

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 101 Your signature below indicates you have:

- (a.) I have read Chapter 6 of Cook: _____.
- (b.) I have attempted homeworks from Salas and Hille as listed below: _____.

The following homeworks from the text are good rudimentary skill problems. These are not collected or graded. However, they are all usually odd problems thus there are answers given within Salas, Hille and Etgen's text:

- § 16.1 #'s 1, 17
- § 16.2 #'s 13
- § 16.3 #'s 3, 9, 15, 21, 27, 31, 41, 51, 57
- § 16.4 #'s 9, 11, 21, 27
- § 16.5 #'s 1, 15
- § 16.6 #'s 5, 7
- § 16.7 #'s 1, 5, 9, 13, 21, 29, 35, 36, 42, 53
- § 16.8 #'s 1, 5, 9, 11, 15, 17, 25, 31, 35
- § 16.9 #'s 1, 13, 17, 21, 25, 35
- § 16.10 #'s 3, 11, 17, 25, 30

Problem 102 Let a, b be constants. Calculate $\int_0^a \int_0^b (2x + 4y) dx dy$.

Problem 103 Calculate $\int_0^{\pi/2} \int_0^{\pi/2} (\sin(x) + \cos(y)) dx dy$.

Problem 104 Let a, b be constants. Calculate $\int_0^{\ln(a)} \int_0^{\ln(b)} e^{x+y} dx dy$.

Problem 105 Calculate $\int_{-1}^1 \int_0^1 \sin^p(x) \cos^{42}(y) dy dx$ where $p > 2$ is a prime.

Problem 106 Calculate the average of $f(x, y) = x^2 + y^2$ on the unit-square $Q = [0, 1] \times [0, 1]$.

Problem 107 Calculate the average of $f(x, y) = x^2 + y^2$ on the triangle T with vertices $(0, 0)$, $(1, 2)$ and $(2, 0)$.

Problem 108 Suppose $\iint_R f \, dA = \int_0^1 \int_{x^2}^x (1+x) \, dy \, dx$. Calculate the given integral.

Problem 109 For the integral given in the previous problem, explicitly write R as a subset of \mathbb{R}^2 using set-builder notation. In addition, calculate the integral once more with the iteration of the integrals beginning with dx . Draw a picture to explain the inequalities which form the basis for your new set-up to the integral.

Problem 110 Reverse the order of integration in order to calculate the following integral:

$$\int_0^1 \int_y^1 \frac{2}{1+x^4} \, dx \, dy.$$

Problem 111 Let B be the solid region bounded by $x = 0, y = 0, z = 0$ and the plane $2x + 2y - 3z = 1$. Calculate the volume of B .

Problem 112 Let R be the region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and $y = x$ and $y = 3x$. Calculate $\iint_R \sqrt{x^2 + y^2} \, dA$ by changing the integration to polar coordinates.

Problem 113 Find the center of mass of the region Ω inside the circle $r = 2 \sin \theta$ and outside the circle $r = 1$ given mass-density $\sigma(x, y) = y$. (Salas and Hille used λ for density in §16.5)

Problem 114 Let B be the region bounded by $\phi = \pi/2$ and $\phi = \pi/4$. Calculate $\iiint_B x^2 \, dV$.
and $0 \leq \rho \leq 1$

Problem 115 Find the volume bounded by the cylinder $x^2 + y^2 = 1$ and $z = 2 + x + y$ and $z = 1$.

Problem 116 Let B be a ball of radius R centered at the origin. Calculate $\iiint_B e^{-\rho^3} \, dV$.

Problem 117 Let $u = x^2 + y^2$ and $v = x^2 - z^2$ and $w = xyz$ calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Problem 118 Find the volume of a ball of radius R with the top cap of height h is removed. You should assume h is measured along a diameter of the ball and $h < R$.

Problem 119 Calculate $\iint_R \sqrt{x+y} \sin(2x-y) \, dA$ where $R = [0, 1] \times [0, 1]$ by making an appropriate change of variables.

Problem 120 Show your work for Problem #36 of §16.7 in Salas and Hille's Eighth Edition.

