqrt{\frac{21}{2}}\cos\beta$

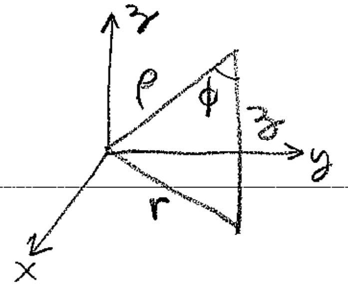
$y = -\frac{1}{2} + \sqrt{\frac{21}{2}}\sin\beta$

$z = 10 - x - y = 11 - \sqrt{\frac{21}{2}}(\cos\beta + \sin\beta)$

Thus, $\boxed{\vec{r}(\beta) = \left\langle \frac{-1}{2} + \sqrt{\frac{21}{2}}\cos\beta, \frac{-1}{2} + \sqrt{\frac{21}{2}}\sin\beta, 11 - \sqrt{\frac{21}{2}}[\cos\beta + \sin\beta] \right\rangle}$

PROBLEM 41 | Convert $4 = \rho \sin\phi$ to cylindrical & cartesian coords.

(a.)



From triangle $\sin\phi = \frac{OPP}{HYP} = \frac{r}{\rho}$ thus,

$r = \rho \sin\phi$

$\therefore \boxed{r = 4}$

(What follows below is already revealed by the picture above)

(b.) $4 = \left(\sqrt{x^2+y^2+z^2}\right)(\sin\phi)$ where $z = \rho\cos\phi$

$= \sqrt{x^2+y^2+z^2}\left[\pm\sqrt{1-\cos^2\phi}\right]$

$= \sqrt{x^2+y^2+z^2}\left[\pm\sqrt{1-\frac{z^2}{\rho^2}}\right]$

$= \sqrt{x^2+y^2+z^2}\left[\pm\sqrt{\frac{\rho^2-z^2}{\rho^2}}\right]$

$= \sqrt{x^2+y^2+z^2}\left[\pm\sqrt{\frac{r^2}{\rho^2}}\right]$

$= \pm\sqrt{r^2}\,\frac{\cancel{\sqrt{\rho^2}}}{\cancel{\sqrt{\rho^2}}}$

$= \pm\sqrt{r^2}$

$= \pm|r| \hookrightarrow \boxed{r = 4}$

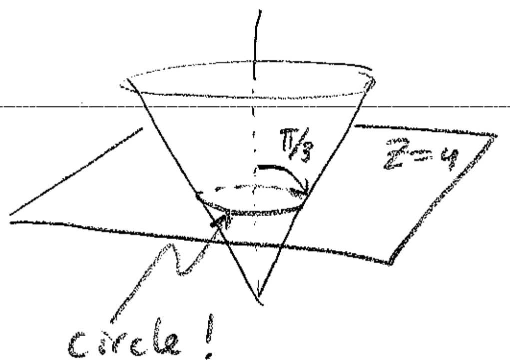
(we allow $0 \le \phi \le \pi$ hence $0 \le \sin\phi \le 1$ cannot have $\rho\sin\phi < 0$ in this convention).

Remark:
It's a pain, but some folks identify
(r, θ) with $(-r, \theta+\pi)$

Anyway, we find $\boxed{\sqrt{x^2+y^2} = 4}$

(didn't need all this work, I'm including it to show you what not to do.)

PROBLEM 42 Find intersection of $\phi = \pi/3$ and $z = 4$ and give a parametrization of this curve



