

Problem 115 Suppose  $\vec{F} = \rho^2 \hat{\rho} + \frac{1}{\rho} \sin(\phi) \hat{\phi}$  find  $f$  such that  $\nabla f = \vec{F}$ .

$$\nabla \vec{F} = \rho^2 \hat{\rho} + \frac{\sin \phi}{\rho} \hat{\phi} = \hat{\rho} \frac{\partial f}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \hat{\theta} \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta}$$

we can compare coefficients of  $\hat{\rho}, \hat{\phi}, \hat{\theta}$   
since these are orthonormal!

$$\frac{\partial f}{\partial \rho} = \rho^2 \quad \text{and} \quad \frac{\partial f}{\partial \phi} = \sin \phi \quad \text{and} \quad \frac{\partial f}{\partial \theta} = 0$$

$$f(\rho, \phi, \theta) = \frac{1}{3} \rho^3 - \cos \phi$$

(can add constant and still obtain  
same gradient)

Problem 116 Suppose  $f(x, y, z) = (x^2 + y^2 + z^2) \tan^{-1}(y/x) + \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$ . Calculate  $\nabla f$ .

$$f(\rho, \phi, \theta) = \rho^2 \theta \phi$$

$$\nabla f = \hat{\rho} \frac{\partial}{\partial \rho} (\rho^2 \theta \phi) + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^2 \theta \phi) + \hat{\theta} \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \theta} (\rho^2 \theta \phi)$$

$$\nabla f = 2\rho \theta \phi \hat{\rho} + \rho \theta \hat{\phi} + \left( \frac{\rho \phi}{\sin \phi} \right) \hat{\theta}$$

**Problem 117** Suppose a formula for  $f(x, y)$  is given. Furthermore, suppose you are asked to calculate  $\frac{\partial f}{\partial r}$  where  $r = \sqrt{x^2 + y^2}$ . Technically, this question is ambiguous. Why? Because you need to know what other variable besides  $r$  is to be used in concert with  $r$ . If we use the usual polar coordinates then  $\tan(\theta) = \frac{y}{x}$  and all is well. We adopt the following (standard) interpretation:

$$f_r = \frac{\partial f}{\partial r} = \frac{\partial}{\partial r} \left[ f(r \cos \theta, r \sin \theta) \right] = \frac{\partial f}{\partial x} \Big|_{(r \cos \theta, r \sin \theta)} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \Big|_{(r \cos \theta, r \sin \theta)} \frac{\partial y}{\partial r}$$

In other words, we define the derivative of  $f$  with respect to some curvilinear coordinate by the derivative of  $f \circ \vec{T}$  where  $\vec{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the coordinate transformation to which the curvilinear coordinate belongs. Denoting  $\vec{T}(r, \theta) = (r \cos \theta, r \sin \theta)$  we define,

$$f_\theta = \frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ (f \circ \vec{T})(r, \theta) \right] = \frac{\partial f}{\partial x} \Big|_{(r \cos \theta, r \sin \theta)} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \Big|_{(r \cos \theta, r \sin \theta)} \frac{\partial y}{\partial \theta}$$

A short calculation reveals that:

$$f_r = f_x \cos \theta + f_y \sin \theta \quad \& \quad f_\theta = -f_x r \sin \theta + f_y r \cos \theta$$

Solve the equations above for  $f_x$  and  $f_y$ .

$$\begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \quad A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

multiply by inverse matrix  $\rightarrow$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{1}{r \cos^2 \theta + r \sin^2 \theta} \begin{pmatrix} r \cos \theta & -\sin \theta \\ r \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos \theta f_r - \sin \theta f_\theta \\ r \sin \theta f_r + \cos \theta f_\theta \end{pmatrix}$$

$$\therefore \boxed{f_x = \cos \theta f_r - \frac{\sin \theta}{r} f_\theta} \quad \& \quad \boxed{f_y = \sin \theta f_r + \frac{\cos \theta}{r} f_\theta}$$

**Problem 118** Recall that  $\nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0$  is called Laplace's equation. In cartesian coordinates, in two dimensions, Laplace's equation reads  $\Phi_{xx} + \Phi_{yy} = 0$ . Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0.$$

(yes, most of this is in the notes, but I'd like to see the rest of the details)

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right) = \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left( \cos \theta \frac{\partial}{\partial r} \Phi - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \Phi \right) \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} \Phi + \cos \theta \sin \theta \left[ \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \Phi - \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \Phi \right] \quad \curvearrowleft - \frac{\sin \theta}{r} \left[ -\sin \theta \frac{\partial}{\partial r} \Phi + \cos \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial r} \Phi \right] \quad \curvearrowleft \\ &\quad \curvearrowleft \rightarrow \frac{\sin \theta}{r} \left[ -\frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \Phi - \frac{\sin \theta}{r} \frac{\partial^2}{\partial \theta^2} \Phi \right] \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} \Phi + \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial}{\partial r} \Phi - \frac{2 \cos \theta \sin \theta}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \Phi + \quad \curvearrowleft \quad (\star) \\ &\quad \curvearrowleft + \frac{\sin^2 \theta}{r} \frac{\partial^2}{\partial r^2} \Phi + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} \Phi \end{aligned}$$

