Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 41 Your signature below indicates you have:

(a.) I have read Chapter 3 and §4.1 – 4.3 of Cook: ____________________.

(b.) I have attempted homeworks from Salas and Hille as listed below:________________.

The following homeworks from the text are good rudimentary skill problems. These are not collected or graded. However, they are all odd problems thus there are answers given within Salas, Hille and Etgen's text:

§ 14.1 #'s 1, 5, 11, 27

§ 14.4 #'s 1, 5, 9, 13, 15, 17, 21, 27, 31, 33, 47

§ 14.5 #'s 1, 3, 5, 9

§ 14.6 #'s 1, 5, 11, 15, 17, 21, 23, 27

§ 15.1 #'s 1, 7, 15, 17, 25, 29

§ 15.2 #'s 1, 5, 11, 19, 25, 33, 37

Problem 42 Prove by a picture that the half-plane $H = \{(x, y) \mid y > 0\}$ is indeed open. Your picture should illustrate why each point in H is an interior point.

Problem 43 Consider the annulus $A = \{(x, y) \mid 1 \leq x^2 + y^2 < 4\}$. Find the boundary of A. Again, include a sketch of every disk centered on the claimed boundary point has points both inside and outside the set.

Problem 44 Let $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ for $(x, y) \neq 0$ and $f(0, 0) = A$. Choose the appropriate value for A such that f is continuous on $\mathbb{R}^2$ and show, relative to that choice, f is indeed continuous on $\mathbb{R}^2$.

Problem 45 Let $f(x, y) = \frac{x^2 y}{x^2+y^2}$ for $(x, y) \neq 0$ and $f(0, 0) = A$. If possible, choose the appropriate value for A such that f is continuous on $\mathbb{R}^2$ and show, relative to that choice, f is indeed continuous on $\mathbb{R}^2$. Or, if no choice of A makes f continuous at $(0, 0)$ explain why by explicitly showing the limit at $(0, 0)$ of f does not exist.

Problem 46 Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for:

(a) Let $f(x, y) = e^x \sin(xy)$.

(b) Let $f(x, y) = \ln\left(e^{x+y}\sqrt{x^2 + y^2}\right)$.

Problem 47 Suppose $g(x,y) = e^{x-y}$. Evaluate $g, g_x, g_y, g_{xx}, g_{xy}, g_{yy}$ at $(0,0)$. What can you say about the values as they compare to the series formed by $e^u = 1 + u + \frac{1}{2}u^2 + \cdots$ with $u = \cancel{x^2 - y^2}$. $x - y$ Use this example to propose[1] the second order Taylor polynomial centered at $(0,0)$.

Problem 48 Let $f(\rho, \theta, \phi) = \tan^2(\rho + \phi\theta^2)$. Calculate $\partial_\theta \partial_\rho \partial_\phi f$.

Problem 49 Show that $u = e^x \cos(y)$ and $v = e^x \sin(y)$ are solutions of $\Phi_{xx} + \Phi_{yy} = 0$. This equation is known as Laplace's Equation and it can also be written as $\nabla^2 \Phi = 0$.

Problem 50 Suppose $f(x_1, x_2, \dots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Calculate $\frac{\partial f}{\partial x_1}$

Instructions for following two problems: Calculate the total differential and gradient for the functions given below. The total differential of $f(x_1, x_2, \dots, x_n)$ is

$$df = (\partial_1 f)dx_1 + (\partial_2 f)dx_2 + \cdots + (\partial_n f)dx_n$$

and the gradient of $f(x_1, x_2, \dots, x_n)$ is $\nabla f = \langle \partial_1 f, \partial_2 f, \dots, \partial_n f \rangle$.

Problem 51 (a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $g(x,y) = [f(x,y)]^2 = x^2 + y^2$

Problem 52 $h(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \cdots x_n^2$.
(notice $h = f^2$ where f is like the function studied in Problem 50 , note the similarity with the preceding problem)

Problem 53 Let $f(x,y) = 3 + xy + x^3$. Consider the point $p = (2,1)$. Find unit-vector(s) based at p which point in the direction in which f changes at the rate of 12 if possible. Also, if possible, find the unit-vectors which point in the direction in which f changes at rate ~~15~~ 5 at p.

