

PROBLEM 70] We can show  $\lim_{(x,y) \rightarrow (0,0)} \left[ \frac{x+y}{x-y} \right]$  d.n.e. by the two path test.

$$\lim_{(x,0) \rightarrow (0,0)} \left[ \frac{x+y}{x-y} \right] = \lim_{x \rightarrow 0} \left[ \frac{x}{x} \right] = 1 \quad (\text{along } x\text{-axis.})$$

$$\lim_{(0,y) \rightarrow (0,0)} \left[ \frac{x+y}{x-y} \right] = \lim_{y \rightarrow 0} \left[ \frac{y}{-y} \right] = -1 \quad (\text{along } y\text{-axis.})$$

Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \left[ \frac{x+y}{x-y} \right]$  d.n.e. since all paths have to agree when limit exists.

PROBLEM 71

$$\begin{aligned} \lim_{(x,y) \rightarrow (B,B)} \left( \frac{x^4 - y^4}{x^2 - y^2} \right) &= \lim_{(x,y) \rightarrow (B,B)} \left( \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 - y^2} \right) \\ &= \lim_{(x,y) \rightarrow (B,B)} (x^2 + y^2) \\ &= B^2 + B^2 \\ &= \boxed{2B^2}. \end{aligned}$$

PROBLEM 72] I suspect  $\lim_{(x,y) \rightarrow (0,0)} \left( \frac{y^2x}{y^4+x^2} \right)$  d.n.e. Consider

$$\underbrace{\lim_{t \rightarrow 0} \left( \frac{m^2 t^3}{m^4 t^4 + t^2} \right)}_{\vec{r}(t) = \langle t, mt \rangle} = \lim_{t \rightarrow 0} \left( \frac{m^2 t}{m^4 t^2 + 1} \right) = 0. \quad (\text{no help.})$$

$\vec{r}(t) = \langle t, mt \rangle$  or line through origin.

Let's see how it does with  $\vec{r}(t) = \langle kt^2, t \rangle$

$$\lim_{t \rightarrow 0} \left( \frac{kt^2 \cdot t^4}{t^4 + k^2 t^4} \right) = \lim_{t \rightarrow 0} \left( \frac{kt^6}{1 + k^2 t^2} \right) = \frac{k}{1+k^2}$$

Take  $k=1$  and we obtain a limit of  $\frac{1}{2} \neq 0$

Therefore, the multivariate limit  $\lim_{(x,y) \rightarrow (0,0)} \left( \frac{y^2x}{y^4+x^2} \right)$  does

not converge.

PROBLEM 73/

$$\lim_{(x,y) \rightarrow (0,0)} \left[ \frac{\sin(x^2+y^2)}{2(x^2+y^2)} \right] = \lim_{r \rightarrow 0} \left( \frac{\sin(r^2)}{2r^2} \right)$$

$$\stackrel{(2)}{\not\exists} \lim_{r \rightarrow 0} \left( \frac{2r \cos(ar)}{4r} \right)$$

$$= \frac{1}{2} \cos(0)$$

$$\therefore \boxed{\frac{1}{2}}$$

PROBLEM 74/ Show  $\lim_{(x,y) \rightarrow (0,0)} \left[ \frac{x-y^2}{x^2+y^2} \right]$  diverges.

$$\lim_{(x,x) \rightarrow (0,0)} \left[ \frac{x-x^2}{x^2+x^2} \right] = \lim_{x \rightarrow 0} \left[ \frac{x-x^2}{2x^2} \right] = \lim_{x \rightarrow 0} \left[ \frac{1-x}{x} \right] \text{ d.n.e.}$$

Since  $\frac{1}{x} \rightarrow \infty$  as  $x \rightarrow 0^+$  and  $\frac{1}{x} \rightarrow -\infty$  as  $x \rightarrow 0^-$ .

Therefore, the limit diverges.

(Converges  $\Rightarrow$  all paths converge)

thus one failing means the limit  $(x,y) \rightarrow (0,0)$  diverges.)