Problem 118

$$\begin{aligned}
 \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\partial}{\partial r} \left( \frac{\partial \Phi}{\partial \theta} \right) \\
 &= (\sin \theta \partial_r + \frac{1}{r} \cos \theta \partial_\theta) \left( \sin \theta \partial_r \Phi + \frac{1}{r} \cos \theta \partial_\theta \Phi \right) \\
 &= \sin^2 \theta \partial_r^2 \Phi + \sin \theta \left[ -\frac{1}{r^2} \cos \theta \partial_\theta \Phi + \frac{1}{r} \cos \theta \partial_r \partial_\theta \Phi \right] \rightarrow \\
 &\quad \rightarrow + \frac{1}{r} \cos \theta \left[ \cos \theta \partial_r \Phi + \sin \theta \partial_\theta \partial_r \Phi \right] \rightarrow \\
 &\quad \rightarrow + \frac{1}{r^2} \cos \theta \left[ -\sin \theta \partial_\theta \Phi + \cos \theta \partial_\theta^2 \Phi \right] \\
 \\ 
 &= \sin^2 \theta \partial_r^2 \Phi - \frac{2 \sin \theta \cos \theta}{r^2} \partial_\theta \Phi + \frac{2 \sin \theta \cos \theta}{r} \partial_r \partial_\theta \Phi \rightarrow \\
 &\quad \rightarrow + \frac{1}{r} \cos^2 \theta \partial_r \Phi + \frac{1}{r^2} \cos^2 \theta \partial_\theta^2 \Phi \quad (\star \star)
 \end{aligned}$$

Now add  $(\star)$  and  $(\star \star)$  notice everything cancels except,

$$\begin{aligned}
 \Phi_{xx} + \Phi_{yy} &= \sin^2 \theta \partial_r^2 \Phi + \frac{1}{r} \cos^2 \theta \partial_r \Phi + \frac{1}{r^2} \cos^2 \theta \partial_\theta^2 \Phi + \\
 &\quad \rightarrow + \cos^2 \theta \partial_r^2 \Phi + \frac{1}{r} \sin^2 \theta \partial_r \Phi + \frac{1}{r^2} \sin^2 \theta \partial_\theta^2 \Phi
 \end{aligned}$$

$$\begin{aligned}
 0 = \Phi_{xx} + \Phi_{yy} &= \partial_r^2 \Phi + \frac{1}{r} \partial_r \Phi + \frac{1}{r^2} \partial_\theta^2 \Phi \\
 &= \Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta\theta} \\
 &= \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = \nabla^2 \Phi = 0
 \end{aligned}$$

**Problem 119** Given the potential functions  $\Phi$  below show they are solutions to Laplace's equations either via computation in cartesian coordinates or polar coordinates.

- (a.)  $\Phi(x, y) = (\sqrt{x^2 + y^2})^0 = 1$  (sorry actually  $\Phi(r, \theta) = r^n$  solves)  
 (b.)  $\Phi(x, y) = \tan^{-1}(y/x)$   
 (c.)  $\Phi(r, \theta) = r^2 \cos \theta \sin \theta$

$$(a.) \quad \nabla_{xx} \Phi + \nabla_{yy} \Phi = 0 + 0 = 0.$$

$$(b.) \quad \Phi(r, \theta) = 0$$

$$\nabla_{rr} \Phi = 0, \quad \nabla_r \Phi = 0, \quad \nabla_{\theta\theta} \Phi = 0$$

$$\Rightarrow \nabla_{rr} \Phi + \frac{1}{r} \nabla_r \Phi + \frac{1}{r^2} \nabla_{\theta\theta} \Phi = 0 + 0 + 0 \neq 0$$

$$(c.) \quad \Phi(x, y) = xy$$

$$\nabla_{xx} \Phi = 0 \quad \text{and} \quad \nabla_{yy} \Phi = 0$$

$$\therefore \nabla_{xx} \Phi + \nabla_{yy} \Phi = 0 + 0 \neq 0$$

**Problem 120** Define hyperbolic coordinates  $h, \phi$  by the following equations

$$x = h \cosh \phi \quad \& \quad y = h \sinh(\phi)$$

Let's study these coordinates by answering the following:

- (a.) solve the equations above for  $h$  and  $\phi$ .  $(\pm 1, 0)$  and  $(-\pm 1, \pm \pi)$
- (b.) Find hyperbolic coordinates for  $(1, 1), (-1, 1), (-1, -1)$  and  $(1, -1)$ . Write a diagram which explains the signs for  $h$  and  $\phi$  in each quadrant.
- (c.) What do are level curves of  $h$  ?
- (d.) What are level curves of  $\phi$ ?