Problem 54 Let $f(x,y) = e^{yx^2}$. Find the rate of change of f at $(0,0)$ in the $\langle a, b \rangle$-direction. If L is a line through the origin which is perpendicular to the directions in which the maximum and minimum rates of change of f at $(0,0)$ occur then what special property do we observe for $f(x,y)$ where $(x,y) \in L$?

Problem 55 Calculate ∇f for each of the functions below:

(a) $f(x,y) = 2x + 3y$

(b) $f(x,y) = exp(-x^2 + 2x - y^2)$

(c) $f(x,y) = \sin(x+y)$

Problem 56 What is the rate of change for each of the functions given in the previous problem at the point $(1,3)$ in the direction of the vector $\langle 1, -1 \rangle$.

Problem 57 Again, concerning the functions given in ~~Problem 62~~ Problem 55, in what directions are the functions locally constant at the point $(1,3)$? (give answers in terms of unit-direction vectors)?

[1]spolier alert; You can check your claim against page 254 of my notes.

Problem 58 Suppose the temperature T is a function of the coordinates x, y in a large plane of battle. Furthermore, suppose the enemy ninja is carefully building a large attack by molding chakra over some time. During the preparation of the attack the enemy is vulnerable to your attack. Knowing this he has obscured your field of vision with multiple smoke bombs. However, the mass of energy building actually heats the ground. Fortunately one of your ninja skills is temperature sensitivity. You extrapolate from the temperature of the ground near your location that the temperature function has the form $T(x, y) = 50 + 2x + 3y$. In what direction should you attack?

Problem 59 Suppose $\vec{F}(x, y, z) = \langle 2xy^2, 2x^2y, 3\rangle$. What scalar function f yields $\vec{F}$ as a gradient vector field? Find f such that $\nabla f = \vec{F}$.
(here we have to work backwards, write down what you want and guess, by the way, the function $-f$ is the pontial energy function for the force field $\vec{F}$.)

Problem 60 Consider x, y the cartesian coordinate functions on $\mathbb{R}^2$. Prove from the limit-based definition of partial derivatives that $\frac{\partial y}{\partial x} = 0$ and $\frac{\partial x}{\partial x} = \cancel{0}$. 1

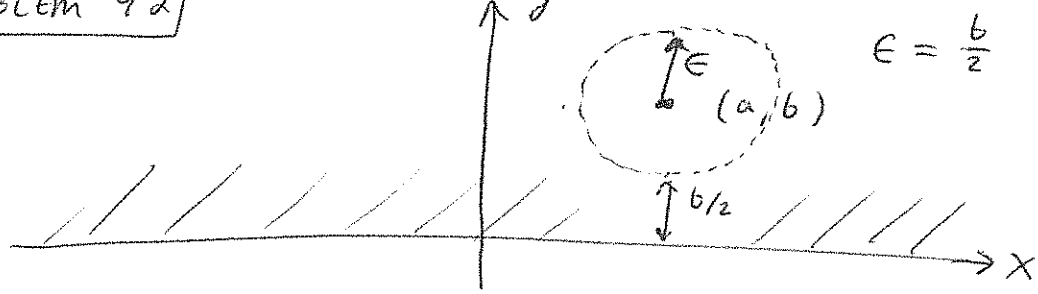
Bonus Problem: [use of technology to solve algebraic and/or transcendental equation that the problem suggests] The temperature in an air conditioned room is set at 65. A ninja with expert ocular jutsu disguises himself in plain sight by bending light near him with his art. However, his art does not extend to the infared spectrum and his body heat leaves a signature variation in the otherwise constant room temperature. In particular,

$$T(x, y, z) = 33 \exp\left[\frac{-(x-3)^2 - (y-4)^2 - (z-1)^2}{10}\right] + 65.$$