Polar approach:

$$\lim_{(x,y) \rightarrow (0,0)} \left[ \frac{x-y^2}{x^2+y^2} \right] = \lim_{r \rightarrow 0} \left[ \frac{r \cos \theta - r^2 \sin^2 \theta}{r^2} \right]$$

$$= \lim_{r \rightarrow 0} \left[ \underbrace{\frac{\cos \theta}{r} - \sin^2 \theta}_{\text{clearly diverges.}} \right]$$

clearly diverges.

1.)  $\frac{1}{r} \rightarrow \infty$  as  $r \rightarrow 0$

2.)  $\cos \theta, \sin^2 \theta$  multiply-valued at  $r=0$ .

PROBLEM 75

$$\text{Let } f(x, y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ A & (x, y) = 0 \end{cases}$$

Choose  $A$  such that  $f$  is continuous at  $(0, 0)$ .

Polar coordinates:  $\lim_{r \rightarrow 0} \left( \frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r^2} \right) = \lim_{r \rightarrow 0} (r(\cos^3 \theta + \sin^3 \theta)) = 0.$

Choose  $A = 0$  thus  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0 = f(0, 0)$

which gives the desired continuity at  $(0, 0)$ .

These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed.

**Problem 76** Suppose  $f(x, y) = x \cosh(x + y^2)$ . Calculate  $f_x$  and  $f_y$

$$\boxed{\begin{aligned} f_x &= \cosh(x + y^2) + x \sinh(x + y^2) \\ f_y &= 2y x \sinh(x + y^2) \end{aligned}}$$

**Problem 77** Calculate  $\nabla f$  for each of the functions below:

1.  $f(x, y) = 2x + 3y$

$$\boxed{\nabla f = \langle 2, 3 \rangle}$$

2.  $f(x, y) = \exp(-x^2 + 2x - y^2)$

$$\nabla f = \langle (-2x + 2)e^{-x^2+2x-y^2}, -2ye^{-x^2+2x-y^2} \rangle$$

$$\boxed{\nabla f = 2e^{-x^2+2x-y^2} \langle 1-x, -y \rangle} \leftarrow \text{easier for later calculations.}$$

3.  $f(x, y) = \sin(x + y)$

$$\boxed{\nabla f = \langle \cos(x+y), \cos(x+y) \rangle}$$

**Problem 78** What is the rate of change in the functions given in Problem 77 at the point  $(1, 3)$  in the direction of the vector  $\langle 1, -1 \rangle$ .  $\rightsquigarrow \hat{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$

$$D_{\hat{u}} f(1, 3) = \nabla f(1, 3) \cdot \hat{u}$$

$$1.) D_{\hat{u}} f(1, 3) = \langle 2, 3 \rangle \cdot \left( \frac{1}{\sqrt{2}} \langle 1, -1 \rangle \right) = \frac{2-3}{\sqrt{2}} = \boxed{-\frac{1}{\sqrt{2}}}$$

$$2.) D_{\hat{u}} f(1, 3) = 2e^{-1+2-9} \langle 0, -3 \rangle \cdot \left( \frac{1}{\sqrt{2}} \langle 1, -1 \rangle \right) = \frac{6e^{-8}}{\sqrt{2}} \boxed{\frac{6e^{-8}}{\sqrt{2}}}$$

$$3.) D_{\hat{u}} f(1, 3) = \langle \cos 4, \cos 4 \rangle \cdot \left( \frac{1}{\sqrt{2}} \langle 1, -1 \rangle \right) = \boxed{0}$$

**Problem 79** Again, concerning the functions given in Problem 77, in what directions are the functions locally constant at the point  $(1,3)$ ? (give answers in terms of unit-direction vectors)?

$f$  is constant where  $D_{\hat{u}} f(1,3) = (\nabla f)(1,3) \cdot \hat{u} = 0$ .

$$1.) \langle 2, 3 \rangle \cdot \langle a, b \rangle = 0 \Rightarrow 2a + 3b = 0$$

$$\text{Then } a = -\frac{3b}{2} \Rightarrow \left(-\frac{3b}{2}\right)^2 + b^2 = \frac{13b^2}{4} = 1 \therefore b = \pm \frac{2}{\sqrt{13}}$$

$$\text{and } a = \mp \frac{3}{\sqrt{13}} \text{ so } \boxed{\hat{u} = \pm \left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle}$$

$$2.) \boxed{\hat{u} = \langle \pm 1, 0 \rangle} \quad (\text{by almost same calculation})$$

$$3.) \boxed{\hat{u} = \pm \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$$

divided by  $\sqrt{2}$  to normalize  
to length one.