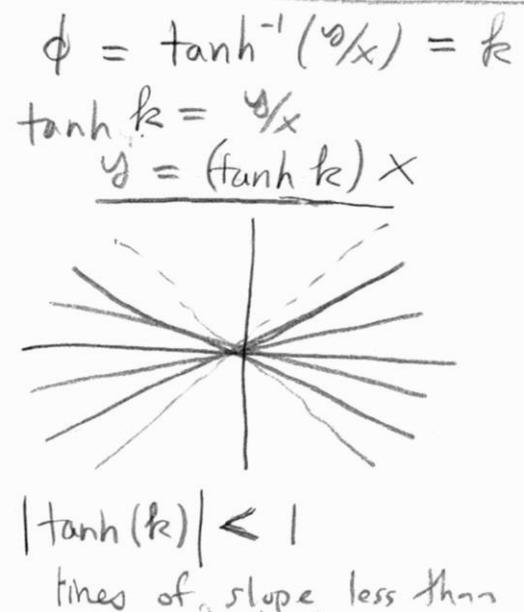
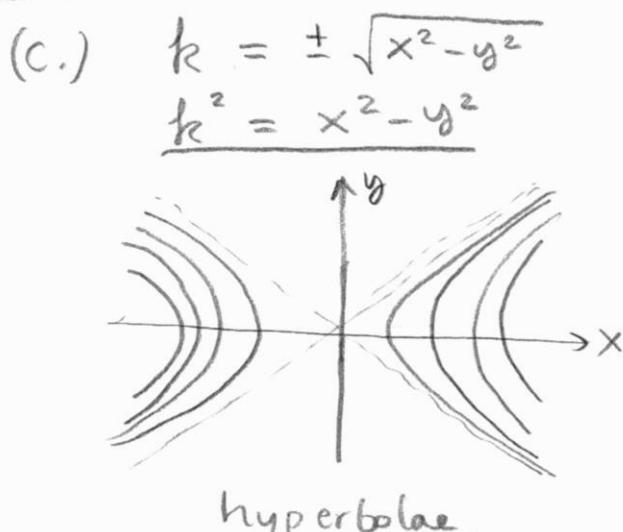
$$(a.) \tanh \phi = \frac{\sinh \phi}{\cosh \phi} = \frac{h \sinh \phi}{h \cosh \phi} = \frac{y}{x}$$

$\phi = \tanh^{-1}(y/x)$

$$\begin{aligned} x^2 - y^2 &= h^2 \cosh^2 \phi - h^2 \sinh^2 \phi && \text{choose } (+) \text{ for } x > 0 \\ &= h^2 (\cosh^2 \phi - \sinh^2 \phi) && \checkmark \quad (-) \text{ for } x < 0 \\ &= h^2 \quad \Rightarrow \quad h = \pm \sqrt{x^2 - y^2} \end{aligned}$$

$$(b.) \left. \begin{array}{l} x = 2 \\ y = 1 \end{array} \right\} \Rightarrow \phi = \tanh^{-1}(1/2) = 0.549 = \phi \quad \left. \begin{array}{l} 0.549 = \phi \\ \sqrt{3} = h \end{array} \right\} \text{for } (2, 1)$$

$$\left. \begin{array}{l} x = -2 \\ y = -1 \end{array} \right\} \Rightarrow \phi = \tanh^{-1}(-1/2) = -0.549 = \phi \quad \left. \begin{array}{l} -0.549 = \phi \\ -\sqrt{3} = h \end{array} \right\} \text{for } (-2, -1)$$



**Problem 121** Continuing the study from the previous problem,

- (a.) find functions  $A, B, C, D$  of hyperbolic coordinates  $h, \phi$  which give unit-vectors

$$\hat{h} = A\hat{x} + B\hat{y} \quad \& \quad \hat{\phi} = C\hat{x} + D\hat{y}$$

- (b.) derive a formula for  $\nabla f$  in terms of hyperbolic coordinate derivatives, however express it in terms of  $\hat{x}$  and  $\hat{y}$ . That is find  $\nabla f = \langle f_x, f_y \rangle$  but express  $f_x$  and  $f_y$  in terms of the hyperbolic coordinates and derivatives.