Shino searches for the cloaked ninja by sending insect scouts which are capable of sensing a change in temperature as minute as 0.1 degree per meter. How close do the scout insects have to get before they sense the hidden ninja? (also, where is the hidden ninja and what is his body temperature on the basis of the given T which is in meters and degrees Farenheight)

Mission 3 Solution

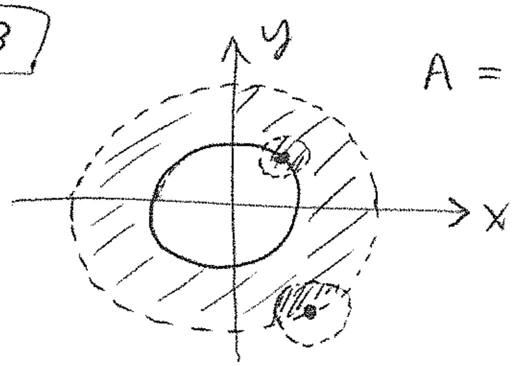
Problem 42



$\epsilon = \frac{b}{2}$

No matter which point, we can always use dish radius $b/2$ for point (a,b). Clearly the open dish $D_{b/2}(a,b) \subset H = \{(x,y) \mid y > 0\}$ hence (a,b) is <u>interior point</u> to H. This proves H is open as each point of H is interior.

Problem 43



$A = \{(x,y) \mid 1 \le x^2+y^2 < 4\}$

clearly dishs at pts. of $x^2+y^2=4$ or $x^2+y^2=1$ are on the boundary.

$$\boxed{\partial A = \{(x,y) \mid x^2+y^2=1 \text{ or } x^2+y^2=4\}}$$

Problem 44

$$f(x,y) = \begin{cases} A & : (x,y) = (0,0) \\ \dfrac{\sin(x^2+y^2)}{x^2+y^2} & : (x,y) \neq (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \left[\frac{\sin(x^2+y^2)}{x^2+y^2}\right] = \lim_{r\to 0}\left[\frac{\sin(r^2)}{r^2}\right]$$

$$\underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{r\to 0}\left[\frac{2r\cos(r^2)}{2r}\right]$$

$$= \cos(0)$$

$$= 1.$$

Thus, choose $A=1$ to make $f(x,y)$ continuous at $(0,0)$. For $(x,y) \neq (0,0)$ the function is clearly continuous.

PROBLEM 45 Let $\underbrace{f(x,y) = \dfrac{x^2y}{x^2+y^2}}_{\text{for } (x,y)\neq(0,0).}$ and $f(0,0)=A$

my initial thought was to use 2 path d.n.e. argument

Consider path $y=x$ then we have

$$\lim_{(x,x)\to(0,0)}\left(\frac{x^2x}{x^2+x^2}\right) = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim(x) = 0.$$

Well, upon further thought, I think this limit exists, so I'll use polar substitution

$$\lim_{(x,y)\to(0,0)}\left[\frac{x^2y}{x^2+y^2}\right] = \lim_{r\to 0}\left[\frac{(r^2\cos^2\theta)(r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}\right]$$

$$= \lim_{r\to 0}\left[\frac{r^3\sin\theta\cos^2\theta}{r^2}\right]$$

$$= \lim_{r\to 0}\left[r\sin\theta\cos^2\theta\right]$$

$$= 0.$$

Thus use $\boxed{A=0}$ and we obtain $f(x,y)$ continuous on $\mathbb{R}^2$.

PROBLEM 46

(a.) $$\frac{\partial}{\partial x}\left(e^x\sin(xy)\right) = e^x\sin(xy) + e^x\cos(xy)\frac{\partial}{\partial x}(xy)$$
$$= \boxed{e^x\sin(xy) + ye^x\cos(xy)}$$

$$\frac{\partial}{\partial y}\left(e^x\sin(xy)\right) = e^x\frac{\partial}{\partial y}[\sin(xy)]$$
$$= e^x\cos(xy)\frac{\partial}{\partial y}[xy]$$
$$= \boxed{xe^x\cos(xy)}$$