Mission 6 Solution:

PROBLEM 102

$$\begin{aligned}
 \int_0^a \int_0^b (2x + 4y) dx dy &= \int_0^a \left(x^2 + 4xy \Big|_{x=0}^{x=b} \right) dy \\
 &= \int_0^a (b^2 + 4by) dy \\
 &= (b^2 y + 2by^2) \Big|_0^a \\
 &= \boxed{ab^2 + 2ba^2}
 \end{aligned}$$

Alternatively:

$$\begin{aligned}
 \int_0^a \int_0^b (2x + 4y) dx dy &= \int_0^a dy \int_0^b 2x dx + \int_0^b dx \int_0^a 4y dy \\
 &= \underline{ab^2 + b \cdot 2a^2}.
 \end{aligned}$$

PROBLEM 103

$$\begin{aligned}
 \int_0^{\pi/2} \int_0^{\pi/2} (\sin x + \cos y) dx dy &= \int_0^{\pi/2} \left[-\cos(x) + x \cos(y) \right] \Big|_0^{\pi/2} dy \\
 &= \int_0^{\pi/2} \left(-\cos \frac{\pi}{2} + \cos(0) + \frac{\pi}{2} \cos(y) - 0 \right) dy \\
 &= \int_0^{\pi/2} \left(1 + \frac{\pi}{2} \cos(y) \right) dy \\
 &= \frac{\pi}{2} + \frac{\pi}{2} \sin(y) \Big|_0^{\pi/2} \\
 &= \frac{\pi}{2} + \frac{\pi}{2} \\
 &= \boxed{\pi}
 \end{aligned}$$

PROBLEM 104

Note, $e^{x+y} = e^x e^y$ so we can use the factoring trick,

$$\begin{aligned} \int_0^{\ln(a)} \int_0^{\ln(b)} e^{x+y} dx dy &= \int_0^{\ln(a)} e^y dy \int_0^{\ln(b)} e^x dx \\ &= (e^{\ln(a)} - e^0)(e^{\ln(b)} - e^0) \\ &= \boxed{(a-1)(b-1)} \end{aligned}$$

PROBLEM 105 $p > 2$ and p is prime.

$$\int_{-1}^1 \int_0^1 \sin^p(x) \cos^{q_2}(y) dy dx = \int_{-1}^1 \sin^p(x) dx \int_0^1 \cos^{q_2}(y) dy$$

Note, p prime $\Rightarrow p$ is odd $\Rightarrow \sin^p(x)$ is odd function
 $p > 2$

Hence $\int_{-1}^1 \sin^p(x) dx = 0$ and we find $\int_{-1}^1 \int_0^1 \sin^p(x) \cos^{q_2}(y) dy dx = 0$.

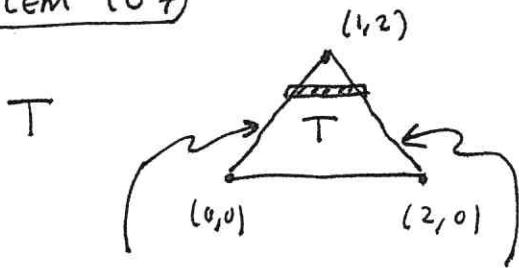
PROBLEM 106

Calculate the average of $f(x, y) = x^2 + y^2$ on $Q = [0, 1]^2$

Note area (Q) = 1 thus,

$$\begin{aligned} f_{\text{avg}}(Q) &= \frac{1}{1} \iint_Q f dA \\ &= \int_0^1 \int_0^1 (x^2 + y^2) dx dy \\ &= \int_0^1 \left(\frac{1}{3}x^3 + xy^2 \right) \Big|_0^1 dy \\ &= \int_0^1 \left(\frac{1}{3} + y^2 \right) dy \\ &= \frac{1}{3} + \frac{1}{3} \\ &= \boxed{\frac{2}{3}} \end{aligned}$$

