MATH 231(DUE TUESDAY BY 5PM OUTSIDE MY OFFICE) PROBLEM SET 6.

These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed.

Problem 126 Apply the method of Lagrange multipliers to solve the following problem: Let a, b be constants. Maximize xy on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Identify objective function $f(x,y) = xy$ and constraint $g(x,y) = b^2x^2 + a^2y^2 = a^2b^2$

Apply the method of Lagrange,

$$\nabla f = \lambda \nabla g \implies \langle y, x \rangle = \lambda \langle 2b^2x, 2a^2y \rangle$$

Thus, $y = 2\lambda b^2 x$ and $x = 2\lambda a^2 y$ solving for λ,

$$\lambda = \frac{y}{2b^2x} = \frac{x}{2a^2y} \quad \left(\text{for } x, y \neq 0, \text{ and } x=0, y=0 \text{ not useful here...}\right)$$

Hence, $y^2 = \frac{b^2}{a^2}x^2$ and substituting into $g(x,y) = a^2b^2$ to obtain $b^2x^2 + a^2\left(\frac{b^2}{a^2}x^2\right) = 2b^2x^2 = a^2b^2 \implies x^2 = \frac{a^2}{2} \implies x = \frac{\pm|a|}{\sqrt{2}}$.

therefore, $y^2 = \frac{b^2}{2} \implies y = \frac{\pm|b|}{\sqrt{2}}$. $\boxed{\text{Assume } a, b > 0}$ (otherwise could use $|a|, |b|$ to frame answer)

then we find four points of interest $(\pm a/\sqrt{2}, \pm b/\sqrt{2})$

$$f(a/\sqrt{2}, b/\sqrt{2}) = \frac{ab}{2}$$
$$f(-a/\sqrt{2}, b/\sqrt{2}) = \frac{-ab}{2}$$
$$f(-a/\sqrt{2}, -b/\sqrt{2}) = \frac{ab}{2}$$
$$f(a/\sqrt{2}, -b/\sqrt{2}) = -ab/2.$$

$\boxed{\text{maximum value } ab/2 \text{ attained at } \pm(a/\sqrt{2}, b/\sqrt{2})}$

$$f(\pm|a|/\sqrt{2}, \pm|b|/\sqrt{2}) = \frac{|a||b|}{2}$$

Problem 127 Apply the method of Lagrange multipliers to solve the following problem: Find the distance from $(1, 0)$ to the parabola $x^2 = 4y$.

Objective $f(x,y) = (x-1)^2 + y^2$ and constraint $g(x,y) = x^2 - 4y = 0$

Apply the method of Lagrange,

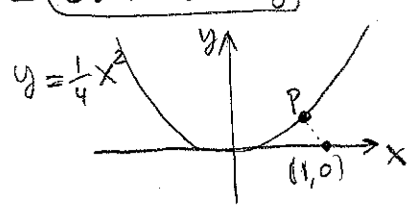
$$\nabla f = \lambda \nabla g \implies \langle 2(x-1), 2y \rangle = \lambda \langle 2x, -4 \rangle$$

Thus, $\underbrace{2(x-1) = 2\lambda x}$ and $\underbrace{2y = -4\lambda}$.

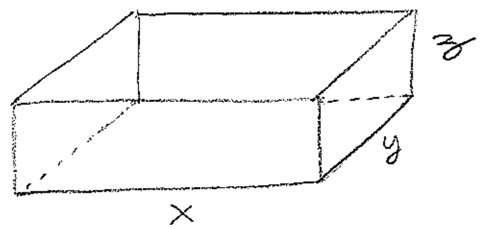
$$\lambda = \frac{x-1}{x} = 1 - \frac{1}{x} = \frac{-y}{2} = \frac{4y}{-8} = \frac{x^2}{-8}$$

Hence, $-8x + 8 = x^3 \implies x^3 + 8x - 8 = 0 \implies \boxed{x \approx 0.9068}$

and $y \cong \frac{1}{4}(0.9068)^2 = \boxed{0.2056 = y} \implies$ $\boxed{(0.9068, 0.2056) = P \text{ Closest point on } x^2 = 4y \text{ to } (1,0). \text{ The distance is } \sqrt{0.0932^2 + 0.2056^2} = 0.23}$