- (c.) derive  $\nabla f$  in purely hyperbolic notation: that is find  $E, F$  such that

$$\nabla f = E\hat{h} + F\hat{\phi}.$$

partial derivative computation is fun... but, what does it mean? We explore this question in the pair of problems below

( $\hat{h}$  on next page)

$$(a.) \nabla \phi = \nabla (\tanh^{-1}(\frac{y}{x}))$$

$$\frac{d}{du} (\tanh^{-1}(u)) = \frac{1}{1-u^2}$$

derive it.

$$\begin{aligned} &= \left\langle \left( \frac{1}{1-u^2} \right) \frac{\partial}{\partial x} \left[ \frac{y}{x} \right], \left( \frac{1}{1-u^2} \right) \frac{\partial}{\partial y} \left[ \frac{y}{x} \right] \right\rangle \\ &= \left\langle \frac{1}{1-y^2/x^2} \left[ \frac{-y}{x^2} \right], \frac{1}{1-y^2/x^2} \left[ \frac{1}{x} \right] \right\rangle \\ &= \left\langle \frac{-y}{x^2-y^2}, \frac{x}{x^2-y^2} \right\rangle \\ &= \left\langle \frac{-y}{h^2}, \frac{x}{h^2} \right\rangle \\ &= \left\langle \frac{-h \sinh \phi}{h^2}, \frac{h \cosh \phi}{h^2} \right\rangle \\ &= \left( -\frac{1}{h} \sinh \phi \right) \hat{x} + \left( \frac{1}{h} \cosh \phi \right) \hat{y} \end{aligned}$$

$$\begin{aligned} \|\nabla \phi\| &= \sqrt{\frac{1}{h^2} (\sinh^2 \phi + \cosh^2 \phi)} \\ &= \frac{1}{h} \sqrt{\frac{1}{4} (e^{2\phi} - 2 + e^{-2\phi}) + \frac{1}{4} (e^{2\phi} + 2 + e^{-2\phi})} \\ &= \frac{1}{h} \sqrt{\frac{1}{2} (e^{2\phi} + e^{-2\phi})} \\ &= \frac{1}{h} \sqrt{\cosh(2\phi)} \end{aligned}$$

$$\hat{\phi} = \frac{\nabla \phi}{\|\nabla \phi\|} = \frac{-\sinh \phi}{\sqrt{\cosh(2\phi)}} \hat{x} + \frac{\cosh \phi}{\sqrt{\cosh(2\phi)}} \hat{y}$$

$$\therefore C = \frac{-\sinh \phi}{\sqrt{\cosh(2\phi)}} \quad \& \quad D = \frac{\cosh \phi}{\sqrt{\cosh(2\phi)}}$$

Problem 121a continued

Calculate  $\nabla h$  and eventually  $\hat{h}$ ,

$$\begin{aligned}\nabla h &= \langle h_x, h_y \rangle, \quad h = \sqrt{x^2 - y^2} \\ &= \left\langle \frac{2x}{2\sqrt{x^2 - y^2}}, \frac{-2y}{2\sqrt{x^2 - y^2}} \right\rangle \\ &= \left\langle \frac{x}{h}, \frac{-y}{h} \right\rangle \quad \star \\ &= \underline{\cosh \phi \hat{x} - \sinh \phi \hat{y}}\end{aligned}$$

Then normalize,

$$\|\nabla h\| = \sqrt{\cosh^2 \phi + \sinh^2 \phi} = \sqrt{\cosh(2\phi)}$$

$$\therefore \hat{h} = \frac{\cosh \phi}{\sqrt{\cosh(2\phi)}} \hat{x} - \frac{\sinh \phi}{\sqrt{\cosh(2\phi)}} \hat{y}$$

Thus,  $A = \frac{\cosh \phi}{\sqrt{\cosh(2\phi)}}$  and  $B = \frac{-\sinh \phi}{\sqrt{\cosh(2\phi)}}$

121b Calculate  $\nabla f$  in the hyperbolic derivatives  $\frac{\partial}{\partial h}$  &  $\frac{\partial}{\partial \phi}$

$$\frac{\partial}{\partial x} = \frac{\partial h}{\partial x} \frac{\partial}{\partial h} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \quad \begin{array}{l} h = \sqrt{x^2 - y^2} \\ \phi = \tanh^{-1}(y/x) \end{array}$$

$$\frac{\partial}{\partial y} = \frac{\partial h}{\partial y} \frac{\partial}{\partial h} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi}$$

Notice  $h_x = \cosh \phi$  and  $h_y = -\sinh \phi$  from work at  $\star$   
whereas  $\phi_x = \frac{1}{h} \sinh \phi$  and  $\phi_y = \frac{1}{h} \cosh \phi$  from work at  $\circlearrowleft$

Therefore,

$$\nabla f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} = \hat{x} \left( \cosh \phi \frac{\partial f}{\partial h} - \frac{\sinh \phi}{h} \frac{\partial f}{\partial \phi} \right) + \hat{y} \left( -\sinh \phi \frac{\partial f}{\partial h} + \frac{\cosh \phi}{h} \frac{\partial f}{\partial \phi} \right)$$

Problem 12/c) derive  $\nabla f$  in purely hyperbolic notation.