$$z = \rho \cos\phi$$

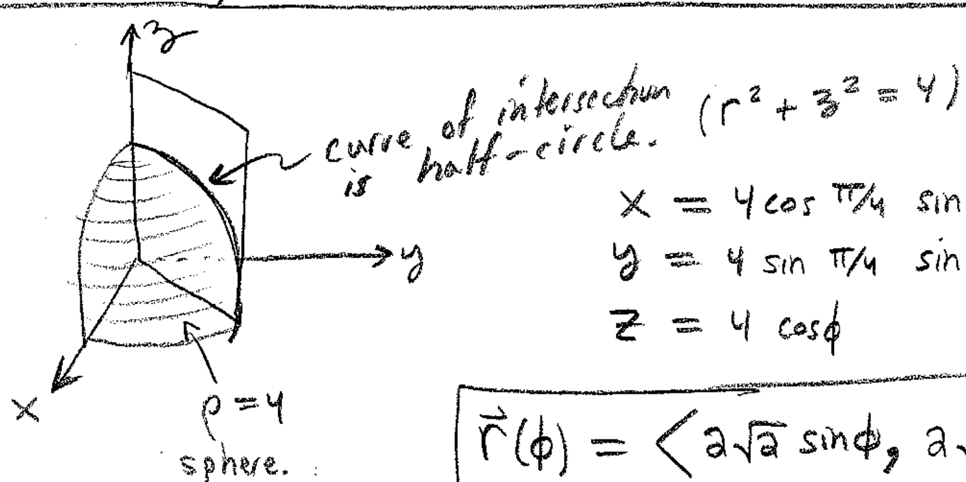
$$4 = \rho \cos(\pi/3) = \rho/2 \Rightarrow \underline{\rho = 8}.$$

$$x = \rho\cos\theta\sin\phi = 8\cos\theta\,(\sqrt{3}/2)$$
$$y = \rho\sin\theta\sin\phi = 8\sin\theta\,(\sqrt{3}/2)$$
$$z = \rho\cos\phi = 4$$

$$\boxed{\vec{r}(\theta) = \langle 4\sqrt{3}\cos\theta,\ 4\sqrt{3}\sin\theta,\ 4\rangle,\quad 0 \le \theta \le 2\pi}$$

__Remark__: this time, θ really is the polar angle. this technique of using spherical coordinates is often useful.

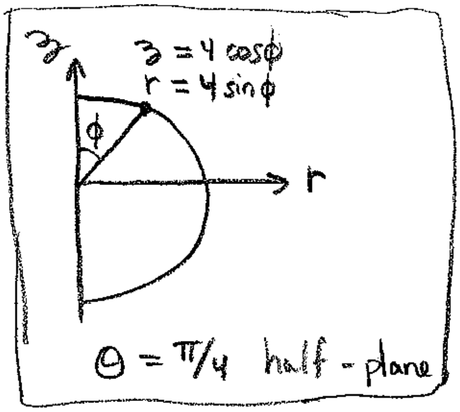
PROBLEM 43 Parametrize intersection of $\theta = \pi/4$ and $\rho = 4$



$$x = 4\cos\tfrac{\pi}{4}\sin\phi$$
$$y = 4\sin\tfrac{\pi}{4}\sin\phi$$
$$z = 4\cos\phi$$

$$\boxed{\begin{gathered}\vec{r}(\phi) = \langle 2\sqrt{2}\sin\phi,\ 2\sqrt{2}\sin\phi,\ 4\cos\phi\rangle \\ 0 < \phi < \pi \quad (\text{or } 0 \le \phi \le \pi)\end{gathered}}$$

(technically $\phi = 0$ & $\phi = \pi$ are where θ is undefined. But, I'd allow = just the same...)



PROBLEM 44 Convert to polar coordinate eq^{n}'s

a.) ellipse $x^2/a^2 + y^2/b^2 = 1$

$$\hookrightarrow \boxed{\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1} \leftarrow \text{ok}$$

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1$$

$$\boxed{r = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}}$$

(assuming $r \geq 0$ convention)

might be useful later.

(b.) $y = 1 - 2x$

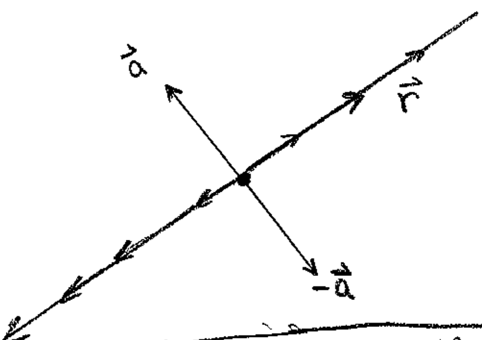
$$\boxed{r\sin\theta = 1 - 2r\cos\theta} \leftarrow \text{ok}$$

$$r(\sin\theta + 2\cos\theta) = 1$$

$$\boxed{r = \frac{1}{\sin\theta + 2\cos\theta}}$$

PROBLEM 45 Characterize the all $\vec{r}$ solving: ($\vec{a} \neq 0$ for both)