Problem 128 Apply the method of Lagrange multipliers to solve the following problem: Suppose the base of a rectangular box costs twice as much per square foot as the sides and the top of the box. If the volume of the box must be $12ft^3$ then what dimensions should we build the box to minimize the cost? *[Please state the dimensions of the base and altitude clearly. Include a picture in your solution to explain the meaning of any variables you introduce, thanks!]*



$$V = xyz = 12$$

$$\underbrace{f(x,y,z)}_{\text{COST}} = \underbrace{2xy}_{\text{BASE}} + \underbrace{2xz + 2yz}_{\text{SIDES}} + \underbrace{xy}_{\text{TOP}}$$

$$f(x,y,z) = 3xy + 2xz + 2yz$$

Apply method of Lagrange,

$$\nabla f = \lambda \nabla V \Rightarrow \langle 3y+2z,\ 3x+2z,\ 2x+2y \rangle = \lambda \langle yz, xz, xy \rangle$$

Must solve simultaneously

$$3y + 2z = \lambda yz \Rightarrow \lambda = \frac{3y+2z}{yz} = 3/z + 2/y$$

$$3x + 2z = \lambda xz \Rightarrow \lambda = \frac{3x+2z}{xz} = 3/z + 2/x$$

$$2x + 2y = \lambda xy \Rightarrow \lambda = \frac{2x+2y}{xy} = \frac{2}{y} + \frac{2}{x}$$

$$xyz = 12$$

Hence $\lambda = \lambda$ yields $\frac{3}{z} + \frac{2}{y} = \frac{2}{y} + \frac{2}{x} \Rightarrow \underline{z = \frac{3}{2}x}$

Likewise $\lambda = \lambda$ yields $\underline{x = y}$ $\therefore$ $xyz = \frac{3x^3}{2} = 12 \Rightarrow x^3 = 8$

Hence, $\boxed{x = 2ft,\ y = 2ft,\ z = 3ft}$

Problem 129 Taking a break from the method of Lagrange. Assume a, b, c are constants: Show that the surfaces $\underbrace{xy = az^2}_{F(x,y,z) = xy - az^2}$, $\underbrace{x^2 + y^2 + z^2 = b}_{G}$ and $\underbrace{z^2 + 2x^2 = c(z^2 + 2y^2)}_{H(x,y,z) = z^2 + 2x^2 - c(z^2+2y^2)}$ are mutually perpendicular.

$$\nabla F(x,y,z) = \langle y, x, -2az \rangle$$

$$\nabla G(x,y,z) = \langle 2x, 2y, 2z \rangle$$

$$\nabla H(x,y,z) = \langle 4x, -4cy, 2z - 2cz \rangle$$

1.) $\nabla F \cdot \nabla G = 2xy + 2xy - 4az^2 = 4xy - 4az^2 \overset{\text{applying } F = 0}{=} 4az^2 - 4az^2 = 0$

2.) $\nabla F \cdot \nabla H = 4xy - 4cxy - 4az^2 + 4acz^2 = \cancel{4az^2} - \cancel{4caz^2} - \cancel{4az^2} + \cancel{4acz^2} = 0$

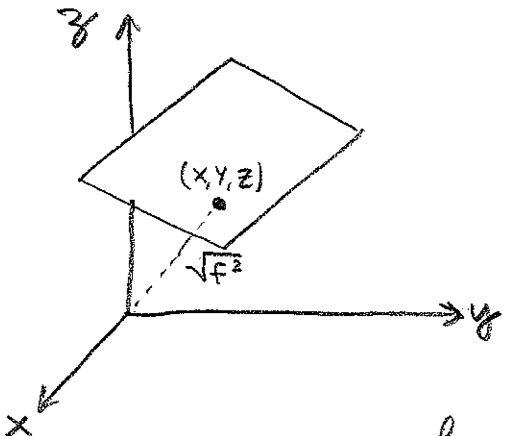
applying $F = 0$.

