

PROBLEM SET 3

PROBLEM 51) Let $\vec{r}(t) = \langle R\cos t, 42, R\sin t \rangle$ for $0 \leq t \leq 2\pi$. Calculate, and simplify, the \vec{T} , \vec{N} and \vec{B} fields. Given $R > 0$.

$$\frac{d\vec{r}}{dt} = \langle -R\sin t, 0, R\cos t \rangle$$

$$\left\| \frac{d\vec{r}}{dt} \right\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2} = \sqrt{R^2} = R = \checkmark \text{ speed.}$$

$$\text{Thus, } \vec{T}(t) = \frac{1}{\| \vec{r}'(t) \|} \frac{d\vec{r}}{dt} = \boxed{\langle -\sin t, 0, \cos t \rangle} = \vec{T}(t)$$

Calculate $\vec{T}'(t) = \langle -\cos t, 0, -\sin t \rangle$ note

$$\| \vec{T}'(t) \| = 1 \text{ hence } \boxed{\vec{N}(t) = \langle -\cos t, 0, -\sin t \rangle}$$

$$\begin{aligned} \vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) = \langle -\sin t, 0, \cos t \rangle \times \langle -\cos t, 0, -\sin t \rangle \\ &= \langle 0, 0, \sin^2 t + \cos^2 t \rangle \\ &= \boxed{\langle 0, 0, 1 \rangle} = \vec{B}(t) \end{aligned}$$

PROBLEM 52) Several approaches to calculate κ for previous problem exist. I'll show two

$$\text{in } (t) \quad \boxed{\kappa(t) = \frac{1}{\sqrt{\| \frac{d\vec{r}}{dt} \|}} = \frac{1}{R} \text{ thus } \kappa = 1/R}$$

known formula for curvature *

$$\text{in } s \quad s(t) = \int_0^t \| \frac{d\vec{r}}{du} \| du = R u \Big|_0^t = Rt \Rightarrow t = s/R$$

Hence $\vec{T}(s) = \langle -\sin(s/R), 0, \cos(s/R) \rangle$ then

$$\frac{d\vec{T}}{ds} = \langle -\frac{1}{R} \cos(s/R), 0, -\frac{1}{R} \sin(s/R) \rangle \text{ and}$$

$$\text{we find } \boxed{\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{1}{R} \text{ thus } \kappa = 1/R}$$

definition of curvature.

(* apply chain-rule to get κ from def*)

PROBLEM 53) find torsion τ for curve given in Problem 51

We can either use unit-speed formula $\tau = \frac{d\vec{\theta}}{ds} \cdot \vec{N}$

or use the chain-rule adjusted formula for arbitrary parameter t ; $\tau(t) = \frac{1}{v} \left(\frac{d\vec{\theta}}{dt} \cdot \vec{N}(t) \right)$

unit-speed approach] Recall $\tilde{T}(s) = \langle -\sin(s/R), 0, \cos(s/R) \rangle$

and note $\tilde{N}(s) = \frac{1}{\|\tilde{T}'(s)\|} \frac{d\tilde{T}}{ds} = \langle -\cos(s/R), 0, -\sin(s/R) \rangle$

Calculate the binormal,

$$\tilde{B}(s) = \tilde{T}(s) \times \tilde{N}(s) = \langle 0, 0, 1 \rangle$$

of course, we might well have found this by substitution into $\vec{B}(t)$. (Equally valid means to find $\vec{\theta}$)

In any event, clearly $\frac{d\tilde{B}}{ds} = 0 \Rightarrow \boxed{\tau = 0}$

(generally the torsion is a function of s or t , but here it vanishes identically.)

time t approach] Recall $\vec{B}(t) = \langle 0, 0, 1 \rangle$

$$\text{thus } \tau = \frac{1}{v} \left(\frac{d\vec{\theta}}{dt} \cdot \vec{N} \right) = \frac{1}{R} (\vec{0} \cdot \vec{N}) = 0.$$

Again we find $\boxed{\tau = 0}$

Remark: I introduced unit-speed curves for the logical unambiguity of κ and τ of a given oriented curve. In practice it may be easier to use $B(t) = \frac{1}{v} \left| \frac{dT}{ds} \right|$ and

$$\tau(t) = \frac{1}{v} \left(\frac{d\vec{\theta}}{dt} \cdot \vec{N}(t) \right). \quad \text{There are many other shortcuts I do not attempt to find or promote here.}$$

PROBLEM 54) Find $\vec{T}, \vec{N}, \vec{B}$ for curve given by
 $x = e^{-t} \cos t, y = e^{-t} \sin t, z = e^{-t}$ for $0 \leq t \leq 4\pi$

$$\vec{r}(t) = e^{-t} \langle \cos t, \sin t, 1 \rangle$$

$$\frac{d\vec{r}}{dt} = -e^{-t} \langle \cos t, \sin t, 1 \rangle + e^{-t} \langle -\sin t, \cos t, 0 \rangle$$

$$\frac{d^2\vec{r}}{dt^2} = e^{-t} \langle -\cos t - \sin t, \cos t - \sin t, -1 \rangle$$

$$v = \left\| \frac{d^2\vec{r}}{dt^2} \right\| = e^{-t} \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2 + 1} \quad (\star)$$

$$= e^{-t} \sqrt{\cancel{\cos^2 t} + 2\cancel{\cos t \sin t} + \cancel{\sin^2 t} + \cancel{\cos^2 t} - 2\cancel{\sin t \cos t} + \cancel{\sin^2 t} + 1} \\ = e^{-t} \sqrt{3}$$

Thus, $\vec{T}(t) = \frac{1}{\sqrt{3}} \langle -\cos t - \sin t, \cos t - \sin t, -1 \rangle$

Consequently, $\vec{T}'(t) = \frac{1}{\sqrt{3}} \langle \sin t - \cos t, -\sin t - \cos t, 0 \rangle$

$$\left\| \vec{T}'(t) \right\| = \frac{1}{\sqrt{3}} \sqrt{(\sin t - \cos t)^2 + (\sin t + \cos t)^2} = \sqrt{\frac{2}{3}}$$