PROBLEM 107



$$\text{area}(T) = \frac{1}{2}(2)(2) = 2.$$

$$0 \leq y \leq 2$$

$$\frac{y}{2} \leq x \leq 2 - \frac{y}{2}$$

$$y = 2x$$

$$x = \frac{y}{2}$$

$$y = -2x + 4$$

$$x = \frac{y-4}{-2} = 2 - \frac{y}{2}$$

$$f_{avg}(T) = \frac{1}{2} \iint_T f dA = \frac{1}{2} \int_0^2 \int_{\frac{y}{2}}^{2-\frac{y}{2}} (x^2 + y^2) dx dy$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{3} x^3 + xy^2 \right) \Big|_{x=\frac{y}{2}}^{x=2-\frac{y}{2}} dy$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{3} (2 - \frac{y}{2})^3 + (2 - \frac{y}{2})y^2 - \frac{1}{3}(\frac{y}{2})^3 - \frac{y}{2}y^2 \right) dy$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{3} (2 - \frac{y}{2})^3 + 2y^2 - \frac{1}{2}y^3 - \frac{1}{24}y^3 - \frac{1}{2}y^3 \right) dy$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{3} (2 - \frac{y}{2})^3 + 2y^2 - \frac{25}{24}y^3 \right) dy$$

$$= \frac{1}{2} \left(\frac{1}{12} (2 - \frac{y}{2})^4 \cdot (-2) + \frac{2}{3}y^3 - \frac{25}{96}y^4 \right) \Big|_0^2$$

$$= \frac{1}{2} \left(\underbrace{-\frac{1}{6} + \frac{2}{3}(8) - \frac{25}{96}(16)}_{y=2} + \underbrace{\frac{1}{6}(16)}_{y=0} \right)$$

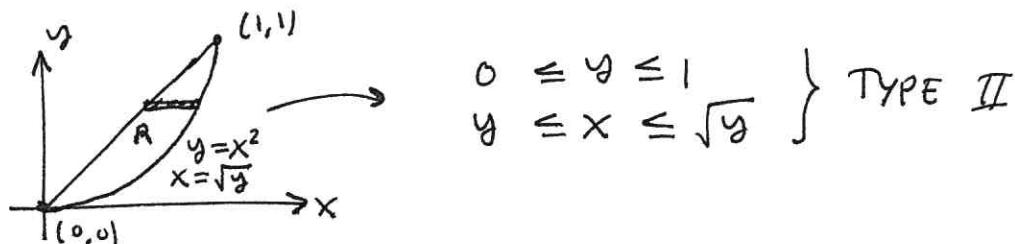
$$= \boxed{\frac{11}{6}}$$

PROBLEM 108

$$\begin{aligned}
 \iint_R f dA &= \int_0^1 \int_{x^2}^x (1+x) dy dx \\
 &= \int_0^1 \left((1+x)y \Big|_{x^2}^x \right) dx \\
 &= \int_0^1 (1+x)(x - x^2) dx \\
 &= \int_0^1 (x + x^2 - x^2 - x^3) dx \\
 &= \frac{1}{2} - \frac{1}{4} \\
 &= \boxed{\frac{1}{4}}
 \end{aligned}$$

PROBLEM 109

$$R : (x, y) \in \mathbb{R}^2 \text{ s.t. } \begin{cases} 0 \leq x \leq 1 \\ x^2 \leq y \leq x \end{cases} \} \text{ TYPE I}$$



$$\begin{aligned}
 \iint_R f dA &= \int_0^1 \int_y^{\sqrt{y}} (1+x) dx dy \\
 &= \int_0^1 \left(x + \frac{1}{2}x^2 \right) \Big|_{x=y}^{x=\sqrt{y}} dy \\
 &= \int_0^1 \left(\sqrt{y} + \frac{1}{2}(\sqrt{y})^2 - y - \frac{1}{2}y^2 \right) dy \\
 &= \int_0^1 \left(\sqrt{y} - \frac{1}{2}y - \frac{1}{2}y^2 \right) dy \\
 &= \frac{2}{3} - \frac{1}{4} - \frac{1}{6} \\
 &= \frac{8-3-2}{12} = \frac{3}{12} = \boxed{\frac{1}{4}}
 \end{aligned}$$