3.) $\nabla G \cdot \nabla H = 8x^2 - 8cy^2 + 4z^2 - 4cz^2$

$= 4(z^2 + 2x^2) - 4c(z^2 + 2y^2)$ use $H = 0$,

$= 0$

Problem 130 Apply the method of Lagrange multipliers to derive a formula for the distance from the plane $ax + by + cz + d = 0$ to the origin. If necessary, break into cases.



$$f(x,y,z) = x^2 + y^2 + z^2$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

Constraint is $g(x,y,z) = ax + by + cz + d = 0$. Apply the method of Lagrange multipliers, assume $abc \neq 0$,

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle a, b, c \rangle \quad \Rightarrow \quad x = \lambda a/2, \quad y = \lambda b/2, \quad z = \lambda c/2$$

Note that

$$ax + by + cz + d = 0 \implies \frac{\lambda}{2}a^2 + \frac{\lambda}{2}b^2 + \frac{\lambda}{2}c^2 + d = 0$$

Thus, $\lambda = \dfrac{-2d}{a^2+b^2+c^2}$ and $a^2+b^2+c^2 \neq 0$ since $\langle a,b,c \rangle \neq \langle 0,0,0 \rangle$ for a plane. (the eq^n $d = 0$ is either having sol^n set $\mathbb{R}^3$ or $\emptyset$ in this context.)

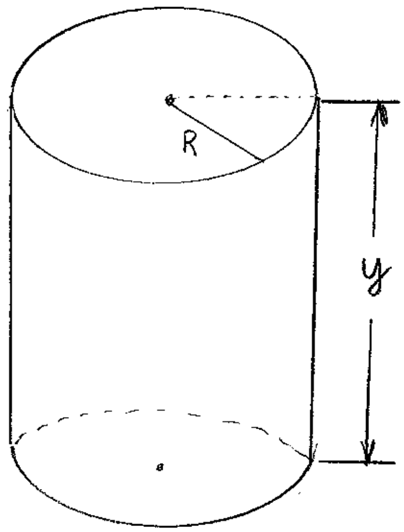
Consequently,

$$x = \frac{-ad}{a^2+b^2+c^2}, \quad y = \frac{-bd}{a^2+b^2+c^2}, \quad z = \frac{-cd}{a^2+b^2+c^2}$$

Closest point is $\dfrac{-d}{a^2+b^2+c^2}(a,b,c)$ hence

$$\boxed{\text{distance to origin} = \frac{d}{\sqrt{a^2+b^2+c^2}}}$$

Problem 131 Suppose you want to design a soda can to contain volume V of soda. If the can must be a right circular cylinder then what radius and height should you use to minimize the cost of producing the can? *assume the cost is directly proportional to the surface area of the can*



$$\underbrace{V = \pi R^2 y = \text{constant}}_{\text{constraint}}.$$

$$f(R,y) = \underbrace{2\pi R^2}_{\text{base \& top areas}} + \underbrace{2\pi R y}_{\text{area of side}} = 2\pi(R^2 + Ry)$$

$$\nabla f = \lambda \nabla V$$

$$\left\langle \frac{\partial f}{\partial R}, \frac{\partial f}{\partial y} \right\rangle = \lambda \left\langle \frac{\partial V}{\partial R}, \frac{\partial V}{\partial y} \right\rangle$$

$$\langle 2\pi(2R+y), 2\pi R \rangle = \lambda \langle 2\pi R y, \pi R^2 \rangle$$

$$4R + 2y = 2\lambda R y$$

$$2R = \lambda R^2 \Rightarrow \underline{\lambda = 2/R}.$$ ($R \neq 0$ clear from context.)