Hence, $\vec{N}(t) = \frac{\vec{T}'(t)}{\| \vec{T}'(t) \|} = \frac{1}{\sqrt{2}} \langle \sin t - \cos t, -\sin t - \cos t, 0 \rangle$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \Rightarrow \frac{1}{\sqrt{6}} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos t - \sin t & \cos t - \sin t & -1 \\ \sin t - \cos t & -\sin t - \cos t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}} \left(\hat{x}(-\sin t - \cos t) - \hat{y}(\sin t - \cos t) + \hat{z} \left((\sin t + \cos t)^2 + (\cos t - \sin t)^2 \right) \right)$$

$$\vec{B}(t) = \frac{1}{\sqrt{6}} \langle -\sin t - \cos t, \cos t - \sin t, 2 \rangle$$

algebra (\star)
same calculation.

You can (and should) check $\vec{T} \cdot \vec{B} = \vec{N} \cdot \vec{B} = 0$.

PROBLEM 55 / Calculate κ for curve given in 54

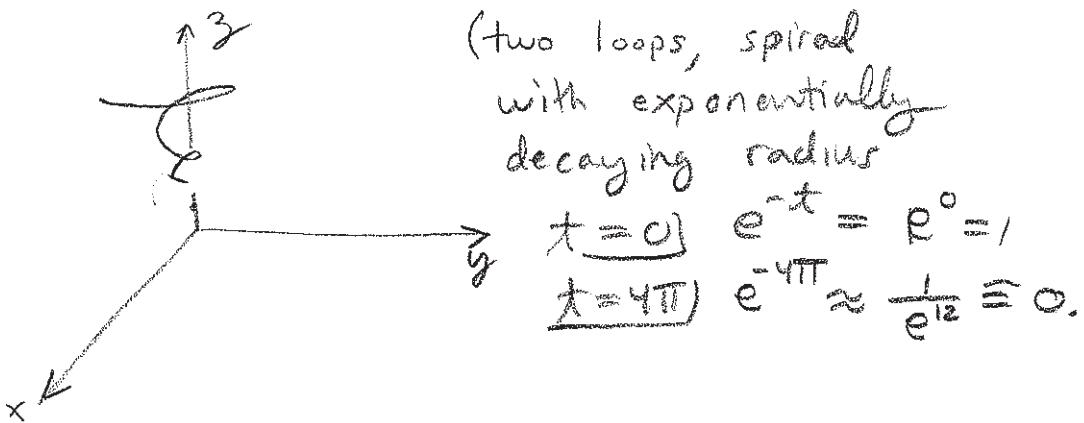
$$\kappa(t) = \frac{1}{\sqrt{\| \frac{d\vec{T}}{dt} \|}} = \frac{1}{e^{-t}\sqrt{3}} \sqrt{\frac{2}{3}} = \boxed{\frac{e^t \sqrt{2}}{\sqrt{3}}} = \kappa(t)$$

(using 54) (torsion as function of t).

PROBLEM 56 / Calculate τ for curve given in 54

$$\begin{aligned} \tau(t) &= \frac{-1}{\sqrt{\| \frac{d\vec{\theta}}{dt} \cdot \vec{N}(t) \|}} \xrightarrow{\text{(using 54)}} \\ &= \frac{-1}{\sqrt{\| \frac{d}{dt} \langle -\sin t - \cos t, \cos t - \sin t, 2 \rangle \|}} \cdot \vec{N}(t) \\ &= \frac{-1}{\sqrt{2} \sqrt{2}} \langle -\cos t + \sin t, -\sin t - \cos t, 0 \rangle \cdot \langle \sin t - \cos t, -\sin t - \cos t, 0 \rangle \\ &= \frac{-1}{\sqrt{2} \sqrt{2}} \left((\sin t - \cos t)^2 + (\sin t + \cos t)^2 + 0 \right) \\ &= \frac{-1}{\sqrt{2} \sqrt{2}} (2) \\ &= \frac{-\sqrt{2}}{e^{-t} \sqrt{3}} \\ &= \boxed{-e^t \sqrt{\frac{2}{3}} = \tau(t)} \end{aligned}$$

By the way, this curve looks something like



Problem 57 Find point on curve $y = \frac{1}{x}$ (for $x > 0$) which maximizes curvature

$$\vec{r}(x) = \langle x, \frac{1}{x}, 0 \rangle$$

$$\frac{d\vec{r}}{dx} = \langle 1, -\frac{1}{x^2}, 0 \rangle \quad \begin{matrix} \text{speed as function} \\ \text{of } x. \end{matrix}$$

$$\left\| \frac{d\vec{r}}{dx} \right\| = \sqrt{1 + \frac{1}{x^4}} = \frac{1}{\sqrt{1 + \frac{1}{x^4}}} = \frac{1}{\sqrt{V(x)}} = V(x)$$

$$\vec{T}(x) = \frac{1}{\left\| \frac{d\vec{r}}{dx} \right\|} \frac{d\vec{r}}{dx} = \underbrace{\frac{1}{\sqrt{1 + \frac{1}{x^4}}}}_{\text{Speed}} \langle 1, -\frac{1}{x^2}, 0 \rangle = \vec{T}(x)$$

$$\vec{T}'(x) = \frac{1}{2} \left(1 + \frac{1}{x^4} \right)^{-3/2} \left(-\frac{4}{x^5} \right) \langle 1, -\frac{1}{x^2}, 0 \rangle +$$

$$+ \left(1 + \frac{1}{x^4} \right)^{-1/2} \langle 0, \frac{2}{x^3}, 0 \rangle$$

$$\vec{T}'(x) = \left\langle \frac{2}{x^5} \left(1 + \frac{1}{x^4} \right)^{-3/2}, \frac{-2}{x^7} \left(1 + \frac{1}{x^4} \right)^{-3/2} + \frac{2}{x^3} \left(1 + \frac{1}{x^4} \right)^{-1/2}, 0 \right\rangle$$