PROBLEM 110

$$\int_0^1 \int_{y^2}^1 \frac{2}{1+x^4} dx dy = \int_0^1 \int_0^x \frac{2 dy}{1+x^4} dx : \text{By diagram notice } 0 \leq x \leq 1, 0 \leq y \leq x \text{ describe region as TYPE I.}$$

$y \leq x \leq 1$
 $0 \leq y \leq 1$

$$= \int_0^1 \frac{2x dx}{1+x^4} : \begin{cases} u = x^2 \\ du = 2x dx \\ u(0) = 1^2 = 1 \\ u(1) = 0^2 = 0 \end{cases}$$

$$= \left[\tan^{-1}(u) \right]_0^1$$

$$= \boxed{\frac{\pi}{4}}$$

$$\left(\begin{array}{l} \tan^{-1}(1) = \pi/4 \\ \tan^{-1}(0) = 0 \end{array} \right)$$

PROBLEM 111

$$2x + 2y - 3z = 1$$

$$3z = 2x + 2y - 1$$

$$z = \frac{1}{3}(2x + 2y - 1)$$

$$z = 0 \rightarrow 2x + 2y = 1$$

$$y = \frac{1}{2} - x$$

$$\rightarrow R: \quad \begin{aligned} 0 \leq x &\leq \frac{1}{2} \\ 0 \leq y &\leq \frac{1}{2} - x \end{aligned}$$

Remark: we learn that $dV = -z dA$ for R as in this problem.

$$\text{Vol} = \int_0^{1/2} \int_0^{\frac{1}{2}-x} \frac{1}{3} (2x + 2y - 1) dy dx$$

$$= \frac{1}{3} \int_0^{1/2} \left[(2x-1)(\frac{1}{2}-x) + (\frac{1}{2}-x)^2 \right] dx$$

$$= \frac{1}{3} \int_0^{1/2} \left(x - \frac{1}{2} - 2x^2 + \cancel{x^2} + \frac{1}{4} - \cancel{x^2} + x^2 \right) dx$$

$$= \frac{1}{3} \int_0^{1/2} \left(x - \frac{1}{4} - x^2 \right) dx$$

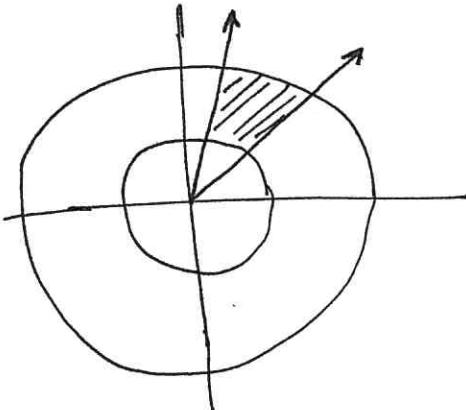
$$= \frac{1}{3} \left[\frac{1}{2} \left(\frac{1}{2}\right)^2 - \frac{1}{4} \left(\frac{1}{2}\right) - \frac{1}{3} \left(\frac{1}{2}\right)^3 \right] = \boxed{-\frac{1}{72}}$$

Thus, my picture is upside down!

$$\text{Vol} = \boxed{\frac{1}{72}}$$

PROBLEM 11a

R bounded by $x^2 + y^2 = 1$, $y^2 + x^2 = 4$, $y = x$, $y = 3x$
 $\theta = \frac{\pi}{4}$ $\Theta_2 = ?$



$$\tan \Theta_2 = \frac{y}{x} = \frac{3x}{x} = 3$$

$$\Theta_2 = \tan^{-1}(3).$$

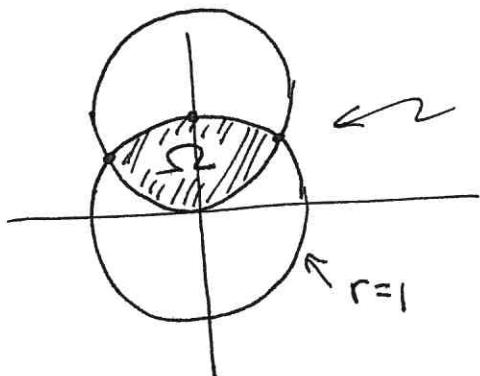
$$\text{we have } 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \tan^{-1}(3)$$