**Problem 80** Find parametrizations for the normal lines through  $(1,3)$  for the functions given in Problem 77. These lines will be perpendicular to the level curve of  $f$  through  $(1,3)$ .

In each case can use  $\vec{r}(t) = (1,3) + t \nabla f(1,3)$ .  
although, for case 2 I drop the exponential constant  
for convenience.

$$1.) \boxed{\vec{r}(t) = (1,3) + t \langle 2, 3 \rangle}$$

$$2.) \boxed{\vec{r}(t) = (1,3) + t \langle 0, -3 \rangle}$$

$$3.) \boxed{\vec{r}(t) = (1,3) + t \langle 1, 1 \rangle}$$

**Problem 81** find all critical points of (use integer notation for (b.) since there are many answers!)

1.  $f(x, y) = \exp(-x^2 + 2x - y^2)$

$$\nabla f(x, y) = \langle 0, 0 \rangle \Rightarrow \underbrace{2e^{-x^2+2x-y^2}}_{\substack{\text{nonzero} \\ \text{always}}} \langle 1-x, -y \rangle = \langle 0, 0 \rangle$$

$$\Rightarrow 1-x=0 \quad -y=0$$

$$\Rightarrow x=1, \quad y=0$$

∴  $(1, 0)$  only critical pt

2.  $f(x, y) = \sin(x + y)$

$$\nabla f(x, y) = \langle \cos(x+y), \cos(x+y) \rangle = \langle 0, 0 \rangle$$

$$\Rightarrow \cos(x+y) = 0 \quad \therefore x+y = \frac{1}{2}(2n+1)\pi \text{ for } n \in \mathbb{Z}.$$

set of all critical points =  $\{ (x, y) \mid y = \frac{1}{2}(2n+1)\pi - x, n \in \mathbb{Z} \}$

**Problem 82** Suppose the temperature  $T$  is a function of the coordinates  $x, y$  in a large plane of battle. Furthermore, suppose the enemy ninja is carefully building a large attack by molding chakra over some time. During the preparation of the attack the enemy is vulnerable to your attack. Knowing this he has obscured the field of vision with multiple smoke bombs. However, the mass of energy building actually heats the ground. Fortunately one of your ninja skills is temperature sensitivity. You extrapolate from the temperature of the ground near your location that the temperature function has the form  $T(x, y) = 50 + x - y$ . In what direction should you attack?

$$\nabla T(x, y) = \langle 1, -1 \rangle$$

$$\Rightarrow \hat{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle \text{ is direction}$$

of fastest increase in  $T$   
and thus the direction to the enemy.

( $\langle 1, -1 \rangle$  also good here  
since I didn't specify unit-vector  
answer)

**Problem 83** [use of technology to solve algebraic and/or transcendental equation that the problem suggests] The temperature in an air conditioned room is set at 65. A ninja with expert ocular jitzu disguises himself in plain sight by bending light near him with his art. However, his art does not extend to the infrared spectrum and his body heat leaves a signature variation in the otherwise constant room temperature. In particular,

$$T(x, y, z) = 33 \exp[-(x-3)^2 - (y-4)^2 - (z-1)^2] + 65.$$

Shino searches for the cloaked ninja by sending insect scouts which are capable of sensing a change in temperature as minute as 0.1 degree per meter. How close do the scout insects have to get before they sense the hidden ninja? (also, where is the hidden ninja and what is his body temperature on the basis of the given  $T$  which is in meters and degrees Farenheight)

$$\begin{aligned}\nabla T(x, y, z) &= \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle \\ &= \left\langle 33e^x \frac{\partial x}{\partial x}, 33e^x \frac{\partial x}{\partial y}, 33e^x \frac{\partial x}{\partial z} \right\rangle \\ &= 33e^x \langle -2(x-3), -2(y-4), -2(z-1) \rangle\end{aligned}$$