$$= \left(1 + \frac{1}{x^4} \right)^{-3/2} \left\langle \frac{2}{x^5}, \frac{-2}{x^7} + \frac{2}{x^3} \left(1 + \frac{1}{x^4} \right), 0 \right\rangle$$

$$= \left(1 + \frac{1}{x^4} \right)^{-3/2} \left\langle \frac{2}{x^5}, \frac{-2}{x^7} + \frac{2}{x^3} + \frac{2}{x^7}, 0 \right\rangle$$

$$= \frac{2}{x^3} \left(1 + \frac{1}{x^4} \right)^{-3/2} \left\langle \frac{1}{x^2}, 1, 0 \right\rangle$$

$$\left\| \vec{T}'(x) \right\| = \frac{2}{x^3} \left(1 + \frac{1}{x^4} \right)^{-3/2} \underbrace{\sqrt{\frac{1}{x^4} + 1}}_{V(x)} \quad \begin{matrix} \text{cancels} \\ \text{nicely.} \end{matrix}$$

$$K(x) = \frac{1}{V(x)} \left\| \frac{d\vec{T}}{dx} \right\| = \frac{2}{x^3} \left(1 + \frac{1}{x^4} \right)^{-3/2}$$

PROBLEM 57 continued

Maximize $\mathbb{E}(x) = \frac{2}{x^3} \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}}$ for $x > 0$.

Apply first derivative test to critical point(s) of $\mathbb{E}(x)$.
First we need to locate the critical pts,

$$\begin{aligned} \frac{d\mathbb{E}}{dx} &= \frac{-6}{x^4} \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} - \frac{6}{x^3} \left(1 + \frac{1}{x^4}\right)^{-\frac{5}{2}} \left(\frac{-4}{x^5}\right) \\ &= \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \left[\frac{-6}{x^4} + \frac{3}{x^3} \left(\frac{4}{x^5}\right) \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \right] \\ &= \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \left[\frac{-6}{x^4} + \frac{12}{x^8} \left[\frac{1}{1 + \frac{1}{x^4}} \right] \right] \\ &= \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \left[\frac{-6}{x^4} + \frac{12}{x^4} \left(\frac{1}{x^4+1} \right) \right] \\ &= \underbrace{\frac{6}{x^4}}_{\substack{\text{positive} \\ \text{for } x \neq 0}} \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \underbrace{\left[-1 + \frac{3}{x^4+1}\right]}_{\substack{\text{must be source} \\ \text{of any critical pt.}}} \end{aligned}$$

$$\begin{aligned} \text{Observe } -1 + \frac{2}{x^4+1} = 0 &\Rightarrow 2 = x^4 + 1 \\ &\Rightarrow x^4 - 1 = 0 \\ &\Rightarrow (x^2 + 1)(x^2 - 1) = 0 \\ &\Rightarrow (x^2 + 1)(x + 1)(x - 1) = 0 \end{aligned}$$

Critical points at $x = \pm 1$. We consider $x > 0$
Hence $x = 1$ is point of interest. Note,

$$\frac{d\mathbb{E}}{dx} = \frac{6}{x^4} \left(1 + \frac{1}{x^4}\right)^{-\frac{3}{2}} \left[\frac{2 - x^4 - 1}{x^4 + 1} \right]$$

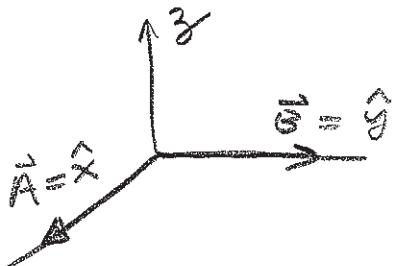
clearly $\mathbb{E}'(1/2) > 0$ whereas $\mathbb{E}'(3/2) < 0$ thus, by
First derivative test we find $\mathbb{E}(1) = 2(2)^{-\frac{3}{2}}$ is max. curvature.

PROBLEM 58 Suppose $\vec{A} \perp \vec{B}$ and $A = B = 1$.

Show that if $\vec{C} = \vec{A} \times \vec{B}$ then $\vec{B} \times \vec{C} = \vec{A}$ and $\vec{C} \times \vec{A} = \vec{B}$.

Andrew's Sol^e: choose coordinates x, y, z such that

(geometric)



\vec{A}, \vec{B} point along x, y -axes.
This is possible since we're given $\vec{A} \perp \vec{B}$ so we know $\theta_{AB} = 90^\circ$.

Moreover, $\vec{C} = \vec{A} \times \vec{B}$ places \vec{C} in $z > 0$ direction and $\|\vec{C}\| = |AB \sin \theta_{AB}| = |1 \cdot 1 \cdot \sin 90^\circ| = 1$ provides that $\vec{C} = \hat{z}$. Thus

$$\vec{B} \times \vec{C} = \hat{y} \times \hat{z} = \hat{x} = \vec{A},$$

$$\vec{C} \times \vec{A} = \hat{z} \times \hat{x} = \hat{y} = \vec{B}.$$

My Sol^e: Once more note $C = \|\vec{A} \times \vec{B}\| = |AB \sin 90^\circ| = 1$.

(algebraic)

$$(\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{C} \cdot \vec{C} = 1 \quad \text{using triple product identity!}$$

Hence $\boxed{\vec{B} \times \vec{C} = \vec{A}}$.

If (note $\|\vec{B} \times \vec{C}\| = |BC \sin \theta_{BC}| = 1 \sin 90^\circ = 1$ so the calculation above shows the component in \vec{A} -direction has length one, but that is the total length of $\vec{B} \times \vec{C}$ so it must be the case that $\vec{B} \times \vec{C} = \vec{A}$.)