(a & b.) $\|\vec{r} - \vec{a}\|^2 = \|\vec{r} + \vec{a}\|^2 \hookrightarrow (\vec{r}-\vec{a})\cdot(\vec{r}-\vec{a}) = (\vec{r}+\vec{a})\cdot(\vec{r}+\vec{a})$



$$\Rightarrow \cancel{\vec{r}\cdot\vec{r}} - \vec{a}\cdot\vec{r} - \vec{r}\cdot\vec{a} + \cancel{\vec{a}\cdot\vec{a}} = \cancel{\vec{r}\cdot\vec{r}} + \vec{a}\cdot\vec{r} + \vec{r}\cdot\vec{a} + \cancel{\vec{a}\cdot\vec{a}}$$

$$\Rightarrow -2\vec{r}\cdot\vec{a} = 2\vec{r}\cdot\vec{a}$$

$$\Rightarrow \vec{r}\cdot\vec{a} = 0$$

$$\Rightarrow \vec{r} \perp \vec{a}.$$

(a.) $\boxed{\text{The set of all } \vec{r} \in \mathbb{R}^2 \text{ such that } \vec{r}\cdot\vec{a} = 0 \text{ is a line through origin with normal line in direction } \vec{a}.}$

(b.) $\boxed{\text{The set of all } \vec{r} \in \mathbb{R}^3 \text{ such that } \vec{r}\cdot\vec{a} = 0 \text{ is a plane through origin with normal vector } \vec{a}}$

PROBLEM 46 Calculate the derivatives.

(a.) $\frac{d}{dt} \langle t^2, e^t, \ln(t) \rangle = \boxed{\langle 2t, e^t, \frac{1}{t} \rangle}.$

(b.) $\frac{d}{dt} \langle \cosh(t^2), \sinh(\ln(t)) \rangle = \boxed{\langle 2t\sinh(t^2), \frac{1}{t}\cosh(\ln(t)) \rangle}$

(c.) $\int \langle 1, t, \sin t \rangle dt = \langle \int dt, \int t\,dt, \int \sin t\,dt \rangle$

$= \boxed{\langle t + c_1, \frac{1}{2}t^2 + c_2, -\cos t + c_2 \rangle}$

PROBLEM 47 Let $\vec{g}, \vec{r}_o, \vec{V}_o$ be constant vectors and suppose $\vec{r}(t) = \vec{r}_o + t\vec{V}_o + \frac{1}{2}t^2\vec{g}$

(a.) $\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\vec{r}_o + t\vec{V}_o + \frac{1}{2}t^2\vec{g}\right)$

$= \underbrace{\frac{d\vec{r}_o}{dt}}_{0} + \vec{V}_o \underbrace{\frac{dt}{dt}}_{1} + \frac{1}{2}\vec{g}\underbrace{\frac{d}{dt}(t^2)}_{2t}$

$= \boxed{\vec{V}_o + t\vec{g}}$ (velocity)

I use $\frac{d\vec{r}_o}{dt} = 0$, $\frac{d\vec{V}_o}{dt} = 0$, $\frac{d\vec{g}}{dt} = 0$ throughout

(b.) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\underbrace{\vec{V}_o}_{0} + t\vec{g}\right) = \boxed{\vec{g}.}$ (acceleration constant)

(c.) $\frac{d^3\vec{r}}{dt} = \frac{d}{dt}\left(\frac{d^2\vec{r}}{dt^2}\right) = \frac{d}{dt}(\vec{g}) = \boxed{\vec{0}.}$ (jerk is ZERO.)

Remark: $\vec{r}(t) = \vec{r}_o + t\vec{V}_o + \frac{1}{2}t^2\vec{g}$ is the equation of motion for a particle with initial position $\vec{r}_o$ and initial velocity $\vec{V}_o$ subject to constant acceleration $\vec{a} = \vec{g}$. We derived $\underbrace{\vec{V}(t) = \vec{V}_o + t\vec{g}}_{\text{velocity at time } t.}$

(We have more to say about physics in the next Problem Set.)

PROBLEM 48 Suppose $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), t/8 \rangle$ for $t \geq 0$.