PROBLEM 46

(b.) $$\frac{\partial}{\partial x}\left[\ln\left(e^{x+y}\sqrt{x^2+y^2}\right)\right] = \frac{\partial}{\partial x}\left[x+y+\tfrac{1}{2}\ln(x^2+y^2)\right]$$
$$= 1+\frac{1}{2}\left(\frac{2x}{x^2+y^2}\right)$$
$$= \boxed{1+\frac{x}{x^2+y^2}}$$

$$\frac{\partial}{\partial y}\left[\ln\left(e^{x+y}\sqrt{x^2+y^2}\right)\right] = \frac{\partial}{\partial y}\left[x+y+\tfrac{1}{2}\ln(x^2+y^2)\right]$$
$$= \boxed{1+\frac{y}{x^2+y^2}}$$

PROBLEM 47

$$g(x,y) = e^{x-y}$$

$g_x = e^{x-y}$	$g_x(0,0) = 1$
$g_y = -e^{x-y}$	$g_y(0,0) = -1$
$g_{xx} = e^{x-y}$	$g_{xx}(0,0) = 1$
$g_{xy} = -e^{x-y}$	$g_{xy}(0,0) = -1$
$g_{yy} = e^{x-y}$	$g_{yy}(0,0) = 1$

$$g(u) = e^u = 1+u+\tfrac{1}{2}u^2+\cdots$$

$u = x-y$ hence,
$$g(x,y) = 1+x-y+\tfrac{1}{2}(x-y)^2+\cdots$$
$$= 1+x-y+\tfrac{1}{2}(x^2-2xy+y^2)+\cdots$$

Maybe you can see the pattern,
$$\Rightarrow g(x,y) = g(0,0)+g_x(0)x+g_y(0)y+\tfrac{1}{2}\left(g_{xx}(0)x^2+2g_{xy}(0)xy+g_{yy}(0)y^2\right)+$$

(we'll derive this later for an arbitrary function of (x,y))

$\boxed{\text{PROBLEM 48}}$

$$\partial_\theta \partial_\rho \partial_\phi f = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\phi}\left[\tan^2(\rho+\phi\theta^2)\right] \qquad \frac{\partial}{\partial\phi}[\rho+\phi\theta^2]=\theta^2$$

$$= \frac{\partial}{\partial\theta}\frac{\partial}{\partial\rho}\left[2\tan(\rho+\phi\theta^2)\sec^2(\rho+\phi\theta^2)\,\theta^2\right]$$

$$= \frac{\partial}{\partial\theta}\Big[2\theta^2\Big(\sec^4(\rho+\phi\theta^2)\frac{\partial}{\partial\rho}(\rho+\phi\theta^2) +$$

$$+\tan(\rho+\phi\theta^2)\cdot 2\sec^2(\rho+\phi\theta^2)\tan(\rho+\phi\theta^2)\frac{\partial}{\partial\rho}(\rho+\phi\theta^2)$$

$$= \frac{\partial}{\partial\theta}\left[2\theta^2(\sec^4(\rho+\phi\theta^2)+2\sec^2(\rho+\phi\theta^2)\tan(\rho+\phi\theta^2)\right]$$

$$= 4\theta(\sec^4(\alpha)+2\sec^2(\alpha)\tan(\alpha))+$$

$$+2\theta^2\Big\{4\sec^3(\alpha)\cdot\sec(\alpha)\tan(\alpha)\frac{\partial}{\partial\theta}(\rho+\phi\theta^2)$$

$$+4\sec\alpha\cdot\sec\alpha\tan\alpha\tan\alpha\frac{\partial}{\partial\theta}(\rho+\phi\theta^2)$$

$$+2\sec^2\alpha\sec^2\alpha\frac{\partial}{\partial\theta}(\rho+\phi\theta^2)\Big\}$$

$$= 4\theta(\sec^4\alpha+2\sec^2\alpha\tan\alpha)$$

$$+2\theta^2\{8\theta\sec^4\alpha\tan\alpha+8\theta\sec^2\alpha\tan^2\alpha+4\theta\sec^4\alpha\}$$

$$= (4\theta+8\theta^3)\sec^4(\rho+\phi\theta^2)+8\theta\sec^2(\rho+\phi\theta^2)\tan(\rho+\phi\theta^2)+$$

$$+16\theta^3\sec^4(\rho+\phi\theta^2)\tan(\rho+\phi\theta^2)+16\theta^3\sec^2(\rho+\phi\theta^2)\tan^2(\rho+\phi\theta^2)$$