Also, $\underbrace{4R + 2y = 2\left(\frac{2}{R}\right) R y}$

$$\Rightarrow 4R + 2y = 4y$$

$$\Rightarrow 2y = 4R$$

$$\Rightarrow y = 2R \Rightarrow V = \pi R^2 (2R) = 2\pi R^3$$

$$\Rightarrow R = \tfrac{1}{2} y \Rightarrow V = \pi \left(\tfrac{y}{2}\right)^2 y = \frac{\pi y^3}{4}$$

Hence $\boxed{R = \sqrt[3]{\frac{V}{2\pi}}}$ and $\boxed{y = \sqrt[3]{\frac{4V}{\pi}}}$

Problem 132 Find any extreme values of xy^2z on the sphere $x^2+y^2+z^2$ $= 1$ (oops.). *note the sphere is compact and the function $f(x,y,z) = xy^2z$ is continuous so this problem will have at least two interesting answers*

$f(x,y,z)$ is objective function, $g(x,y,z) = x^2+y^2+z^2 = 1$

$$\nabla f = \lambda \nabla g$$

$$\langle y^2z,\ 2xyz,\ xy^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$y^2z = 2x\lambda$$
$$2xyz = 2y\lambda$$
$$xy^2 = 2z\lambda$$

$$\lambda = \underbrace{\frac{y^2z}{2x} = \frac{xyz}{y}}\ \underbrace{= \frac{xy^2}{2z}}$$ (assuming $x, y, z \neq 0$)

$y^2z = 2x^2z$ $\Rightarrow$ $y^2 = 2x^2$. ‖ $2xz^2 = xy^2$ $\Rightarrow$ $2z^2 = y^2$. $\Rightarrow$ $z^2 = \frac{1}{2}y^2 = x^2$.

Hence $x^2+y^2+z^2 = x^2+2x^2+x^2 = 4x^2 = 1$ $\Rightarrow x^2 = \frac{1}{4}$

thus $x = \pm 1/2$ and $y = \pm 1/2$, and $z = \pm 1/2$.

Points (a,b,c) such that $a,b,c \in \{1/2, -1/2\}$ are of interest.

$$f(1/2, \pm 1/2, 1/2) = \tfrac{1}{2}\left(\pm\tfrac{1}{2}\right)^2 \tfrac{1}{2} = \tfrac{1}{16}$$
$$f(-1/2, \pm 1/2, -1/2) = \tfrac{-1}{2}\left(\pm\tfrac{1}{2}\right)^2 \left(\tfrac{-1}{2}\right) = \tfrac{1}{16}$$
$$f(1/2, \pm 1/2, -1/2) = f(-1/2, \pm 1/2, 1/2) = \tfrac{-1}{16}$$

$$\boxed{\text{Extreme values of max. } \tfrac{1}{16} \text{ and min. } \tfrac{-1}{16}}$$

What about $x = 0$? If $x = 0$ then $y^2z = 0 \Rightarrow z = 0 \Rightarrow y^2 = 1$ OR $y^2 = 0$ hence $z^2 = 1$. But, then $z = 0$ and $y = 1$ inconsistent with $2xyz = 2y\lambda$ unless $\lambda = 0$, but then the method fails. etc... can rule out useful sol^{n}'s for $x = 0$, $y = 0$ or $z = 0$. We did not lose anything of interest.

Problem 133 Again, breaking from optimization, this problem explores a concept some of you have not yet embraced. Find the point(s) on $x^2 + y^2 + z^2 = 4$ which the curve $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$ intersects.

plug-in.

$$\sin^2 t + \cos^2 t + t^2 = 4$$

$$1 + t^2 = 4$$

$$t^2 = 3$$

$$t = \pm\sqrt{3}$$

$$\boxed{\begin{array}{l} \vec{r}(\sqrt{3}) = \langle \sin\sqrt{3}, \cos\sqrt{3}, \sqrt{3} \rangle \\ \hline \vec{r}(-\sqrt{3}) = \langle -\sin\sqrt{3}, \cos\sqrt{3}, -\sqrt{3} \rangle \end{array}}$$