We seek  $E, F$  such that  $\nabla f = E\hat{h} + F\hat{\phi}$  and the best way to single those out is with dot-products.

$$\begin{aligned}
 E &= (\nabla f) \cdot \hat{h} \\
 &= \left\langle \cosh \phi \frac{\partial f}{\partial h} - \frac{\sinh \phi}{h} \frac{\partial f}{\partial \phi}, -\sinh \phi \frac{\partial f}{\partial h} + \frac{\cosh \phi}{h} \frac{\partial f}{\partial \phi} \right\rangle \cdot \left\langle \frac{\cosh \phi}{\sqrt{\cosh 2\phi}}, \frac{-\sinh \phi}{\sqrt{\cosh 2\phi}} \right\rangle \\
 &= \frac{1}{\sqrt{\cosh 2\phi}} \left( \left[ \cosh \phi \frac{\partial f}{\partial h} - \frac{\sinh \phi}{h} \frac{\partial f}{\partial \phi} \right] \cosh \phi + \left[ -\sinh \phi \frac{\partial f}{\partial h} + \frac{\cosh \phi}{h} \frac{\partial f}{\partial \phi} \right] (-\sinh \phi) \right) \\
 &= \frac{1}{\sqrt{\cosh 2\phi}} \left( \frac{(\cosh^2 \phi + \sinh^2 \phi)}{\partial h} \frac{\partial f}{\partial h} - 2 \sinh \phi \cosh \phi \frac{\partial f}{\partial \phi} \right) \\
 &= \frac{\cosh 2\phi}{\sqrt{\cosh 2\phi}} \frac{\partial f}{\partial h} - \frac{\sinh 2\phi}{h \sqrt{\cosh 2\phi}} \frac{\partial f}{\partial \phi} \quad \begin{cases} \text{note: } 2 \sinh \phi \cosh \phi = - \\ \rightarrow \frac{1}{2}(e^\phi - e^{-\phi})(e^\phi + e^{-\phi}) = \\ \rightarrow \frac{1}{2}(e^{2\phi} - e^{-2\phi}) = \sinh 2\phi \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 F &= \nabla f \cdot \hat{\phi} \\
 &= \left\langle \cosh \phi \frac{\partial f}{\partial h} - \frac{\sinh \phi}{h} \frac{\partial f}{\partial \phi}, -\sinh \phi \frac{\partial f}{\partial h} + \frac{\cosh \phi}{h} \frac{\partial f}{\partial \phi} \right\rangle \cdot \left\langle \frac{-\sinh \phi}{\sqrt{\cosh 2\phi}}, \frac{\cosh \phi}{\sqrt{\cosh 2\phi}} \right\rangle \\
 &= \left[ \cosh \phi \frac{\partial f}{\partial h} - \frac{\sinh \phi}{h} \frac{\partial f}{\partial \phi} \right] \left( \frac{-\sinh \phi}{\sqrt{\cosh 2\phi}} \right) + \left[ -\sinh \phi \frac{\partial f}{\partial h} + \frac{\cosh \phi}{h} \frac{\partial f}{\partial \phi} \right] \left( \frac{\cosh \phi}{\sqrt{\cosh 2\phi}} \right) \\
 &= \frac{1}{\sqrt{\cosh 2\phi}} \left( -2 \cosh \phi \sinh \phi \frac{\partial f}{\partial h} + \frac{1}{h} (\sinh^2 \phi + \cosh^2 \phi) \frac{\partial f}{\partial \phi} \right) \\
 &= \frac{-\sinh 2\phi}{\sqrt{\cosh 2\phi}} \frac{\partial f}{\partial h} + \frac{1}{h} \frac{\cosh(2\phi)}{\sqrt{\cosh(2\phi)}} \frac{\partial f}{\partial \phi}
 \end{aligned}$$

Thus,

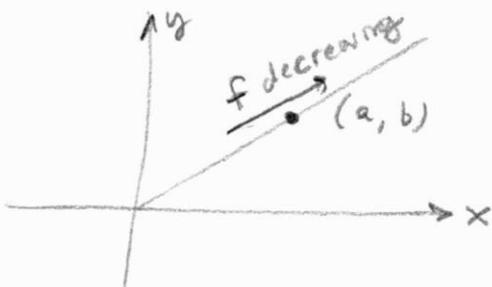
$$\begin{aligned}
 \nabla f &= \frac{1}{\sqrt{\cosh 2\phi}} \left( \cosh(2\phi) \frac{\partial f}{\partial h} - \frac{1}{h} \sinh(2\phi) \frac{\partial f}{\partial \phi} \right) \hat{h} + \\
 &\quad + \frac{1}{\sqrt{\cosh 2\phi}} \left( -\sinh(2\phi) \frac{\partial f}{\partial h} + \frac{1}{h} \cosh(2\phi) \frac{\partial f}{\partial \phi} \right) \hat{\phi}
 \end{aligned}$$