Likewise, $(\vec{C} \times \vec{A}) \cdot \vec{B} = (\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{C} \cdot \vec{C} = 1$

$$\Rightarrow \boxed{\vec{C} \times \vec{A} = \vec{B}} \quad \text{as } \|\vec{C} \times \vec{A}\| = CA \sin \theta_{AC} = 1,$$

PROBLEM 59) Show $C_{12} = -C_{21}$ as given on 112-113
of my 2011 Lecture Notes

We derived that

$$\vec{T}' = C_{12} \vec{N}, \quad \vec{N}' = C_{21} \vec{T} + C_{23} \vec{B}, \quad \vec{B}' = -C_{23} \vec{N}$$

Consider, $\vec{T} = \vec{N} \times \vec{B}$ (apply Problem S8 to $\vec{B} = \vec{T} \times \vec{N}$)

$$\vec{T}' = \vec{N}' \times \vec{B} + \vec{N} \times \vec{B}' \quad (\text{product rule for cross products})$$

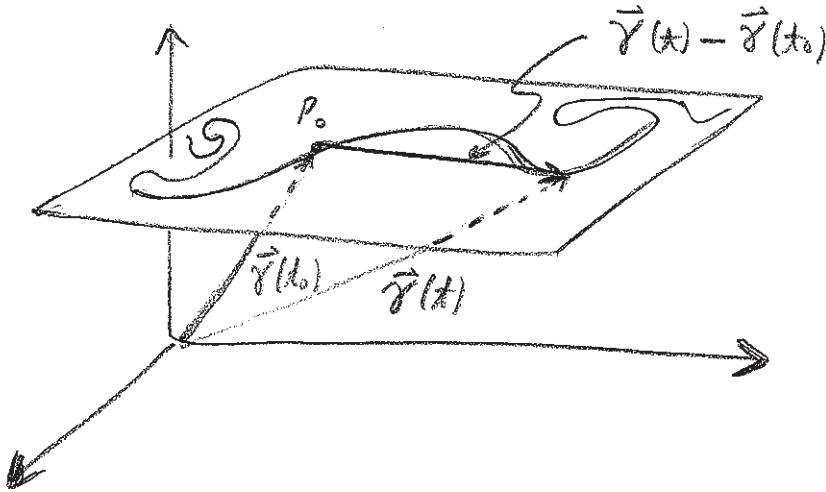
$$\hookrightarrow C_{12} \vec{N} = (C_{21} \vec{T} + C_{23} \vec{B}) \times \vec{B} + \vec{N} \times (-C_{23} \vec{N})$$

$$\Rightarrow C_{12} \vec{N} = C_{21} \vec{T} \times \vec{B} \quad (\text{other terms vanish since } \vec{B} \times \vec{B} = \vec{0} \text{ and } \vec{N} \times \vec{N} = \vec{0})$$

$$\Rightarrow C_{12} \vec{N} = -C_{21} \vec{N} \quad (\text{since } \vec{N} = \vec{B} \times \vec{T} \Rightarrow -\vec{N} = \vec{T} \times \vec{B})$$

$$\Rightarrow \boxed{C_{12} = -C_{21}} \quad (\text{take dot-product of both sides with } \vec{N} \text{ to reveal this equality.})$$

PROBLEM 60) Suppose $\tau(t) = 0 \quad \forall t \in \text{dom}(\vec{\gamma})$ and suppose $t_0 \in \text{dom}(\vec{\gamma})$ hence $\vec{\gamma}(t_0) = P_0 \in \mathbb{R}^3$.



we need to show this vector lies in a plane for all $t \in \text{dom}(\vec{\gamma})$.

This will show the curve is planar.

Continued 2

PROBLEM 60 continued

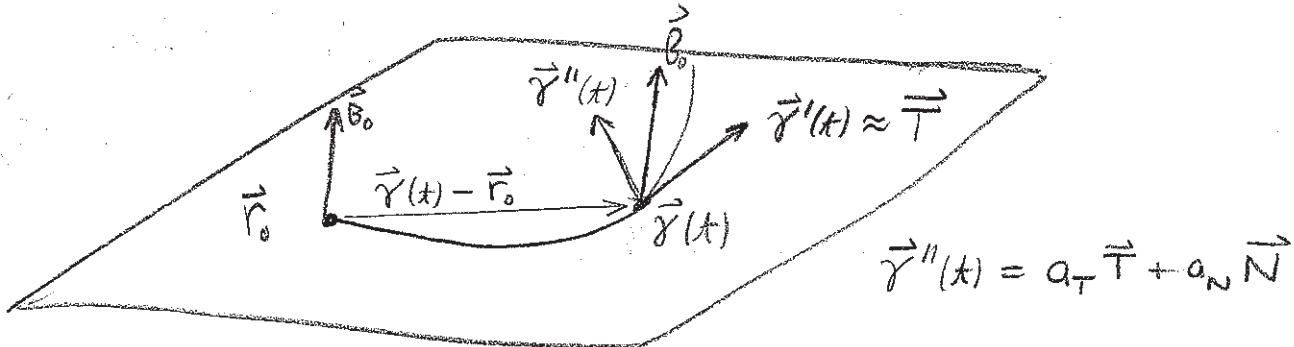
Given $T(t) = \frac{1}{v} \left(\frac{d\vec{B}}{dt} \cdot \vec{N}(t) \right) = 0 \quad \forall t \in \text{dom}(\vec{\gamma}).$

Frenet/Serret gives us $\frac{d\vec{B}}{dt} = -vT\vec{N}$. Thus,

$$\frac{d\vec{B}}{dt} = -v(0)\vec{N} = 0$$

Hence $\vec{B}(t) = \vec{B}(t_0) = \vec{B}_0$ for all $t \in \text{dom}(\vec{\gamma})$.