Problem 134 Consider $f(x, y) = x^3 - 3x - y^2$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

$$f_x = 3x^2 - 3 \qquad f_{xx} = 6x \qquad f_{xy} = 0$$

$$f_y = -2y \qquad f_{yy} = -2$$

Critical pts have $3x^2 - 3 = 0 \Rightarrow 3(x^2-1) = 0$

$3(x-1)(x+1) = 0$

$\therefore\ x = 1, -1$

$\underbrace{-2y = 0}_{y = 0}$

We find $(1, 0)$ and $(-1, 0)$ are critical.

$$D(1,0) = \left(f_{xx} f_{yy} - f_{xy}^2\right)(1,0) = 6(1)(-2) = -12 < 0$$

$\therefore$ $\boxed{\text{f(1,0) is at a saddle point of } z = f(x,y).}$

$$D(-1,0) = \left(f_{xx} f_{yy} - f_{xy}^2\right)(-1,0) = 6(-1)(-2) = 12 > 0$$

and $f_{xx}(-1,0) = -6 < 0$ $\therefore$ $\boxed{\text{f(1,0) is a a local maximum for } z = f(x,y).}$

Problem 135 Consider $f(x, y) = x^2 - y^2$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

$$f_x = 2x \qquad f_{xx} = 2 \qquad f_{xy} = 0$$

$$f_y = -2y \qquad f_{yy} = -2$$

Critical pts. $f_x = f_y = 0 \Rightarrow x = 0,\ y = 0$

$\therefore$ $\boxed{(0,0) \text{ only critical pt.}}$

$$D(0,0) = \left(f_{xx} f_{yy} - f_{xy}^2\right)(0,0)$$
$$= 2(-2)$$
$$= -4 < 0 \quad \therefore$$

$\boxed{f(0,0) \text{ is the value of } f \text{ at saddle point of } z = f(x,y).}$

Problem 136 Consider $f(x,y) = x^3 + y^3 - 3xy$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

$$\nabla f = \langle 3x^2 - 3y,\ 3y^2 - 3x \rangle \underset{\uparrow}{=} \langle 0, 0 \rangle$$

for critical pts.

$$3x^2 = 3y \Rightarrow x^2 = y$$
$$3y^2 = 3x \Rightarrow y^2 = x$$

Hence, $x^4 = y^2 \Rightarrow x^4 = x$
$$\Rightarrow x(x^3 - 1) = 0$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$$\Rightarrow \underbrace{x\ (x-1)}_{\text{yield}}\underbrace{(x^2+x+1)}_{\text{irreducible.}} = 0$$

$x = 0$ & $x = 1 \Rightarrow y = 0$ & $y = 1$ (respective.)

Thus, $\boxed{(0,0) \text{ and } (1,1) \text{ are critical}}$.

$f_{xx} = 6x \qquad f_{xy} = -3$

$f_{yy} = 6y$

$$D(0,0) = (f_{xx} f_{yy} - f_{xy}^2)(0,0) = 36(0) - 9 < 0$$

$\boxed{\text{Therefore, } f(0,0) \text{ is at a saddle point in } z = f(x,y).}$

$$D(1,1) = (36 - 9) = 27 > 0$$

and $f_{xx}(1,1) = 6 > 0 \quad \therefore \boxed{f(1,1) \text{ is local min}}$

Problem 136+i An armored government agent decides to investigate a disproportionate use of electricity in a gated estate. Foolishly entering without a warrant he find himself at the mercy of Ron Swanson (at $(1,0,0)$), Dwight Schrute (at $(-1,1,0)$) and Kakashi (in a tree at $(1,1,3)$). Supposing Ron Swanson inflicts damage at a rate of 5 units inversely proportional from the square of his distance to the agent, and Dwight inflicts constant damage at a rate of 3 in a sphere of radius 2. If Kakashi inflicts a damage at a rate of 5 units directly proportional to the square of his distance from his location (because if you flee it only gets worse the further you run as he attacks you retreating) then where should you assume a defensive position as you call for back-up? What location minimizes your damage rate? Assume the ground is level and you have no jet-pack and/or antigravity devices.