**Problem 122** Let  $(a, b) \in \mathbb{R}^2$  be a particular point. Explain geometrically the meaning of the equations given below:

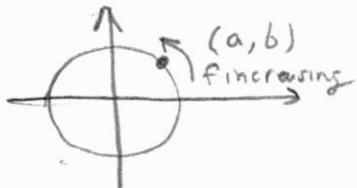
- (a.)  $\frac{\partial f}{\partial r}(a, b) = -1$
- (b.)  $\frac{\partial f}{\partial \theta}(a, b) = 1$
- (c.)  $\frac{\partial f}{\partial \phi}(a, b) = 0$  (same notation as in previous pair of problems)

As an example:  $\frac{\partial f}{\partial x}(a, b) = 0$  indicates that the function stays constant along the line passing through  $(a, b)$  on which  $y$  is held fixed at value  $b$  (parametrically  $f$  is constant along the path  $t \rightarrow (a + ta, b)$  near  $t = 0$ ).

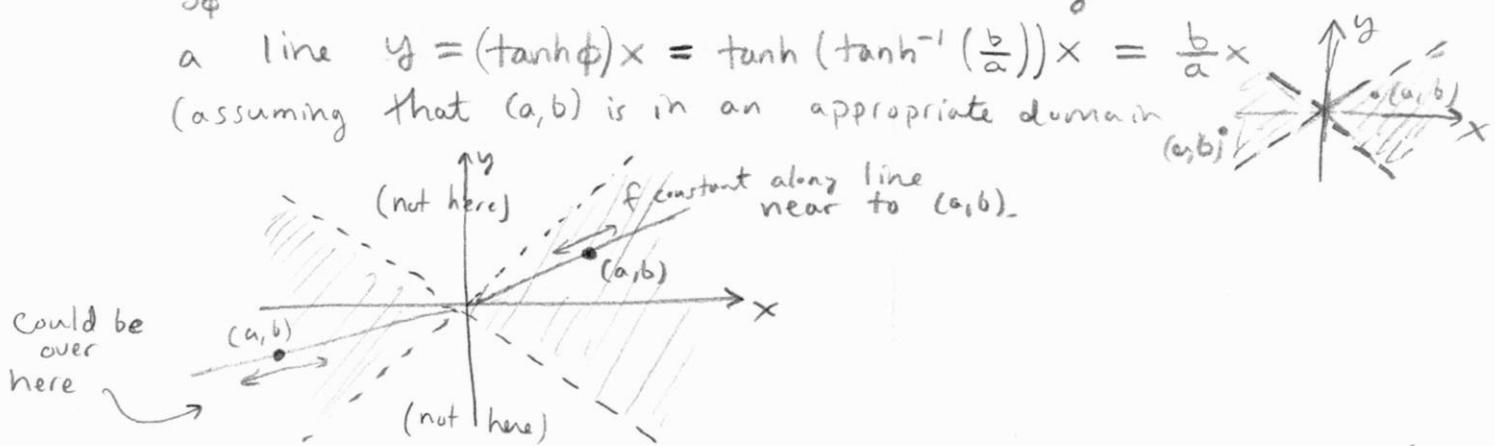
(a.)  $\frac{\partial f}{\partial r}(a, b) = -1$  means  $f$  decreases at rate of 1 along a ray through the origin which passes through  $(a, b)$



(b.)  $\frac{\partial f}{\partial \theta}(a, b) = 1$  means  $f$  increases along a circular arc through  $(a, b)$



(c.)  $\frac{\partial f}{\partial \phi}(a, b) = 0$  means  $f$  is constant along a line  $y = (\tanh \phi)x = \tanh(\tanh^{-1}(\frac{b}{a}))x = \frac{b}{a}x$  (assuming that  $(a, b)$  is in an appropriate domain)



(my arguments are best justified by 3rd page of this problem set.)

**Problem 123** Joshua asked if  $\frac{\partial}{\partial(xy)}$  had meaning. I would say yes. In fact, it has many meanings.

- Define  $u = xy$  and  $v = y/x$  for  $(x, y) \in (0, \infty)^2$ . Find inverse transformations. That is, solve for  $x = x(u, v)$  and  $y = y(u, v)$  in view of the definition just given and comment on the level curves of  $u, v$  (if they are a named curve then name them).
- Explain what  $\frac{\partial f}{\partial u} = 0$  means for a function  $f$  at a given point. (use meaning suggested from part (a.))
- Define  $u = xy$  and  $w = y$  for  $(x, y) \in (0, \infty)^2$ . Find inverse transformations. That is, solve for  $x = x(u, w)$  and  $y = y(u, w)$  in view of the definition just given and comment on the level curves of  $u, w$  (if they are a named curve then name them).
- Explain what  $\frac{\partial f}{\partial u} = 0$  means for a function  $f$  at a given point. (use meaning suggested from part (c.)) (it is **not** a directional derivative in the traditional sense of the term.)