Notice  $(3, 4, 1)$  gives  $\nabla T(3, 4, 1) = \langle 0, 0, 0 \rangle$  and this makes  $(3, 4, 1)$  a critical point. A moment's reflection on  $T(x, y, z)$  formula reveals  $(3, 4, 1)$  gives global max. The ninja is at  $(3, 4, 1)$  with  $T(3, 4, 1) = 98^\circ \text{F}$

The insects sense  $|D_u T| = |\nabla T \cdot \hat{u}| \geq 0.1 \Rightarrow$  once we find  $(x, y, z)$  with  $\|\nabla T(x, y, z)\| = 0.1$  then the insects sense the hidden ninja.

$$\|\nabla T(x, y, z)\| = 66e^x \sqrt{(x-3)^2 + (y-4)^2 + (z-1)^2} = 0.1$$

**Problem 84** Suppose  $A, B, C$  are constants. Calculate all nonzero partial derivatives for  $z = Ax^2 + Bxy + Cy^2$ .

Let  $x = 3 + p \cos \theta \sin \phi$ ,  $y = 4 + p \sin \theta \sin \phi$ ,  $z = 1 + p \cos \phi$   
then  $\|\nabla T(x, y, z)\| = 66e^{-p^2} p = 0.1$

Note for  $p \approx 0$  we have  $e^{-p^2} \approx 1$  so solve  $66p = 0.1$  and obtain  $p \approx \frac{0.1}{66} \Rightarrow$  insects within distance

$$\approx \frac{0.1}{66} \text{ find the ninja}$$

$$\begin{aligned}\partial_x z &= 2Ax + By \\ \partial_y z &= Bx + 2Cy\end{aligned}$$

$$\begin{aligned}\partial_{xy} z &= B \\ \partial_{xx} z &= 2A \\ \partial_{yy} z &= 2C\end{aligned}$$

However, the sol<sup>k</sup>  
 $\boxed{p \approx 2.7}$  is better  
(Wolfram Alpha)

**Problem 85** Assume  $g, h$  are differentiable functions on  $\mathbb{R}$ . Calculate  $f_x$  and  $f_{xy}$  for

$$f(x, y) = xg(x^2 + y^2) + h(x)$$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} (xg(x^2 + y^2)) + \frac{\partial h}{\partial x} \\ &= g(x^2 + y^2) + xg'(x^2 + y^2) \frac{\partial}{\partial x}(x^2 + y^2) + h'(x) \frac{\partial x}{\partial x} \\ &= [g(x^2 + y^2) + 2x^2 g'(x^2 + y^2) + h'(x)] = f_x \end{aligned}$$

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} (xg(x^2 + y^2)) + \frac{\partial h}{\partial y} \leftarrow h'(x) \frac{\partial x}{\partial y} = 0. \\ &= [2xy g'(x^2 + y^2)] \end{aligned}$$

**Problem 86** The ideal gas law states that  $P = kT/V$  for a volume  $V$  of gas at temperature  $T$  and pressure  $P$ . Show that

$$V \frac{\partial P}{\partial V} = -P \quad \text{and} \quad V \frac{\partial P}{\partial V} + T \frac{\partial P}{\partial T} = 0$$

$$\frac{\partial P}{\partial V} = \frac{\partial}{\partial V} \left( \frac{kT}{V} \right) = -\frac{kT}{V^2} = -\frac{P}{V} \Rightarrow V \frac{\partial P}{\partial V} = -P$$

$$\frac{\partial P}{\partial T} = \frac{\partial}{\partial T} \left( \frac{kT}{V} \right) = \frac{k}{V} = \frac{P}{T} \Rightarrow T \frac{\partial P}{\partial T} = P.$$

$$\therefore V \frac{\partial P}{\partial V} + T \frac{\partial P}{\partial T} = 0.$$

**Problem 87** The operation of  $\nabla = \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j}$  takes in a function  $f$  with domain in  $\mathbb{R}^n$  and creates a vector field  $\nabla f$  which assigns an  $n$ -vector at each point in  $\mathbb{R}^n$ . This operation has several nice properties to prove here: for differentiable real-valued functions  $f, g$  and constant  $c$ ,