① ②

$\hookrightarrow \boxed{z=0}$ *

Let $f(x,y,z)$ denote damage rate for gov. agent.

$$* \hookrightarrow \; f(x,y) = \underbrace{\frac{5}{(x-1)^2+y^2}}_{①} + \underbrace{5\left[(x-1)^2+(y-1)^2+9\right]}_{②}$$

We'll account for dwight at the end.

$$\nabla f = \left\langle \frac{-10(x-1)}{\left((x-1)^2+y^2\right)^2} + 10(x-1),\ \frac{-10y}{\left((x-1)^2+y^2\right)^2} + 10(y-1) \right\rangle$$

$$\frac{x-1}{\left[(x-1)^2+y^2\right]^2} = x-1 \quad \& \quad \frac{y}{\left[(x-1)^2+y^2\right]^2} = y-1$$

$\underline{x \neq 1}$ $\quad \left[(x-1)^2+y^2\right]^2 = 1 \Rightarrow y = y-1 \Rightarrow 0 = -1 \Rightarrow$ no sol^{n}!

However, $x = 1 \Rightarrow \quad y/y^4 = y-1 \Rightarrow y = y^5 - y^4$

$\Rightarrow y^5 - y^4 - y = 0$

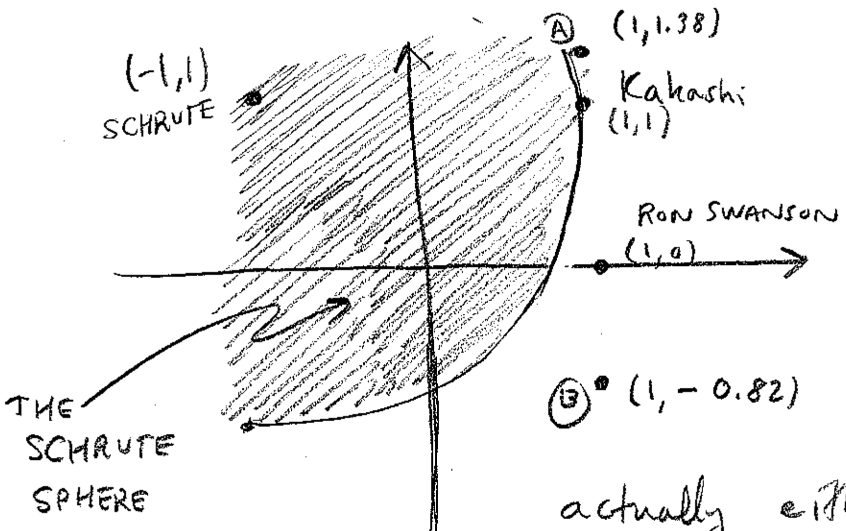
$\Rightarrow y(y^4 - y^3 - 1) = 0$

$\Rightarrow y = 0$

or

$y = -0.82$

$y = 1.38$



actually either Ⓐ or Ⓑ will work. So, which to choose depends on where the agent starts.

Problem 137 Find global extrema for $f(x,y) = exp(x^2 - 2x + y^2 - 6y)$ on the closed region bounded by $x^2/4 + y^2/16 = 1$.

$$f(x,y) = e^{(x-1)^2 + (y-3)^2 - 10}$$
$$= \frac{1}{e^{10}}\left(1 + (x-1)^2 + (y-3)^2 + \cdots\right)$$

Thus, $\boxed{(1,3) \text{ is critical point}}$

(Common sense tells me this is the only one since this is analogus to $z = e^{r^2}$)

$$f(1,3) = \frac{1}{e^{10}} \quad \text{(possible extrema.)}$$
$$= 0.0000454\ldots$$

Consider the boundary via the method of Lagrange,

$$\nabla f = \lambda \nabla g$$

$$f\langle 2(x-1), 2(y-3)\rangle = \lambda\langle \tfrac{1}{2}x, \tfrac{1}{8}y\rangle$$

$$\left.\begin{aligned} 2f(x-1) &= \lambda x/2 \\ 2f(y-3) &= \lambda y/8 \end{aligned}\right\} \xrightarrow{\%} \frac{x-1}{y-3} = \frac{8x}{2y} = \frac{4x}{y}$$