<u>Remark</u>: why. why. what have I done.

$$\partial_\theta\partial_\rho\partial_\phi f = \partial_\theta\partial_\phi\partial_\rho f \qquad \text{easier}$$

last is good.

PROBLEM 49

$$u = e^x \cos(y) \quad\Longrightarrow\quad \begin{cases} u_x = e^x \cos(y) \Rightarrow u_{xx} = e^x \cos(y) \\ u_y = -e^x \sin(y) \Rightarrow u_{yy} = -e^x \cos(y) \\ \text{thus } u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0. \end{cases}$$

$$v = e^x \sin(y)$$

Likewise $v_{xx} = e^x \sin(y)$ and $v_{yy} = -e^x \sin(y)$

Hence $v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0.$

Thus $u = e^x \cos y$, $v = e^x \sin y$ are solutions to Laplaces Eq^n

PROBLEM 50 $f(x_1, \dots, x_n) = \sqrt{x_1^2 + \dots + x_n^2}$

$$\frac{\partial f}{\partial x_1} = \frac{1}{2\sqrt{x_1^2 + \dots + x_n^2}} \frac{\partial}{\partial x_1}\left(x_1^2 + \dots + x_n^2\right) = \frac{x_1}{\sqrt{x_1^2 + \dots + x_n^2}} = \boxed{\frac{x_1}{f}}$$

PROBLEM 51 Calculate df and ∇f for,

(a.) $f(x,y) = \sqrt{x^2+y^2}$

Note $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$ and $\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\therefore \quad \boxed{df = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy}$$

$$\boxed{\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle}.$$

see over for (b.)

(b) $\boxed{df = \underbrace{e^{-x^2+2x-y^2}}_{\text{factored out.}}\left((2-2x)\,dx - 2y\,dy\right)}$

oops. bonus sol^{n}s. sorry.

$$\boxed{\nabla f = e^{-x^2+2x-y^2}\langle 2-2x, -2y\rangle}$$

(c) $f(x,y) = \sin(x+y)$ $\longrightarrow$ $\boxed{df = \cos(x+y)\,dx + \cos(x+y)\,dy}$

$\longrightarrow$ $\boxed{\nabla f = \langle \cos(x+y), \cos(x+y)\rangle}$

could factor out $\cos(x+y)$

PROBLEM 51 continued

(b.) $g(x,y) = x^2+y^2$ $\to$ $\boxed{\nabla g = \langle 2x, 2y\rangle}$, $\boxed{dg = 2x\,dx + 2y\,dy}$

comment

Alternatively, since $f^2 = g$ we can look at this by chain-rule,

$$dg = 2f\,df = 2\sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy\right) = 2x\,dx + 2y\,dy.$$

But, the interesting way to think here is that $dg = 2x\,dx + 2y\,dy$ is <u>easy</u> to calculate and $df = \frac{1}{2f}dg = \frac{1}{2f}(2x\,dx + 2y\,dy) = \frac{x}{f}dx + \frac{y}{f}dy$.

$\boxed{\text{PROBLEM 52}}$ $h = x_1^2 + x_2^2 + \cdots + x_n^2$

$$\boxed{dh = 2x_1\,dx_1 + 2x_2\,dx_2 + \cdots + 2x_n\,dx_n}$$

$$\boxed{\nabla h = \langle 2x_1, 2x_2, \ldots, 2x_n\rangle}$$

Comment

If $f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ then $h = f^2$ and we can derive df from dh with ease:

$$df = \frac{1}{2f}dh \Rightarrow \left\langle \frac{x_1}{f}, \frac{x_2}{f}, \ldots, \frac{x_n}{f}\right\rangle = \nabla f$$

$$\& \ \frac{1}{f}(x_1\,dx_1 + x_2\,dx_2 + \cdots + x_n\,dx_n) = df.$$

<u>Remark</u>: to grader: the comments are not req^{d} from students. These are for the interested reader of this sol^{n}.