$$\begin{aligned} \iint_R \sqrt{x^2 + y^2} dA &= \int_1^2 \int_{\pi/4}^{\tan^{-1}(3)} r r dr d\theta \\ &= \left(\tan^{-1}(3) - \frac{\pi}{4} \right) \int_1^2 r^2 dr \\ &= \left(\tan^{-1}(3) - \frac{\pi}{4} \right) \left(\frac{8}{3} - \frac{1}{3} \right) \\ &= \boxed{\frac{7}{3} \left(\tan^{-1}(3) - \frac{\pi}{4} \right)} \end{aligned}$$

PROBLEM 113

$\sigma = \frac{dm}{dA} = y$ for region Ω bounded by $r = 2\sin\theta$
 $r = 1$

$$\begin{aligned} \text{Notice } r^2 &= 2r\sin\theta \rightarrow x^2 + y^2 = 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$



Goal: find center of mass for Ω .

Note: $r = 1$ intersects $r = 2\sin\theta$
where $1 = 2\sin\theta \rightarrow \sin\theta = \frac{1}{2}$
that gives $\theta = \pi/6$ and $\theta = 5\pi/6$

(OOPS! OUTSIDE $r = 1$, THANKFULLY)
See ↗

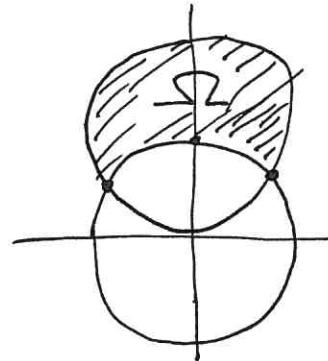
PROBLEM 113 Find the c.m. for

Σ is inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$. Assume $\sigma = \frac{dm}{dA} = y$.

$$r = 2\sin\theta$$

$$\Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y$$

$$\Rightarrow x^2 + (y-1)^2 = 1$$



Pts. of intersection: $1 = 2\sin\theta$

$$\Rightarrow \theta = \frac{\pi}{6} \text{ and } \theta = \frac{5\pi}{6}.$$

Σ : $1 \leq r \leq 2\sin\theta$ for $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

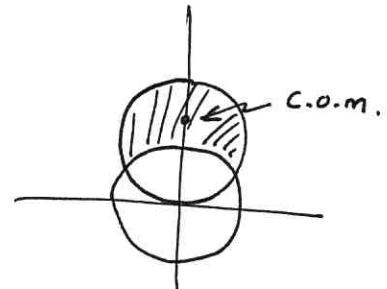
Thus calculate the mass,

$$\begin{aligned} M &= \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^2 \sin\theta dr d\theta \\ &= \int_{\pi/6}^{5\pi/6} \sin\theta \left(\frac{8\sin^3\theta}{3} - \frac{1}{3} \right) d\theta \\ &= \frac{8}{3} \int_{\pi/6}^{5\pi/6} \sin^4\theta d\theta - \frac{1}{3} \int_{\pi/6}^{5\pi/6} \sin\theta d\theta \\ &= \frac{8}{3} \int_{\pi/6}^{5\pi/6} \frac{1}{4} (1 - \cos 2\theta)^2 d\theta - \frac{\cos\theta}{3} \Big|_{\pi/6}^{5\pi/6} \\ &= \frac{8}{12} \int_{\pi/6}^{5\pi/6} (1 - 2\cos 2\theta + \cos^2(2\theta)) d\theta - \frac{1}{3} \cos \frac{5\pi}{6} + \frac{1}{3} \cos \frac{\pi}{6} \\ &= \frac{2}{3} \int_{\pi/6}^{5\pi/6} (1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta) d\theta + \underbrace{\frac{\sqrt{3}}{2(3)}}_{\frac{\sqrt{3}}{3}} + \underbrace{\frac{\sqrt{3}}{2(3)}}_{-\frac{1}{8}\sin(\frac{4\pi}{3})} \\ &= \frac{1}{\sqrt{3}} + \frac{2}{3} \left[\frac{4\pi}{6} - \frac{2\sin \frac{10\pi}{6}}{2} + \frac{2\pi}{6} + \frac{1}{2} \frac{\sin(\frac{20\pi}{6})}{4} \right] + \underbrace{\frac{2}{2} \sin(\frac{5\pi}{6})}_{\frac{\sqrt{3}}{3}} - \underbrace{\frac{1}{8} \sin(\frac{4\pi}{3})}_{-\frac{1}{8}\sin(\frac{4\pi}{3})} \\ &= \frac{\sqrt{3}}{4} + \frac{2\pi}{3} \approx \underline{2.5274}. \end{aligned}$$