Thus, $xy - y = 4xy - 12x$

$$-3xy - y = -12x$$
$$y(1+3x) = 12x \longrightarrow \underbrace{y = \frac{12x}{1+3x}}_{\text{nice.}}$$

$$\frac{x^2}{4} + \frac{1}{16}\left(\frac{12x}{1+3x}\right)^2 = 1$$

Wolfram Alpha! $\to x_1 = -0.1664$, $\to x_2 = 1.2333$

$$4x^2 + y^2 = 16 \Rightarrow y_1 = \pm\sqrt{16 - 4(0.1664)^2} \cong \pm 3.986$$
$$\Rightarrow y_2 = \pm\sqrt{16 - 4(1.2333)^2} \cong \pm 3.149$$

Evaluate $f(-0.1664, 3.986) = 0.000468$ $\quad f(1.2333, 3.149) = 0.000049$

$f(-0.1664, -3.986) = 2.78 \times 10^{17}$ $\quad f(1.2333, -3.149) = 1.26 \times 10^{12}$

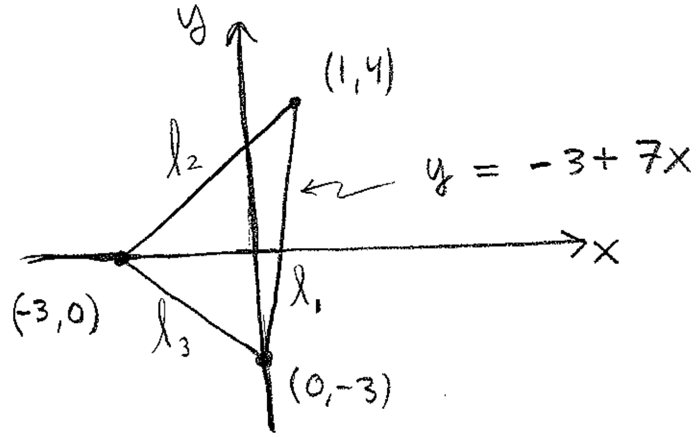
$\boxed{\text{Thus } 2.78 \times 10^{17} \text{ is the max attained at } (-0.1664, -3.986) \text{ whereas } f(1,3) = e^{-10} \text{ is global minimum. (not surprising!)}}$

Problem 138 Find the maximum and minimum values for $f(x,y) = x^2+y^2-1$ on the region bounded by the triangle with vertices $(-3,0)$, $(1,4)$ and $(0,-3)$.

$$\nabla f = \langle 2x, 2y \rangle = \langle 0,0 \rangle \Rightarrow (0,0) \text{ only critical pt.}$$

$\boxed{f(0,0) = -1}$ ← save for later

Consult explicit restriction on other parts.



l_1 $x = t$, $y = -3+7t$, $0 \le t \le 1$

$g(t) = f(t, -3+7t) = t^2 + (7t-3)^2 - 1 = t^2 + 49t^2 - 42t + 8$

Hence $g'(t) = 100t - 42 = 0 \Rightarrow \underline{t = 0.42}$ (within $dom(g) = [0,1]$)

We should save $g(0.42) = \boxed{f(0.42, -0.06) \doteq -0.82}$ ← possible extrema.

l_2 $x = -3+4t$, $y = 4t$, $0 \le t \le 1$

$g(t) = f(4t-3, 4t) = (4t-3)^2 + (4t)^2 - 1 = 16t^2 - 24t + 9 + 16t^2 - 1$

Hence $g'(t) = 64t - 24 = 0 \Rightarrow t = 24/64 = 3/8 = 0.375 \in dom(g)$.

Now, evaluate, $g(0.375) = f(-1.5, 1.5) = 2.25 + 2.25 - 1 = \boxed{3.5}$ ← possible extrema.

l_3 $x = -3+3t$, $y = -3t$, $0 \le t \le 1$

$g(t) = f(3t-3, -3t) = (3(t-1))^2 + (-3t)^2 - 1 = 9(t-1)^2 + 9t^2 - 1$

$g'(t) = 18(t-1) + 18t = 0 \Rightarrow 36t = 18 \Rightarrow t = 0.5 \in dom(g)$.