PROBLEM 53 (in response to class discussion we modified the $15 \to 5$ (BAD IDEA ☹)

Let $f(x,y) = 3 + xy + x^3$.
Consider the point $P = (2,1)$
Find $\hat{u} = \langle a,b \rangle$ for which $(D_{\hat{u}} f)(P) = \begin{cases} 12 \\ 5 \end{cases}$

To find rate of change for $f(x,y)$ at point $(2,1)$ in direction $\hat{u} = \langle a,b \rangle$ we need to calculate $(\nabla f)(2,1) \cdot \langle a,b \rangle$. Note,

$$(\nabla f)(x,y) = \left\langle \frac{\partial}{\partial x}(3+xy+x^3), \frac{\partial}{\partial y}(3+xy+x^3) \right\rangle$$

$$(\nabla f)(x,y) = \langle y + 3x^2, x \rangle$$

Hence,

$$(\nabla f)(2,1) = \langle 1 + 3(2)^2, 2 \rangle = \langle 13, 2 \rangle.$$

Now the 1^{st} thing we should do is to see if rate 5 or 12 is possible. If not we save ourselves the trouble of the next two pages (!). Note:

$$\|(\nabla f)(2,1)\| = \|\langle 13,2 \rangle\| = \sqrt{169+4} \cong \underline{13.15}$$

Therefore, my original problem statement was HALF the work of the modified version, Sadness.

(if we had left 15 then the answer for rate 15 was just, nope, not possible max rate 13.15.)

PROBLEM S3 continued

A rate of change of 5 requires we solve

$$13a + 2b = 5 \longrightarrow b = \frac{5 - 13a}{2}$$
$$a^2 + b^2 = 1$$

$$a^2 + \frac{(5-13a)^2}{4} = 1$$

$$4a^2 + 25 - 130a + 169a^2 = 4$$

$$173a^2 - 130a + 21 = 0$$

$$a = \frac{130 \pm \sqrt{(130)^2 - 4(173)(21)}}{2(173)} = \frac{130 \pm 48.662}{2(173)}$$

$(+)$ $\quad a = 0.5164 \Rightarrow b = \frac{5-13a}{2} = \frac{5-13(0.5164)}{2} = -0.8566$

$(-)$ $\quad a = 0.2351 \Rightarrow b = \frac{5-13(0.2351)}{2} = 0.9719$

Thus $D_{\hat{u}} f(2,1) = 5$ for $\hat{u} \cong \langle 0.5164, -0.8566 \rangle$
or $\hat{u} = \langle 0.2351, 0.9719 \rangle$

PROBLEM 53 continued

$$13a + 2b = 12 \longrightarrow b = \frac{12 - 13a}{2}$$

$$a^2 + b^2 = 1$$

$$a^2 + \frac{(12 - 13a)^2}{4} = 1$$

$$4a^2 + 144 - 312a + 169a^2 = 4$$

$$173a^2 - 312a + 140 = 0$$

$$a = \frac{312 \pm \sqrt{(312)^2 - 4(173)(140)}}{2(173)} \cong \frac{312 \pm 21.54}{346}$$

$$a = 0.964 \quad \text{or} \quad a = 0.839$$

$$b = \frac{12 - 13(0.964)}{2} = 0.266 \quad \text{or} \quad b = \frac{12 - 13(0.839)}{2} = 0.543$$

Therefore, to obtain change rate 5 at (1,2) we may ~~either~~ go in one of the following directions:

$$\boxed{\begin{array}{l} \hat{u} \equiv \langle 0.964, 0.266 \rangle \\ \hat{u} = \langle 0.839, 0.543 \rangle \end{array}} \longrightarrow (D_{\hat{u}} f)(2,1) = 5.$$

PROBLEM 54 Let $f(x,y) = e^{yx^2}$.