PROBLEM 113

Notice $x_{cm} = 0$ by symmetry of both Σ and $\sigma(x,y) = y$.
On the other hand, y_{cm} requires calculation, $\sigma y = y^2 = r^2 \sin^2 \theta$

$$\begin{aligned} y_{cm} &= \frac{1}{M} \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^3 \sin^2 \theta \, dr \, d\theta \\ &= \frac{1}{M} \int_{\pi/6}^{5\pi/6} \frac{\sin^2 \theta}{4} (16 \sin^4 \theta - 1) \, d\theta \\ &= \frac{1}{M} \int_{\pi/6}^{5\pi/6} (4 \sin^6 \theta - \frac{1}{4} \sin^2 \theta) \, d\theta \\ &= \frac{1}{M} \left(\frac{11\sqrt{3}}{16} + \frac{3\pi}{4} \right) \approx \frac{3.547}{M} = \frac{3.547}{2.5274} \approx \underline{1.403} \end{aligned}$$



(reasonable.)

Hence, $(0, 1.403)$ is \approx the c.o.m.

or, to be precise, $(0, \underbrace{\left(\frac{11\sqrt{3}}{16} + \frac{3\pi}{4} \right)}_{\text{from Wolfram Alpha}} / \left(\frac{\sqrt{3}}{4} + \frac{2\pi}{3} \right))$

(thanks to Wolfram Alpha $\ddot{\cup}$) $\rightarrow (0, \frac{9}{8} \left[1 + \frac{13}{9+8\sqrt{3}\pi} \right])$

PROBLEM 114 Let B be bounded by $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \rho \leq 1$
 Calculate $\iiint_B x^2 dV$. $[x^2 dV = (\rho^2 \cos^2 \theta \sin^2 \phi)(\rho^2 \sin \phi d\rho d\theta d\phi)]$

$$\int_0^1 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \rho^4 \cos^2 \theta \sin^3 \phi d\phi d\theta d\rho =$$

$$= \left(\int_0^1 \rho^4 d\rho \right) \left(\int_0^{2\pi} \cos^2 \theta d\theta \right) \left(\int_{\pi/4}^{\pi/2} (1 - \cos^2 \phi) \sin \phi d\phi \right) \quad u = \cos \phi \\ u(\pi/4) = 1/\sqrt{2} \\ u(\pi/2) = 0$$

$$= \left(\frac{1}{5} \right) \left(\frac{1}{2} \underbrace{\int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta}_{2\pi} \right) \left(\int_{1/\sqrt{2}}^0 (1 - u^2) (-du) \right)$$

$$= \frac{\pi}{5} \int_0^{1/\sqrt{2}} (1 - u^2) du$$

$$= \frac{\pi}{5} \left(\frac{1}{\sqrt{2}} - \frac{1}{3} \left(\frac{1}{\sqrt{2}} \right)^3 \right)$$

$$= \frac{\pi}{5\sqrt{2}} \left(1 - \frac{1}{6} \right)$$

$$= \frac{\pi}{5\sqrt{2}} \cdot \frac{5}{6}$$

$$= \boxed{\frac{\pi}{6\sqrt{2}}}$$

PROBLEM 115 find volume bounded by $x^2 + y^2 = 1$

and $z = 2+x+y$ and $z = 1$.