We suspect this forms a normal to the plane of motion.



$$\vec{\gamma}'' \times \vec{\gamma}' = (a_T \vec{T} + a_N \vec{N}) \times (v \vec{T}) = a_N v \vec{N} \times \vec{T}$$

$$\vec{\gamma}'' \times \vec{\gamma}' = -a_N v \vec{B} \quad (\star)$$

We seek to show $(\vec{\gamma}(t) - \vec{r}_0) \cdot \vec{B}_0 = 0 \quad \forall t \in \text{dom}(\vec{\gamma}).$

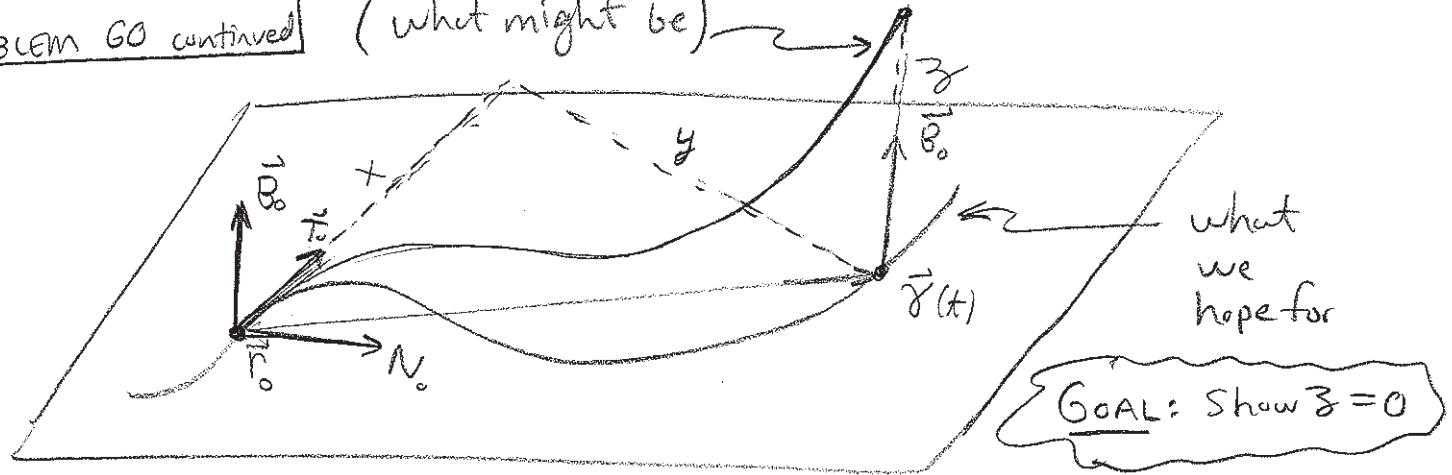
Given (\star) we see it is sufficient to show

$$(\vec{\gamma}(t) - \vec{r}_0) \cdot (\vec{\gamma}''(t_0) \times \vec{\gamma}'(t_0)) = 0.$$

(I pursue a different approach \Rightarrow
I leave these here since
there may be a short-cut
to see in later years...)

PROBLEM GO continued

(what might be)



$$\vec{\gamma}(t) = x \vec{T}_0 + y \vec{N}_0 + z \vec{B}_0$$

Where x, y, z are functions of t . Let $\vec{\gamma}(t_0) = (x_0, y_0, z_0)$. Also $\vec{T}_0 = \vec{T}(t_0)$, $\vec{N}_0 = \vec{N}(t_0)$, $\vec{B}_0 = \vec{B}(t_0)$. (constant vectors)

Differentiate,

$$\vec{\gamma}'(t) = \dot{x} \vec{T}_0 + \dot{y} \vec{N}_0 + \dot{z} \vec{B}_0$$

$$\|\vec{\gamma}'(t)\|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \Rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$\vec{T}(t) = \frac{1}{v} (\dot{x} \vec{T}_0 + \dot{y} \vec{N}_0 + \dot{z} \vec{B}_0)$$

$$\vec{T}'(t) = -\frac{\dot{v}}{v} \frac{1}{v} (\dot{x} \vec{T}_0 + \dot{y} \vec{N}_0 + \dot{z} \vec{B}_0) + \frac{1}{v} (\ddot{x} \vec{T}_0 + \ddot{y} \vec{N}_0 + \ddot{z} \vec{B}_0)$$

$$\vec{T}'(t) = -\frac{\dot{v}}{v} \vec{T}(t) + \frac{1}{v} (\ddot{x} \vec{T}_0 + \ddot{y} \vec{N}_0 + \ddot{z} \vec{B}_0)$$

$$\|\vec{T}'(t)\|^2 = \left(\frac{\ddot{x}v - \dot{x}\dot{v}}{v^2} \right)^2 + \left(\frac{\ddot{y}v - \dot{y}\dot{v}}{v^2} \right)^2 + \left(\frac{\ddot{z}v - \dot{z}\dot{v}}{v^2} \right)^2 = v^2 \kappa^2$$

Recalling $\kappa(t) = \frac{1}{v} \|\vec{T}'(t)\| \hookrightarrow \|\vec{T}'(t)\|^2 = v^2 \kappa^2$. Hence,

$$\vec{N}(t) = \frac{1}{v \kappa} \left[\left(\frac{\ddot{x}v - \dot{x}\dot{v}}{v^2} \right) \vec{T}_0 + \left(\frac{\ddot{y}v - \dot{y}\dot{v}}{v^2} \right) \vec{N}_0 + \left(\frac{\ddot{z}v - \dot{z}\dot{v}}{v^2} \right) \vec{B}_0 \right]$$

Let's denote $\vec{N}(t) = \lambda_x \vec{T}_0 + \lambda_y \vec{N}_0 + \lambda_z \vec{B}_0 \underbrace{\lambda_3}_{\text{etc...}}$