(a.) How many revolutions does helix make as it goes from $z = 0$ to $z = 1$?

$$0 \leq z \leq 1 \implies 0 \leq t/8 \leq 1$$
$$\implies 0 \leq t \leq 8$$
$$\implies 0 \leq \pi t \leq 8\pi$$

Thus πt goes through $0 \xrightarrow[1]{} 2\pi \xrightarrow[2]{} 4\pi \xrightarrow[3]{} 6\pi \xrightarrow[4]{} 8\pi$ that gives $\boxed{\text{4 complete revs. around the } z\text{-axis}}$

(b.) $$\boxed{\frac{d\vec{r}}{dt} = \left\langle -\pi \sin(\pi t), \pi \cos(\pi t), 1/8 \right\rangle}$$

(c.) Find arclength of this curve from $z=0$ to $z=1$

We learned in (a.) that $0 \leq t \leq 8$.

$$ds = \left\| \frac{d\vec{r}}{dt} \right\| dt = \sqrt{\pi^2 \sin^2 \pi t + \pi^2 \cos^2 \pi t + \frac{1}{64}}\, dt$$

$$ds = \sqrt{\pi^2 + \frac{1}{64}}\, dt$$

$$S = \int_0^8 \sqrt{\pi^2 + \frac{1}{64}}\, dt = \boxed{8\sqrt{\pi^2 + \frac{1}{64}}}$$

PROBLEM 49 Find arclength of $\vec{r}(t) = 2\cos t\,\hat{x} + t\,\hat{y} + 2\sin t\,\hat{z}$ for $0 \leq t \leq 4\pi$. Also find $s = s(t)$ and reparametrize for s

$$\frac{d\vec{r}}{dt} = -2\sin t\,\hat{x} + \hat{y} + 2\cos t\,\hat{z}$$

$$\left\|\frac{d\vec{r}}{dt}\right\| = \sqrt{4\sin^2 t + 1 + 4\cos^2 t} = \sqrt{1+4} = \sqrt{5}$$

$$s(t) = \int_0^t \sqrt{5}\,d\tau = \sqrt{5}\,\tau\Big|_0^t = \sqrt{5}\,t$$

Thus $s(4\pi) = \boxed{4\pi\sqrt{5}}$ ← total arclength

$\boxed{s(t) = t\sqrt{5}}$ ← arclength fnct. of time.

Note $t = s/\sqrt{5}$ and substitute to find

$$\boxed{\vec{r}(t(s)) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\hat{x} + \frac{s}{\sqrt{5}}\hat{y} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\hat{z}}$$

PROBLEM 50 Find arclength from $t=0$ to time t for,

(a.) $\vec{r}(t) = \langle e^{-t}, 1 - e^{-t} \rangle$

$$\frac{d\vec{r}}{dt} = \langle -e^{-t}, e^{-t} \rangle \hookrightarrow ds = \sqrt{(-e^{-t})^2 + (e^{-t})^2}\,dt$$

$$= \sqrt{2(e^{-t})^2}\,dt$$

$$= e^{-t}\sqrt{2}\,dt$$

$$s(t) = \int_0^t \left\|\frac{d\vec{r}}{d\tau}\right\| d\tau$$

$$= \int_0^t e^{-\tau}\sqrt{2}\,d\tau$$

$$= \sqrt{2}\,e^{-\tau}\Big|_0^t$$

$$= \boxed{\sqrt{2}\,(e^{-t} - 1) = s(t)}$$

PROBLEM 50 continued

(b.) $\vec{r}(t) = \langle 2-3t,\ 1+t,\ -4t \rangle$

a.k.a. $x = 2-3t \Rightarrow dx = -3dt$

$y = 1+t \Rightarrow dy = dt$

$z = -4t \Rightarrow dz = -4dt$

$$ds = \sqrt{dx^2+dy^2+dz^2}$$
$$= \sqrt{9dt^2+dt^2+16dt^2}$$
$$= \sqrt{26}\,dt$$

justified below in general

$$s(t) = \int_0^t \sqrt{26}\,d\tau = \sqrt{26}\,\tau\Big|_0^t = t\sqrt{26}.$$

$$\boxed{s(t) = t\sqrt{26}}$$

Remark: the notation $dx = \frac{dx}{dt}dt$ $dy = \frac{dy}{dt}dt$ and $dz = \frac{dz}{dt}dt$ to calculate

$$ds = \sqrt{dx^2+dy^2+dz^2}$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2dt^2 + \left(\frac{dy}{dt}\right)^2dt^2 + \left(\frac{dz}{dt}\right)^2dt^2}$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \sqrt{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}\,dt$$
$$= \left\| \frac{d\vec{r}}{dt} \right\| dt.$$