Notice $2+x+y$ is at most $2+\sqrt{2}$ and at smallest $2-\sqrt{2}$ both of which are above $z = 1$. Hence,

$$\begin{aligned} \text{Vol} &= \int_0^{2\pi} \int_0^1 \int_{1}^{2+r\cos\theta+r\sin\theta} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r + r^2 \cos\theta + r^2 \sin\theta - r) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta \\ &= (2\pi)(\frac{1}{2}) \\ &= \boxed{\pi}. \end{aligned}$$

↗ think about where there go to in the θ -integration.

PROBLEM 116 B : ball of radius R

$$\begin{aligned} \iiint_B e^{-\rho^3} dV &= \int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 e^{-\rho^3} \overbrace{d\rho \cdot \sin\phi}^{\text{oops.}} d\phi \, d\theta \\ &= \left(\int_0^R \rho^2 e^{-\rho^3} d\rho \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\pi \sin\phi d\phi \right) \\ &= \left(\frac{-1}{3} e^{-\rho^3} \Big|_0^R \right) (2\pi) (2) \\ &= -\frac{4\pi}{3} (e^{-R^3} - 1) \\ &= \boxed{\frac{4\pi}{3} (1 - e^{-R^3})} \end{aligned}$$

(P117) Let $u = x^2 + y^2$ and $v = x^2 - z^2$
 and $w = xyz$ calculate $\frac{\partial(x, y, z)}{\partial(u, v, w)}$

Note, $\left[\frac{\partial(x, y, z)}{\partial(u, v, w)} \right]^{-1} = \frac{\partial(u, v, w)}{\partial(x, y, z)}$ ↪ fun fact brought to you courtesy of ADVANCED CALCULUS.

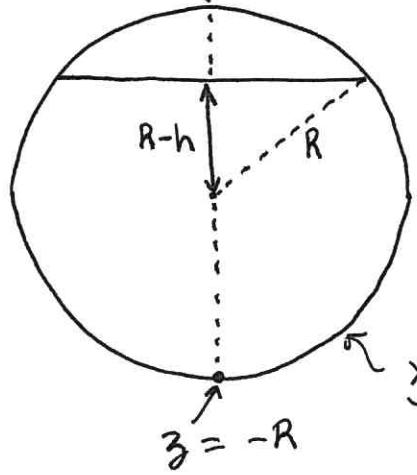
$$= \det \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

$$= \det \begin{array}{c|c|c} 2x & 2y & 0 \\ 2x & 0 & -2z \\ yz & xz & xy \end{array}$$

$$= 2x(2z(xz)) - 2y[2x(xy) + 2z(yz)] \\ = 4x^2z^2 - 4x^2y^2 - 4y^2z^2.$$

↪
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{4(x^2z^2 - x^2y^2 - y^2z^2)}$$

PROBLEM 118 Find volume of sphere with cap of height h chopped off.



Cylindrical coordinates are nice choice, we saw in lecture that sphericals are annoying here.

$$\underbrace{x^2 + y^2 + z^2}_{r^2 + z^2} = R^2$$

$$r^2 + z^2 = R^2$$

$$r = \sqrt{R^2 - z^2}$$

true for whole sphere's surface.

We have $0 \leq \theta \leq 2\pi$, $-R \leq z \leq R-h$ and $0 \leq r \leq \sqrt{R^2 - z^2}$

$$\begin{aligned} \text{Vol} &= \int_{-R}^{R-h} \int_0^{\sqrt{R^2-z^2}} \int_0^{2\pi} r d\theta dr dz \\ &= \int_{-R}^{R-h} \int_0^{\sqrt{R^2-z^2}} 2\pi r dr dz \\ &= \pi \int_{-R}^{R-h} \left(r^2 \Big|_0^{\sqrt{R^2-z^2}} \right) dz \\ &= \pi \int_{-R}^{R-h} (R^2 - z^2) dz \\ &= \pi \left(R^2 z - \frac{1}{3} z^3 \right) \Big|_{-R}^{R-h} \\ &= \boxed{\pi \left[R^2 (2R-h) - \frac{1}{3} ((R-h)^3 + R^3) \right]} \end{aligned}$$