(a.)  $\nabla(f + g) = \nabla f + \nabla g$

$$\begin{aligned} \nabla(f + g) &= \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j} (f + g) \\ &= \sum_{j=1}^n \hat{x}_j \left( \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j} \right) \\ &= \sum_{j=1}^n \hat{x}_j \frac{\partial f}{\partial x_j} + \sum_{j=1}^n \hat{x}_j \frac{\partial g}{\partial x_j} \\ &= \nabla f + \nabla g. \end{aligned}$$

$$(b.) \nabla(cf) = c\nabla f$$

$$\begin{aligned}\nabla(cf) &= \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j}(cf) \\ &= \sum_{j=1}^n c \hat{x}_j \frac{\partial f}{\partial x_j} \\ &= c \sum_{j=1}^n \hat{x}_j \frac{\partial f}{\partial x_j} \\ &= \underline{c \nabla f}.\end{aligned}$$

$$(c.) \nabla(fg) = g\nabla f + f\nabla g$$

$$\begin{aligned}\nabla(fg) &= \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j}(fg) \\ &= \sum_{j=1}^n \hat{x}_j \left( \frac{\partial f}{\partial x_j} g + f \frac{\partial g}{\partial x_j} \right) \\ &= \left( \sum_{j=1}^n \hat{x}_j \frac{\partial f}{\partial x_j} \right) g + f \left( \sum_{j=1}^n \hat{x}_j \frac{\partial g}{\partial x_j} \right) \\ &= \underline{(\nabla f)g + f \nabla g}.\end{aligned}$$

**Problem 88** Set-aside the polar coordinate notation. Define in  $\mathbb{R}^n$  the spherical radius by

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \|\vec{r}\| = \sqrt{\vec{r} \cdot \vec{r}}$$

Show that:

$$(a.) \nabla r = \frac{1}{r} \vec{r}$$

$$\begin{aligned}\nabla r &= \sum_{j=1}^n \hat{x}_j \frac{\partial r}{\partial x_j} \\ &= \sum_{j=1}^n \hat{x}_j \left( \frac{x_j}{r} \right) \\ &= \frac{1}{r} \sum_{j=1}^n \hat{x}_j \hat{x}_j \\ &= \underline{\frac{1}{r} \vec{r}} = \hat{r} \cdot \underline{(useful \text{ later})}.\end{aligned}$$

$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ 
 $\frac{\partial r}{\partial x_j} = \frac{1}{2\sqrt{x_1^2 + \cdots + x_n^2}} \frac{\partial}{\partial x_j} (x_1^2 + x_2^2 + \cdots + x_n^2)$ 
 $= \frac{2x_j}{2r}$ 
 $= x_j/r$

$$(b.) \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}.$$

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) \\ &= \sum_{j=1}^n \hat{x}_j \left(-\frac{1}{r^2} \frac{\partial r}{\partial x_j}\right) \quad \text{by } * \text{ from (a.)} \\ &= -\frac{1}{r^2} \sum_{j=1}^n \frac{\hat{x}_j}{r} \hat{x}_j \\ &= -\frac{1}{r^3} \sum_{j=1}^n x_j \hat{x}_j = \underline{-\frac{1}{r^3} \vec{r}} \quad (\text{for the interested})\end{aligned}$$

**Problem 89** A "power" rule? Show  $\nabla r^n = nr^{n-1}\hat{r}$ .

$$\begin{aligned}\nabla(r^n) &= \sum_{j=1}^n \hat{x}_j \frac{\partial}{\partial x_j}(r^n) \\ &= \sum_{j=1}^n \hat{x}_j nr^{n-1} \frac{\partial r}{\partial x_j} \quad \text{by } * \text{ from (a.)} \\ &= \sum_{j=1}^n \hat{x}_j nr^{n-1} \frac{\hat{x}_j}{r} \\ &= nr^{n-1} \left(\frac{1}{r} \sum_{j=1}^n x_j \hat{x}_j\right) \\ &= \underline{nr^{n-1} \hat{r}}.\end{aligned}$$

**Problem 90** Find the gradient of:

$$1. f(x, y, z, w) = x + y^2 + z^3 + w^4$$

$$\boxed{\nabla f = \langle 1, 2y, 3z^2, 4w^3 \rangle}$$

$$2. f(x, y, z) = xyz \ln(x+y+z)$$

$$\boxed{\nabla f = \left\langle yz \ln(x+y+z) + \frac{xyz}{x+y+z}, xz \ln(x+y+z) + \frac{xyz}{x+y+z}, xy \ln(x+y+z) + \frac{xyz}{x+y+z} \right\rangle}$$