Find rate of change of f at $(0,0)$ in $\langle a,b \rangle$-direction. If L is a line through origin which is $\perp$ to directions of min/max rates of change of f at $(0,0)$ occur then what special property do we observe for $f(x,y)$ where $(x,y) \in L$

Note, $(\nabla f)(x,y) = \left\langle \frac{\partial}{\partial x}(e^{yx^2}), \frac{\partial}{\partial y}(e^{yx^2}) \right\rangle = \langle 2xy, x^2 \rangle e^{yx^2}$

thus $(\nabla f)(0,0) = \langle 0, 0 \rangle$. CURSES. EVERYBODY GETS 100 % ON THIS ONE.

//

Let $f(x,y) = 3 + 2x - y + x^2 + y^2$. Same question.

$$\nabla f = \langle 2 + 2x, -1 + 2y \rangle$$

$$(\nabla f)(0,0) = \langle 2, -1 \rangle$$

To find $\perp$ direction simply use $\langle 1, 2 \rangle$
note $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle = 2 - 2 = 0$.

Thus $L: \vec{r}(t) = t\langle 1, 2 \rangle = \langle t, 2t \rangle$

Notice $f(\vec{r}(t)) = f(t, 2t) =$

$$= 3 + 2t - 2t + t^2 + (2t)^2$$

$$= 3 + 5t^2.$$

For t very small note $f(x,y) \approx 3$ along L.

($f(x,y)$ is constant in the direction $\perp$ to ∇f at a given point. As we found, there is no first order change in f near the point)

PROBLEM 55

(a.) $\nabla f = \nabla(2x+3y) = \langle 2, 3 \rangle$

(b) $\nabla f = \nabla\left(e^{-x^2+2x-y^2}\right) = e^{-x^2+2x-y^2}\,\nabla(-x^2+2x-y^2)$

$$\Rightarrow \boxed{\nabla f = e^{-x^2+2x-y^2}\langle -2x+2, -2y\rangle}$$

(c.) $\nabla f = \nabla(\sin(x+y)) = \cos(x+y)\,\nabla(x+y)$

$= \cos(x+y)\langle 1, 1\rangle.$ also good.

$= \boxed{\langle \cos(x+y), \cos(x+y)\rangle}$

Remark: I hope you see how the chain rule makes life easier for parts (b.) & (c.) above.

(from PROBLEM 55)

PROBLEM 56 rate of change for f at (1,3) in $\langle 1,-1\rangle$-direction

$\hat{u} = \frac{\langle 1,-1\rangle}{\sqrt{2}}$.

(a.) $(D_{\hat{u}} f)(1,3) = \langle 2,3\rangle \cdot \langle 1,-1\rangle \frac{1}{\sqrt{2}}$

$= \frac{2-3}{\sqrt{2}}$

$= \frac{-1}{\sqrt{2}} = -0.7071$.

(b.) $(D_{\hat{u}} f)(1,3) = (\nabla f)(1,3) \cdot \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$ using 55b

$= \frac{\exp(-1+2-9)}{\sqrt{2}}\langle -2+2, -6\rangle \cdot \langle 1,-1\rangle$

$= \boxed{\frac{6\exp(-8)}{\sqrt{2}} \cong 0.001423}$

(c.) $(D_{\hat{u}} f)(1,3) = \frac{1}{\sqrt{2}}\langle \cos(4), \cos(4)\rangle \cdot \langle 1,-1\rangle = \boxed{0.}$

$\boxed{\text{PROBLEM 57}}$ well, 54 was supposed to set this up, the idea is simply the $\perp$ direction to $\nabla f(P)$ gives direction along which $f(x,y)$ stays constant. Generally, for $\langle a,b\rangle$ use $\langle b,-a\rangle$ to obtain $\perp$-vector. Note, $\langle a,b\rangle \cdot \langle b,-a\rangle = ab - ba = 0$. So, use this for $\nabla f = \langle a,b\rangle$,

(a.) $f(x,y) = 2x+3y$ has $(\nabla f)(1,3) = \langle 2,3\rangle$
Hence L with direction $\langle 3,-2\rangle$ gives $f(x,y)$ <u>locally</u> constant.