Notice, as $h \rightarrow 0$ we have

$$\text{Vol} \rightarrow \pi [2R^3 - \frac{1}{3} 2R^3] = \pi R^3 \left(\frac{6}{3} - \frac{2}{3} \right) = \frac{4}{3} \pi R^3$$



PROBLEM 119

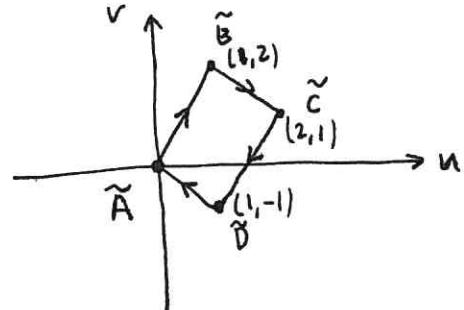
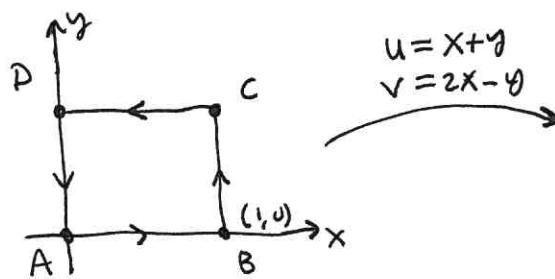
$$\iint_{[0,1] \times [0,1]} \sqrt{x+y} \sin(2x-y) dA$$

$$\begin{aligned} u &= x+y \\ v &= 2x-y \end{aligned} \quad \left\{ \begin{array}{l} u+v = 3x \quad \therefore x = \frac{1}{3}(u+v) \\ y = u-x = u - \frac{1}{3}(u+v) = \frac{1}{3}(2u-v) = y \end{array} \right.$$

$$\text{Hence, } \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \frac{-1}{9} - \frac{2}{9} = \frac{-1}{3}$$

To modify the $R = [0,1] \times [0,1]$ to u,v -word.

We notice the substitution is linear hence, lines go to lines and we can simply track the vertices of the square



	P	\tilde{P}	
A	(0, 0)	(0, 0)	\tilde{A}
B	(1, 0)	(1, 2)	\tilde{B}
C	(1, 1)	(2, 1)	\tilde{C}
D	(0, 1)	(1, -1)	\tilde{D}
	(x,y)	(u,v)	

$$\tilde{AB}: v = 2u$$

$$\tilde{BC}: v = -u + 3$$

$$\tilde{DC}: v = 2u - 3$$

$$\tilde{AD}: v = -u$$

By change of variables theorem,

$$\iint_R \sqrt{x+y} \sin(2x-y) dA = \iint_{\tilde{R}} \sqrt{u} \sin(v) \frac{dudv}{3} + \iint_{\tilde{R}} \sqrt{u} \sin(v) \frac{dudv}{3}$$

$$= \boxed{0.412109}$$

(wolfram alpha) (note, the pt here was more about the set-up than

finding this main point here.)

PROBLEM 120 #36 of §16.7 of 8th Ed. of Sclar and Hille

Show that if $(\bar{x}, \bar{y}, \bar{z})$ is centroid of a solid T then

$$\iiint_T (x_i - \bar{x}_i) dx dy dz = 0 \quad \text{for } x_1 = x, x_2 = y, x_3 = z.$$

In §16.6 we are given a convenient characterization of the centroid $(\bar{x}, \bar{y}, \bar{z})$ for T

$$\bar{x} V = \iiint_T x dV$$

$$\bar{y} V = \iiint_T y dV$$

$$\bar{z} V = \iiint_T z dV$$

where $V = \iiint_T dV$ is volume of T . Thus,

$$\bar{x} \iiint_T dV = \iiint_T x dV \Rightarrow \iiint_T \bar{x} dV - \iiint_T x dV = 0$$

$$\Rightarrow \iiint_T (\bar{x} - x) dV = 0$$

$$\Rightarrow \iiint_T (x - \bar{x}) dxdydz = 0$$

The argument for y, z is very similar.