The previous problem is important for applications. Think about this, what variable is most interesting to your model? It is important to be able to write the equations which describe the model in terms of those variables. On the other hand, it may be simple to express the physics of the model in cartesian coordinates. Hopefully these problems give you an idea about how to translate from one formalism to the other and vice-versa.

(a.)  $u = xy \quad \left\{ \begin{array}{l} \text{solve for } x, y > 0 \\ v = y/x \end{array} \right.$

Divide eq's:  $\frac{u}{v} = \frac{xy}{y/x} = x^2 \Rightarrow x = \sqrt{u/v}$

Multiply eq's:  $uv = (xy)(y/x) = y^2 \Rightarrow y = \sqrt{uv}$

u-Level curves  $u = xy \Rightarrow y = u/x$



for  $xy > 0$  only the  $u > 0$  curves are relevant.

v-Level curves  $v = y/x \Rightarrow y = vx$



- (b.)  $\left(\frac{\partial f}{\partial u}\right)_v = 0$  means  $f$  is constant along a path which holds  $v$ -fixed. So, this says  $f$  is constant along a line through the origin close to the given point.

(b.) If I had asked  $\frac{\partial f}{\partial v} = 0$  then the meaning would have been a bit more interesting. If  $\frac{\partial f}{\partial v} = 0$  then  $f$  is constant along curve holding  $u$ -fixed and these are hyperbolas seen in (a).

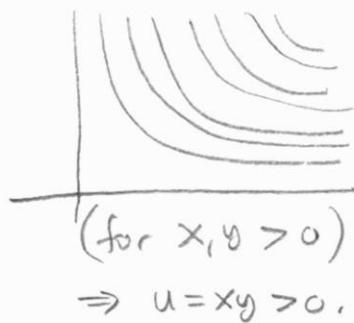
(c.) Let  $u = xy$  solve for  $x$  and  $y$  with  $x, y > 0$   
 $w = y$

Well,  $y = w$ . That's obvious enough  $\textcircled{2}$ .

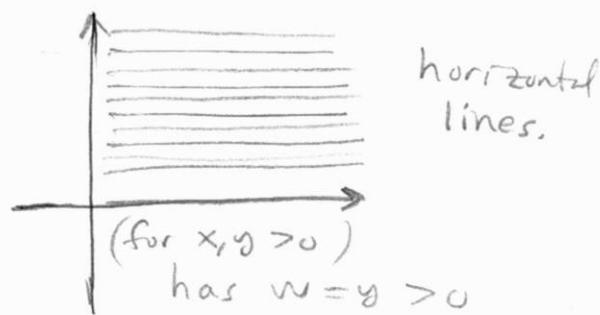
Next,  $x = u/y = u/w$  hence  $x = u/w$

$u$ -level curves are as before,

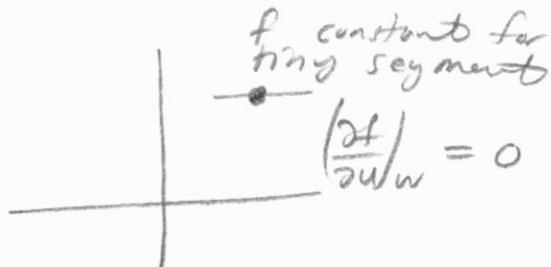
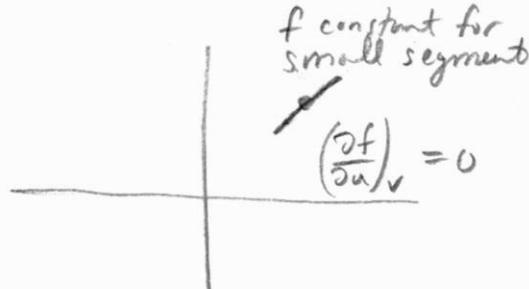
half-hyperbolas



$w$ -level curves



(d.)  $\left(\frac{\partial f}{\partial u}\right)_w = 0$  means  $f$  constant along curve through point which holds  $w$ -fixed as  $u$ -varies. Actually, gives change of  $f$  restricted to horizontal line in this context.



THE  
POINT?

→ (the variable paired with  $u$  plays an important role in defining  $\frac{\partial f}{\partial u}$ .)