PROBLEM 60 CONT'D

$$\begin{aligned}\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\ &= \left(\frac{\dot{x}}{v} \vec{T}_0 + \frac{\dot{y}}{v} \vec{N}_0 + \frac{\dot{z}}{v} \vec{B}_0 \right) \times \left(\lambda_x \vec{T}_0 + \lambda_y \vec{N}_0 + \lambda_z \vec{B}_0 \right) \\ &= \left(\frac{\dot{y}\lambda_z - \dot{z}\lambda_y}{v} \right) \vec{T}_0 + \left(\frac{\dot{z}\lambda_x - \dot{x}\lambda_z}{v} \right) \vec{N}_0 + \left(\frac{\dot{x}\lambda_y - \dot{y}\lambda_x}{v} \right) \vec{B}_0.\end{aligned}$$

But, we've shown $\vec{B}(t) = \vec{B}_0$ hence we obtain,

$$\begin{aligned}\dot{y}\lambda_z - \dot{z}\lambda_y &= 0 & \lambda_x &= \frac{\ddot{x}v - \dot{x}\dot{v}}{v^2} \\ \dot{z}\lambda_x - \dot{x}\lambda_z &= 0 & \lambda_y &= \frac{\ddot{y}v - \dot{y}\dot{v}}{v^2} \\ \dot{x}\lambda_y - \dot{y}\lambda_x &= 1 & \lambda_z &= \frac{\ddot{z}v - \dot{z}\dot{v}}{v^2}\end{aligned}$$

Well, nothing pops out yet. Let's write it all out,

$$\boxed{\vec{T}_0} \quad \dot{y} \left(\frac{\ddot{z}v - \dot{z}\dot{v}}{v^2} \right) - \dot{z} \left(\frac{\ddot{y}v - \dot{y}\dot{v}}{v^2} \right) = 0$$

$$\cancel{v\dot{y}\ddot{z} - \dot{y}\dot{z}\dot{v}} - \cancel{\dot{z}\dot{y}\dot{v}} - \dot{z}\ddot{y}v + \cancel{\dot{z}\dot{y}\dot{v}} = 0$$
$$\boxed{\dot{y}\ddot{z} - \dot{z}\dot{y} = 0} \quad \text{--- I}$$

$$\boxed{\vec{N}_0} \quad \dot{z} \left(\frac{\ddot{x}v - \dot{x}\dot{v}}{v^2} \right) - \dot{x} \left(\frac{\ddot{z}v - \dot{z}\dot{v}}{v^2} \right) = 0$$

$$\cancel{v\dot{z}\ddot{x} - \dot{z}\dot{x}\dot{v}} - \cancel{\dot{x}\dot{z}\dot{v}} + \dot{x}\ddot{z}v + \cancel{\dot{x}\dot{z}\dot{v}} = 0$$
$$\boxed{\dot{z}\ddot{x} - \dot{x}\dot{z} = 0} \quad \text{--- II}$$

$$\boxed{\vec{B}_0} \quad \dot{x} \left(\frac{\ddot{y}v - \dot{y}\dot{v}}{v^2} \right) - \dot{y} \left(\frac{\ddot{x}v - \dot{x}\dot{v}}{v^2} \right) = 1$$

$$\cancel{\dot{x}\dot{y}v - \dot{y}\dot{x}\dot{v}} - \cancel{\dot{y}\dot{x}\dot{v}} + \dot{x}\ddot{y}v + \cancel{\dot{y}\dot{x}\dot{v}} = v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$
$$\boxed{(\dot{x}\ddot{y} - \dot{y}\dot{x})v = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2} \quad \text{--- III}$$

$$\boxed{\dot{x}\ddot{y} - \dot{y}\dot{x} = v} \quad \text{--- IV}$$

PROBLEM 60 concluded

$$\textcircled{I} \dot{x} \Rightarrow \dot{x}\ddot{y}\ddot{z} = \dot{z}\dot{x}\ddot{y}$$

$$\textcircled{II} \dot{y} \Rightarrow \dot{x}\ddot{y}\ddot{z} = \dot{z}\dot{y}\ddot{x}$$

Subtracting equations,

$$\dot{z}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = 0$$

Apply \textcircled{IV} and find $\dot{z}v = 0$.

If $v = 0$ then the motion is clearly planar.

Otherwise $\dot{z} = 0$. Integrate $\dot{z}(t) = C$, but

$\dot{z}(t_0) = \dot{z}_0$, by construction $\therefore \dot{z}(t) = \dot{z}_0 \quad \forall t \in \text{dom}(\vec{F})$.

We've shown that

$$\overrightarrow{\gamma}(t) = x(t)\overrightarrow{T_0} + y(t)\overrightarrow{N_0}.$$

Now it's trivial to show

$$(\overrightarrow{\gamma}(t) - \overrightarrow{\gamma}(t_0)) \cdot \overrightarrow{B_0} \Rightarrow$$

$$\Rightarrow [(x(t) - x_0)\overrightarrow{T_0} + (y(t) - y_0)\overrightarrow{N_0}] \cdot \overrightarrow{B_0} = 0.$$

Thus $t \mapsto \overrightarrow{\gamma}(t)$ is a curve in
the plane with normal $\overrightarrow{B_0}$.

PROBLEM 61] Suppose $\vec{F}(x, y, z) = F(\rho)\hat{\rho}$. Show that $\vec{F} = m\vec{a}$ implies planar motion

Let $\vec{r}(t)$ denote the position of m which is governed by the law $\vec{F} = m\vec{a} = m \frac{d^2\vec{r}}{dt^2}$ where $\vec{F}(x, y, z) = F(\rho)\hat{\rho}$. Furthermore $\vec{r} = \rho\hat{\rho}$ in our current notation as $\rho = \sqrt{x^2 + y^2 + z^2}$ hence $\vec{F}(x, y, z) = \frac{F(\rho)}{\rho}\rho\hat{\rho} = \frac{F(\rho)}{\rho}\vec{r} = m \frac{d^2\vec{r}}{dt^2}$.