**Problem 91** Suppose Paccun speeds towards the base of a valley with paraboloid shape given by the equation  $z = x^2 + 3y^2$ . What is the direction of steepest descent at the point  $(1, 1, 4)$ ?

$$\nabla z = \langle 2x, 6y \rangle$$

Descent greatest when  $\nabla z$  decreases fastest

$$\Rightarrow \text{in direction of } -\nabla z(1,1) = -\langle 2, 6 \rangle = \boxed{\langle -2, -6 \rangle}.$$

$$\text{Or, as a unit-vector, } \boxed{\frac{1}{\sqrt{40}} \langle -2, -6 \rangle}$$

$$\left( D_{\hat{u}} \nabla z \right)(1,1) = \nabla z(1,1) \cdot \hat{u} \Leftarrow \begin{matrix} \hat{u} \text{ antiparallel} \\ \text{to } \nabla z(1,1) \text{ largest} \\ \text{decrease} \end{matrix}$$

**Problem 92** Let  $f(x, y) = x^3 - xy$ . Let  $A = (0, 1)$  and  $B = (1, 3)$ . Find a point  $C$  on the line-segment  $\overline{AB}$  such that  $f(B) - f(A) (= \nabla(C) \cdot (B - A))$ . (this illustrates a mean-value theorem which is known for real-valued functions of several variables)

$$\boxed{(\nabla f)(c)} \quad f(B) - f(A) = f(1, 3) - f(0, 1) = 1 - 3 - 0 = -2.$$

$$\nabla f = \langle 3x^2 - y, -x \rangle, B - A = \langle 1, 2 \rangle$$

Let  $C = (x, y)$  and note we want

$$-2 = \langle 3x^2 - y, -x \rangle \cdot \langle 1, 2 \rangle$$

$$\Rightarrow -2 = \underline{3x^2 - y - 2x} \cdot (I)$$

Note,  $(x, y)$  is on  $\overline{AB}$  hence  $\exists t \in \mathbb{R}$  such that

$$\vec{r}(t) = (0, 1) + t \langle 1, 2 \rangle = \langle t, 1+2t \rangle = \langle x, y \rangle$$

Substitute into (I.)

$$-2 = 3(t)^2 - (1+2t) - 2t$$

$$3t^2 - 4t + 1 = 0$$

$$t = \frac{4 \pm \sqrt{16 - 12}}{6} = \frac{4 \pm 2}{6} = \frac{6}{6} \text{ or } \frac{2}{6}$$

We want  $0 < t < 1$  for  $\vec{r}(t)$  on  $\overline{AB}$  so choose  $\nearrow$

$$C = \vec{r}\left(\frac{1}{3}\right) = \boxed{\left(\frac{1}{3}, \frac{5}{3}\right)}$$

**Problem 93** Suppose  $\vec{F}(x, y, z) = \langle 2xy^2, 2x^2y, 3 \rangle$ . What scalar function  $f$  yields  $\vec{F}$  as a gradient vector field? Find  $f$  such that  $\nabla f = \vec{F}$ .

(here we have to work backwards, write down what you want and guess, by the way, the function  $-f$  is the potential energy function for the force field  $\vec{F}$ .)

Need  $\frac{\partial f}{\partial x} = 2xy^2$  AND  $\frac{\partial f}{\partial y} = 2x^2y$  AND  $\frac{\partial f}{\partial z} = 3$

$$\underbrace{\frac{\partial f}{\partial x} = 2xy^2}_{f(x,y,z) = x^2y^2 + c_1}$$

$$\underbrace{\frac{\partial f}{\partial y} = 2x^2y}_{f(x,y,z) = x^2y^2 + c_2}$$

$$\underbrace{\frac{\partial f}{\partial z} = 3}_{f(x,y,z) = 3z + c_3}$$



$$f(x, y, z) = x^2y^2 + 3z + C$$

**Problem 94** A basic wave equation is

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}.$$

The wave can be viewed as a graph in the  $xy$ -plane which animates with time  $t$ . Calculate appropriate partial derivatives to test if the functions give a solution to the given wave equation. ( $c_1, c_2, v \neq 0$  are constants)