(b.) $f(x,y) = e^{-x^2+2x-y^2}$ has $(\nabla f)(1,3) = e^{-8}\langle 0,-6\rangle$
Hence L with direction $\langle 1,0\rangle$ gives f locally constant.

(c.) $f(x,y) = \sin(x+y)$ has $(\nabla f)(1,3) = \langle \cos(4), \cos(4)\rangle$
Hence L with direction $\langle 1,-1\rangle$ gives f locally constant.

<u>Details on (c.)</u>

$$\vec{r}(t) = (1,3) + t\langle 1,-1\rangle$$

$$\begin{aligned} f(\vec{r}(t)) = f(1+t,\ 3-3t) &= \sin(1+t+3-3t) \\ &= \sin(4-4t) \\ &= \sin 4 \underbrace{\cos(-4t)}_{1} - \cos 4 \underbrace{\sin(4t)}_{0} \end{aligned}$$

for small t

Thus $f(\vec{r}(t)) \longrightarrow \sin(4)$ for $\vec{r}(t)$ with t-small.

PROBLEM 58 NINJA PROBLEM, see end of Chapter PROBLEM 82.

To find max-rate of change in temperature T we consider $T(x,y) = 50 + 2x + 3y$ has

$$\nabla T = \langle 2, 3 \rangle$$

Hence we attack in $\boxed{\langle 2,3 \rangle - \text{direction}}$

PROBLEM 59 $\vec{F}(x,y,z) = \langle 2xy^2, 2x^2y, 3 \rangle$ find f such that $\nabla f = \vec{F}$.

Need $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2xy^2, 2x^2y, 3 \rangle$

$$\frac{\partial f}{\partial x} = 2xy^2 \implies f(x,y) = x^2y^2 + C_1(y,z) \quad \text{I.}$$

$$\frac{\partial f}{\partial y} = 2x^2y \implies f(x,y) = x^2y^2 + C_2(x,z) \quad \text{II.}$$

$$\frac{\partial f}{\partial z} = 3 \implies f(x,y) = 3z + C_3(x,y) \quad \text{III.}$$

We'll see a systematic method soon in lecture, but, for now we just guess on the basis of I, II and III we use

$$\boxed{f(x,y,z) = x^2y^2 + 3z} \quad \left(\text{easy to check } \nabla f = \vec{F}\right)$$

The question remains: is this the only possible f? well, surely ~~x^2y~~ $x^2y^2 + 3z + C$ also works. Is that all? We will find answers to these questions as the course continues.

PROBLEM 60 Prove $\frac{\partial y}{\partial x} = 0$ and $\frac{\partial x}{\partial x} = 1$ for $\mathbb{R}^2$ coordinate functions x & y

1.) Let $f(x,y) = y$ and consider,

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0}\left[\frac{f(a+h,b) - f(a,b)}{h}\right]$$: def^{n} of partial derivative.

$$= \lim_{h \to 0}\left[\frac{b-b}{h}\right]$$

$$= \lim_{h \to 0}[0]$$

$$= 0.$$

This holds for arbitrary $(a,b) \in \mathbb{R}^2$ hence $\underline{\frac{\partial y}{\partial x} = 0}$ //

2.) Let $g(x,y) = x$ and consider,

$$\frac{\partial g}{\partial x}(a,b) = \lim_{h \to 0}\left[\frac{g(a+h,b) - g(a,b)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{a+h-a}{h}\right]$$

$$= \lim_{h \to 0}[1]$$

$$= 1.$$

But, (a,b) is arbitrary hence $\frac{\partial g}{\partial x} = 1$ which shows $\underline{\frac{\partial x}{\partial x} = 1}$ //

__Bonus__: see sol^{n} to NINJA problem 83 in end of Chapter 4 problems. The sol^{n} is at least partly posted on my website ☺.