We define $\vec{v} = \frac{d\vec{r}}{dt}$ as usual. Note,

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times (m\vec{v})) &= \frac{d\vec{r}}{dt} \times (m\vec{v}) + \vec{r} \times \left(m \frac{d\vec{v}}{dt}\right) \\ &= m\vec{v} \times \vec{v} + \vec{r} \times \left(\frac{F(\rho)}{\rho}\vec{r}\right) \\ &= \frac{F(\rho)}{\rho}\vec{r} \times \vec{r} \\ &= 0. \quad (\star) \end{aligned}$$

In physics $\vec{l} = \vec{r} \times \vec{p}$ is the angular momentum. We showed that $\frac{d\vec{l}}{dt} = 0$ for a central force. This means $\vec{r} \times (m\vec{v}) = \vec{l}_0$ for a fixed vector \vec{l}_0 . We suspect $\vec{l}_0 \parallel \vec{B}(t)$ which shows the motion is planar.

$$\vec{T}(t) = \frac{1}{v}\vec{v} \quad \hookrightarrow \quad \vec{T}'(t) = \frac{-1}{v^2} \frac{dv}{dt} \vec{v} + \frac{1}{v} \frac{d\vec{v}}{dt}$$

$$\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|} \vec{T}'(t) = \frac{1}{\tau'} \left(\frac{-1}{v^2} \frac{dv}{dt} \vec{v} + \frac{1}{v} \frac{F(\rho)}{mp} \vec{r} \right)$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{v} \vec{v} \times \left(\frac{-1}{\tau' v^2} \frac{dv}{dt} \vec{v} + \frac{F(\rho)}{\tau' v m p} \vec{r} \right)$$

$$\vec{B}(t) = \frac{F(\rho)}{\tau' v^2 m^2 p} (m\vec{v} \times \vec{r}) = \frac{-F(\rho)}{\tau' m^2 v^2 p} \vec{l}_0 \Rightarrow \frac{d\vec{B}}{dt} = 0.$$

Continued 

PROBLEM 61

$$\vec{B}(t) = \frac{-F(p)}{T'm^2v^2\rho} \vec{L}_o$$

where $T' = \|\vec{T}'(t)\|$ and \vec{L}_o is a fixed vector.

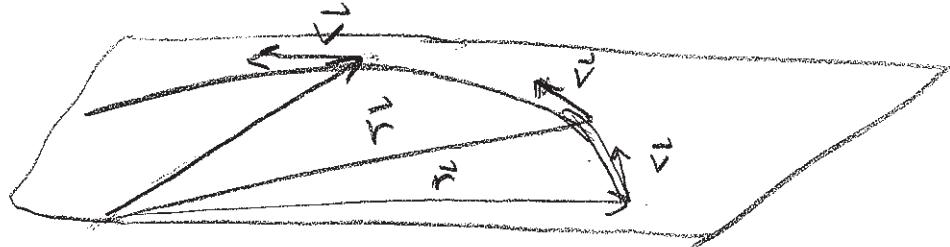
We know $\vec{B} \cdot \vec{B} = 1$ by its construction hence

$$\vec{B}' \cdot \vec{B} = 0 \Rightarrow \left(\frac{-F(p)}{T'm^2v^2\rho} \right)' \vec{L}_o \cdot \left(\frac{-F(p)}{T'm^2v^2\rho} \vec{L}_o \right) = 0$$

Hence, $\frac{F(p)}{T'm^2v^2\rho}$ = constant. We have thus

shown $\vec{B}(t) = \vec{B}_o \Rightarrow T = 0 \Rightarrow$ planar motion.
(using Problem 60)

(*) ← I gave pretty good credit if you
 got this far since $\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$ shows
 $\vec{r} \times \vec{v} = \frac{\vec{L}_o}{m} = \text{constant}$



and geometrically it's somewhat obvious this
 shows the motion lies in a plane.

PROBLEM 62) Place xy -coordinates in plane of motion, write

$$\vec{F} = -\frac{GmM}{r^2} \hat{r} = -\frac{GmM}{r^3} \vec{r}$$

Then observe that

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}(m\vec{r} \times \vec{v}) = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) \\ &= m\vec{v} \times \vec{v} + \vec{r} \times (m\frac{d\vec{v}}{dt}) \\ &= \vec{r} \times \left(-\frac{GmM}{r^3} \vec{r}\right) \\ &= -\frac{GmM}{r^3} \vec{r} \times \vec{r} \\ &= 0. \text{ Therefore, } \vec{L} = \text{constant.}\end{aligned}$$

(oops, I did 62 in 61 in a less fun situation... most folks who care about these things simply assume the motion is planar and go forth from that powerful assumption. Perhaps you can appreciate why they neglect the detail.)

PROBLEM 63) The question amounts to a modified intersection problem. Find intersection(s) of $\vec{r}_1(t) = \langle -10+t, 1+t \rangle$ and $\vec{r}_2(t) = \langle 20-4t, 6+t \rangle$ and then return to ponder the given genin/jonin question. When seeking intersections it is wise to use cartesian and/or swap notation for at least one $\vec{r}_2(t) \rightarrow \vec{r}_2(\tau) = \langle 20-4\tau, 6+\tau \rangle$ (*don't assume* $t = \tau$)

$$\begin{aligned}20-4\tau &= -10+t \quad \text{subtract} \rightarrow 14-5\tau = -11 \\ 6+\tau &= 1+t \quad \Rightarrow -5\tau = -25 \\ &\Rightarrow \boxed{\tau = 5}\end{aligned}$$

Then $6+\tau = 1+t \Rightarrow t = \tau + 5 = 5+5 \Rightarrow \boxed{t = 10}$

The jonin at $\tau=5$ arrives at $\vec{r}_2(5) = \langle 0, 11 \rangle$. However, the genin only gets there at $t=0$, $\vec{r}_1(10) = \langle 0, 11 \rangle$. Fortunately the genin was slow enough to luckily avoid the jonin.