(a.)  $y(x, t) = x - vt$

$$y_t = -v$$

$$y_{tt} = 0$$

$$y_x = 1$$

$$y_{xx} = 0$$

Thus,  $\underline{\frac{\partial^2 y}{\partial t^2}} = v^2 \frac{\partial^2 y}{\partial x^2}$  (0 = 0)

not too exciting...

but true.

$$(b.) y(x, t) = c_1 \sin(x - vt) + c_2 \cos(x - vt)$$

$$y_t = -c_1 v \cos(x - vt) + c_2 v \sin(x - vt)$$

$$y_{tt} = -c_1 v^2 \sin(x - vt) - c_2 v^2 \cos(x - vt) = -v^2 y$$

$$y_x = c_1 \cos(x - vt) - c_2 \sin(x - vt)$$

$$y_{xx} = -c_1 \sin(x - vt) - c_2 \cos(x - vt) = -y$$

$$\text{Thus } \frac{\partial^2 y}{\partial t^2} = -v^2 y$$

$$\text{AND } v^2 \frac{\partial^2 y}{\partial x^2} = v^2 (-y)$$

$$\therefore \underline{\frac{\partial^2 y}{\partial t^2}} = \underline{v^2 \frac{\partial^2 y}{\partial x^2}}.$$

$$(c.) y(x, t) = \sin(vt) \sin(x)$$

$$y_t = v \cos(vt) \sin x$$

$$y_{tt} = -v^2 \sin(vt) \sin x = -v^2 y$$

$$y_x = \sin vt \cos x$$

$$y_{xx} = -\sin vt \sin x = -y$$

$$-v^2 y = \underline{\frac{\partial^2 y}{\partial t^2}} = \underline{v^2 \frac{\partial^2 y}{\partial x^2}}.$$

**Problem 95** Show that  $u = e^x \cos(y)$  and  $v = e^x \sin(y)$  solves Laplace's equation  $\Phi_{xx} + \Phi_{yy} = 0$ .

$$\underline{\text{For } u:} \quad u_x = e^x \cos y \quad u_y = -e^x \sin y \\ u_{xx} = e^x \cos y \quad u_{yy} = -e^x \cos y \quad \therefore u_{xx} + u_{yy} = 0.$$

$$\underline{\text{For } v:} \quad v_x = e^x \sin y \quad v_y = e^x \cos y \\ v_{xx} = e^x \sin y \quad v_{yy} = -e^x \sin y \quad \therefore v_{xx} + v_{yy} = 0.$$

**Problem 96** Find the best linear approximation of each object at the given point. Also, write either an equation or a parametrization of the tangent space in each case ("space" could mean line, surface, space curve or other things...)

(a.)  $f(x) = x^2$  at  $a = 2$ ,  $f'(x) = 2x$ .

$$L_f(x) = f(a) + f'(a)(x-a) \quad \underline{a=2} \\ = \boxed{4 + 4(x-2)}$$

(b.)  $f(x, y) = x^2 - 2xy$  at  $(3, 4)$ ,  $f(3, 4) = 9 - 2(3)(4) = 9 - 24 = -15$ .

$$\nabla f = \langle 2x - 2y, -2x \rangle$$

$$\nabla f(3, 4) = \langle 6 - 8, -6 \rangle = \langle -2, -6 \rangle$$

$$\boxed{L_f(x, y) = -15 - 2(x-3) - 6(y-4)}$$

(c.)  $\vec{r}(t) = \langle t, 3, t^2 2^t \rangle$  at  $t = 0$

$$\vec{r}(0) = \langle 0, 3, 0 \rangle$$

$$\vec{r}'(t) = \langle 1, 0, 2t 2^t + t^2 \ln(2) 2^t \rangle$$

$$\vec{r}'(0) = \langle 1, 0, 0 \rangle$$

$$\boxed{\vec{l}(t) = \langle 0, 3, 0 \rangle + t \langle 1, 0, 0 \rangle}$$

**Problem 97** Given that  $x = u^2 + v^2$  and  $y = 3uv$  and  $z = 3 \sin(uv)$  and  $w = ze^{xy}$  calculate  $w_u$  and  $w_v$  and finally  $w_z$ .