Beyond the qualitative comments made thus far,

When  $x = \sqrt{u/v}$  and  $y = \sqrt{uv}$  we find that

$$\begin{aligned}\frac{\partial f}{\partial u} &= \underset{\text{def}}{=} \frac{\partial}{\partial u} \left[ f(\sqrt{u/v}, \sqrt{uv}) \right] && \left( \begin{array}{l} \text{just like} \\ \text{polar} \\ \text{coordinates} \end{array} \right) \\ &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} && \left( \begin{array}{l} \frac{\partial f}{\partial r} \text{ discussed} \\ \text{in earlier} \\ \text{problem} \end{array} \right)\end{aligned}$$

$$= f_x(\sqrt{u/v}, \sqrt{uv}) \frac{1}{2\sqrt{uv}} + f_y(\sqrt{u/v}, \sqrt{uv}) \frac{v}{2\sqrt{uv}}$$

this is how to explicitly calculate  $\frac{\partial f}{\partial u}$  in this context

On the other hand, when  $x = u/w$  and  $y = w$  we have a different calculation for  $\frac{\partial f}{\partial u}$ ,

$$\begin{aligned}\frac{\partial f}{\partial u} &= \underset{\text{def}}{=} \frac{\partial}{\partial u} \left[ f\left(\frac{u}{w}, w\right) \right] \\ &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \\ &= f_x\left(\frac{u}{w}, w\right) \frac{1}{w} + f_y\left(\frac{u}{w}, w\right) (0) \\ &= f_x\left(\frac{u}{w}, w\right) \frac{1}{w}\end{aligned}$$

you can see how  $\frac{\partial f}{\partial u} = 0 \Rightarrow \frac{\partial f}{\partial x} = 0$

$\Rightarrow f(x, y) = g(y)$  (constant in  $x$ )

which says  $f$  is constant along horizontal lines.

Remark: it's better to denote  $\left(\frac{\partial f}{\partial u}\right)_v$  and  $\left(\frac{\partial f}{\partial u}\right)_w$

when danger of confusion is possible. These are clearly different objects.

**Problem 124** Suppose that the temperature  $T$  in the  $xy$ -plane changes according to

$$\frac{\partial T}{\partial x} = 8x - 4y \quad \& \quad \frac{\partial T}{\partial y} = 8y - 4x.$$

Find the maximum and minimum temperatures of  $T$  on the unit circle  $x^2 + y^2 = 1$ . This time use the method of Lagrange multipliers. Hopefully we find agreement with Problem 107.

$$\nabla T = \lambda \nabla g \quad \text{where } g(x, y) = x^2 + y^2 = 1.$$

$$\langle 8x - 4y, 8y - 4x \rangle = \lambda \langle 2x, 2y \rangle$$

$$4x - 2y = \lambda x \Rightarrow (4 - \lambda)x - 2y = 0$$

$$4y - 2x = \lambda y \Rightarrow (4 - \lambda)y - 2x = 0$$

$$\text{Hence, } (4 - \lambda)x - 2y = (4 - \lambda)y - 2x$$

$$\Rightarrow 4 - \lambda = -2 \text{ and } -2 = 4 - \lambda \text{ thus } \lambda = 6$$

$$\text{Thus } 4x - 2y = 6x \Rightarrow y = -x \Rightarrow x^2 + (-x)^2 = 1$$

$$\text{so } x = \pm 1/\sqrt{2} \Rightarrow y = \pm 1/\sqrt{2}. \text{ Extrema for } T$$

$$\text{are found at } \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

**Problem 125** Use the method of Lagrange multipliers to find the point on the plane  $x + 2y - 3z = 10$  which is closest to the point  $(8, 8, 8)$ .

$$f(x, y, z) = d^2 = (x - 8)^2 + (y - 8)^2 + (z - 8)^2$$

$$g(x, y, z) = x + 2y - 3z = 10$$

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x - 8), 2(y - 8), 2(z - 8) \rangle = \lambda \langle 1, 2, -3 \rangle$$

$$2x - 16 = \lambda \Rightarrow 12x - 96 = 6\lambda$$

$$2y - 16 = 2\lambda \Rightarrow 6y - 48 = 6\lambda$$

$$2z - 16 = -3\lambda \Rightarrow 4z - 32 = -6\lambda$$

$$\text{Thus, } 12x - 96 = 6y - 48 = 32 - 4z.$$

$$\Rightarrow y = \frac{12x - 48}{6} = 2x - 8. \text{ and } z = \frac{12x - 128}{-4} = 32 - 3x$$

$$\text{Hence } x + 2y - 3z = x + 2(2x - 8) - 3(32 - 3x) = 10$$

$$\therefore x + 4x - 16 - 96 + 9x = 10 \Rightarrow 14x = 122$$

$$x = \frac{122}{14}, y = \frac{244}{14} - 8, z = 32 - \frac{366}{14} \text{ aka.}$$

$$\left( \frac{61}{7}, \frac{66}{7}, \frac{41}{7} \right)$$