PROBLEM 64 Suppose $\vec{V}(t) = \langle t, 3, t \cosh(t^2) \rangle$ for a particle initially at $(1, 2, 3)$. Find \vec{a}, \vec{r} for $t \geq 0$

Acceleration is easy,

$$\vec{a} = \frac{d\vec{V}}{dt} = \boxed{\langle 1, 0, \cosh(t^2) + 2t^2 \sinh(t^2) \rangle}$$

Position requires a bit more thought,

$$\begin{aligned}\vec{r}(t) &= \int \vec{V}(t) dt = \int \langle t, 3, t \cosh(t^2) \rangle dt \\ &= \langle \int t dt, \int 3 dt, \int t \cosh(t^2) dt \rangle \\ &= \langle \frac{1}{2}t^2 + C_1, 3t + C_2, \frac{1}{2} \sinh(t^2) + C_3 \rangle\end{aligned}$$

We know $\vec{r}(0) = (1, 2, 3) = \langle C_1, C_2, C_3 \rangle$ hence,

$$\boxed{\vec{r}(t) = \langle \frac{1}{2}t^2 + 1, 3t + 2, \frac{1}{2} \sinh(t^2) + 3 \rangle}$$

PROBLEM 65 Suppose $\vec{a} = 3\hat{x}$ and $\vec{r}(0) = \vec{r}_0$ and $\vec{v}(0) = \vec{v}_0$.

Find $\vec{v}(t)$ and $\vec{r}(t)$

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} = 3\hat{x} &\Rightarrow d\vec{v} = 3\hat{x} dt \\ &\Rightarrow \int_{\vec{v}_0}^{\vec{v}(t)} d\vec{v} = \int_0^t 3\hat{x} dt \\ &\Rightarrow \vec{v}(t) - \vec{v}_0 = 3t\hat{x} \\ &\Rightarrow \boxed{\vec{v}(t) = \vec{v}_0 + 3t\hat{x}}\end{aligned}$$

$$\begin{aligned}\text{Likewise } \vec{v} = \frac{d\vec{r}}{dt} &\Rightarrow d\vec{r} = \vec{v}(t) dt = (\vec{v}_0 + 3t\hat{x}) dt \\ &\Rightarrow \int_{\vec{r}_0}^{\vec{r}(t)} d\vec{r} = \int_0^t (\vec{v}_0 + 3t\hat{x}) dt \\ &\Rightarrow \vec{r}(t) - \vec{r}_0 = t\vec{v}_0 + \frac{3}{2}t^2\hat{x} \\ &\Rightarrow \boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}_0 + \frac{3}{2}t^2\hat{x}}\end{aligned}$$

Problem 66) Suppose $\vec{r}(t) = \langle 2^t, \ln(t), \sqrt{t^2+1} \rangle$.

Find tangent line to curve at $(8, \ln(3), \sqrt{10})$.

Note $2^t = 8$, $\ln(t) = \ln(3)$, $\sqrt{t^2+1} = 10$ all have
so $t = 3$. We seek tangent at $\vec{r}(3)$ to $\vec{r}(t)$.

Note $\vec{r}'(t) = \langle \ln(2)2^t, \frac{1}{t}, \frac{t}{\sqrt{t^2+1}} \rangle$

$$\Rightarrow \vec{r}'(3) = \langle 8\ln(2), \frac{1}{3}, \frac{3}{\sqrt{10}} \rangle$$

Thus $\boxed{\vec{l}(t) = (8, \ln(3), \sqrt{10}) + t \langle 8\ln(2), \frac{1}{3}, \frac{3}{\sqrt{10}} \rangle}$

vector-parametric equations for
the desired tangent line.

PROBLEM 67)

$$\int \sin^2(x) dx = \int \frac{1}{2}(1 - \cos(2x)) dx = \boxed{\frac{x}{2} - \frac{1}{4}\sin(2x) + C}$$

$$\begin{aligned} \int \sin^3(x) dx &= \int \sin^2 x \sin x dx \\ &= \int (1 - \cos^2 x) \sin x dx \end{aligned}$$

$$= \int (1 - u^2)(-du) \quad \text{letting } \begin{cases} u = \cos x \\ du = -\sin x dx \end{cases}$$

$$= \int (u^2 - 1) du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \boxed{\frac{1}{3}\cos^3 x - \cos x + C} \quad \text{defn of cosh.}$$

Problem 68) Note $\cosh(2\ln(t)) = \cosh(\ln(t^2)) = \sqrt{\frac{1}{2}(e^{\ln(t^2)} + e^{-\ln(t^2)})}$
 $= \frac{1}{2}(t^2 + e^{\ln(\sqrt{t^2})})$
 $= \frac{1}{2}(t^2 + \frac{1}{t^2})$

$$\frac{d}{dt}[2t \cosh(2\ln(t))] = \frac{d}{dt}\left[t^3 + \frac{1}{t}\right] = \boxed{3t^2 - \frac{1}{t^2}}$$

PROBLEM 69 $\vec{r}(t) = \langle 2^{-t}, 3\sin t, 4t \rangle$ for $t \geq 0$.

Set-up $s(t)$ for $t \geq 0$. Also find bound on speed

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -\ln(2)2^{-t}, 3\cos t, 4 \rangle$$

$$V(t) = \sqrt{(\ln(2)2^{-t})^2 + 9\cos^2 t + 16}$$

$$\Rightarrow s(t) = \int_0^t \sqrt{(\ln(2)2^{-u})^2 + 9\cos^2 u + 16} du \quad (\text{distance traveled})$$

Speed Limit?

$$V(t) = \sqrt{\left[\frac{\ln(2)}{2^t}\right]^2 + [3\cos t]^2 + [4]^2}$$

$$V(0) = \sqrt{(\ln(2))^2 + (3)^2 + (4)^2} = 5.04$$

Clearly this is your max speed since $\frac{1}{2^t}$ decreases as $t \rightarrow \infty$ and $|3\cos t| \leq 3$ thus we have max. at $t=0$.

So, No, the honest hover cop should leave you in peace for your < 5.04 mps commute.