$$w = w(x, y, z)$$

$$\begin{aligned} w_u &= \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\ &= \underline{yz e^{xy}(2u) + xze^{xy}(3v) + e^{xy} 3 \cos(uv) \cdot v} \\ w_v &= \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\ &= \underline{yz(2v) + xze^{xy}(3u) + e^{xy} 3 \cos(uv) \cdot u} \\ w_z &= \frac{\partial w}{\partial z} = \frac{\partial}{\partial z} (ze^{xy}) = \boxed{e^{xy}} \end{aligned}$$

where  $x = u^2 + v^2$   
 $y = 3uv$   
 $z = 3 \sin(uv)$

**Problem 98** Calculate the Jacobian matrix of  $\vec{r}(u, v) = \langle u^2 + v^2, 3uv, 3 \sin(uv) \rangle$  and that of  $f(x, y, z) = ze^{xy}$ . Multiply these matrices and identify how this relates to the previous problem.

$$\vec{r}: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow J_{\vec{r}} \text{ is } 3 \times 2$$

$$\vec{r}' = \left[ \begin{array}{c|c} \frac{\partial \vec{r}}{\partial u} & \frac{\partial \vec{r}}{\partial v} \end{array} \right] = \left[ \begin{array}{c|c} 2u & 2v \\ 3v & 3u \\ 3v \cos(uv) & 3u \cos(uv) \end{array} \right]$$

$$f' = \nabla f = [yz e^{xy}, xze^{xy}, e^{xy}]$$

$$w' = (f \circ \vec{r})' = f' \vec{r}' = [yz e^{xy}, xze^{xy}, e^{xy}] \left[ \begin{array}{c|c} 2u & 2v \\ 3v & 3u \\ 3v \cos(uv) & 3u \cos(uv) \end{array} \right]$$

$$\left[ \frac{\partial w}{\partial u}, \frac{\partial w}{\partial v} \right] = \underline{[yz e^{xy}(2u) + xze^{xy}(3v) + e^{xy}(3v \cos(uv))]}$$

$$\underline{yz e^{xy}(2v) + xze^{xy}(3u) + e^{xy} 3u \cos uv}$$

$\frac{\partial w}{\partial v}$  matches!

**Problem 99** Suppose  $z = xy$  and  $x = \sinh[g(t)]$  and  $y = h(t^2)$  for some differentiable functions  $g, h$ . Calculate  $dz/dt$  by the chain rule(s).

$$\begin{aligned}
 \frac{dz}{dt} &= \frac{d}{dt}(xy) \\
 &= \frac{dx}{dt}y + x\frac{dy}{dt} \\
 &= \frac{d}{dt}[\sinh(g(t))]y + x\frac{d}{dt}[h(t^2)] \\
 &= y \cosh(g(t)) \frac{dg}{dt} + x h'(t^2) \cdot 2t \\
 &= \boxed{h(t^2) \cosh(g(t)) g'(t) + \sinh(g(t)) h'(t^2) \cdot 2t}
 \end{aligned}$$

$$x^2 + 4y^2 - z^2 = 1$$

**Problem 100** Suppose a car speeds over a hill with equation  $x^2 + 4y^2 + z^2 = 1$ . If at the point with  $x = 1m$  and  $y = 0.5m$  the car has an  $x$ -velocity of  $10m/s$  and a  $y$ -velocity of  $20m/s$  then what  $z$ -velocity does the car have? (assume the car stays on hill)

$$\begin{aligned}
 -z^2 &= 1 - x^2 - 4y^2 \\
 &= 1 - 1 - 4(0.25) \\
 &= -1 \\
 \Rightarrow z &= \pm 1m
 \end{aligned}$$

$$\begin{aligned}
 \dot{z} &= \frac{x\dot{x} + 4y\dot{y}}{z} \\
 &= \frac{(1m)(10\text{ m/s}) + 4(0.5m)(20\text{ m/s})}{1m} \\
 &= 10\text{ m/s} + 40\text{ m/s} \\
 &= \boxed{50\text{ m/s}}
 \end{aligned}$$