Evaluate, $g(0.5) = f(-1.5, -1.5) = \boxed{3.5}$ ← possible extrema.

We have all the information we need to apply the closed set test. Over the triangular region we find $f(0,0) = -1$ is the absolute min. and $f(-1.5, \pm 1.5) = 3.5$ yields the absolute maximum.

Problem 139 Find the maximum and minimum values for $f(x,y) = x^4 - 2x^2 + y^2 - 2$ on the closed disk with boundary $x^2 + y^2 = 9$.

$$\nabla f = \langle 4x^3 - 4x,\ 2y \rangle = \langle 0, 0 \rangle \quad \text{for critical pts.}$$

$$4x(x^2-1) = 4(x-1)(x+1)x = 0 \Rightarrow x = -1, 0, 1$$

$$2y = 0 \Rightarrow y = 0 \qquad \text{all in } x^2+y^2 < 9.$$

We find critical points $(-1,0)$, $(0,0)$ and $(1,0)$. Evaluate for use in extreme value theorem (a.k.a closed set test)

$$f(-1,0) = 1-2+0-2 = \boxed{-3 = f(1,0)} \quad \text{possible extrema}$$

$$\boxed{f(0,0) = -2}$$

Next, we use the method of Lagrange to analyze boundary,

$$\langle 4x(x^2-1),\ 2y \rangle = \lambda \langle 2x, 2y \rangle$$

$$4x(x^2-1) = 2\lambda x$$

$$2y = 2\lambda y \Rightarrow (1-\lambda)y = 0 \quad \to \lambda = 1, \quad \to y = 0$$

① If $y = 0$ then $x = \pm 3$ hence we need

$4(\pm 3)(9-1) = 2\lambda(\pm 3) \Rightarrow \pm 8(12) = \pm 6\lambda \Rightarrow \lambda = \pm 16$

certainly consistent, we should check $(3,0)$ and $(-3,0)$

$$\boxed{f(3,0) = 81 - 2(9) - 2 = 61 = f(-3,0)} \quad \text{possible extrema}$$

② If $\lambda = 1$ then

$$4x(x^2-1) = 2x$$

$$4x^3 - 4x - 2x = 0$$

$$4x^3 - 6x = 0$$

$$x(4x^2 - 6) = 0$$

$\to x_1 = 0$, $\to x^2 = 6/4$ $\to x_2 = \sqrt{3/2}$, $\to x_3 = -\sqrt{3/2}$

$x_1 = 0 \Rightarrow y_1 = \pm 3 \Rightarrow (0, \pm 3)$ points of interest.

$$\boxed{f(0,\pm 3) = 0 - 0 + 9 - 2 = 7} \quad \text{possible extrema}$$

$$x_{2,3} = \pm\sqrt{\tfrac{3}{2}} \Rightarrow y_{2,3} = \pm\sqrt{9 - 3/2} = \pm\sqrt{15/2}$$

$$\boxed{f\left(\pm\sqrt{\tfrac{3}{2}},\ \pm\sqrt{\tfrac{15}{2}}\right) = \left(\tfrac{3}{2}\right)^2 - 2\left(\tfrac{3}{2}\right) + \tfrac{15}{2} - 2 = 4.75}$$

Thus, by the closed set test, we find

$$\boxed{\begin{array}{l} f(\pm 3, 0) = 61 \text{ is absolute max} \\ f(\pm 1, 0) = -3 \text{ is absolute min.} \end{array}}$$

Problem 140 Find the multivariate power series expansion for $f(x,y) = ye^x \sin(y)$ centered at $(0,0)$

$$f(x,y) = y\left(1 + x + \cdots\right)\left(y - \tfrac{1}{3!}y^3 + \cdots\right)$$
$$= y\left(y + xy + \cdots\right)$$
$$= \boxed{y^2 + xy^2 + \cdots}$$

(to 3rd order, of course, we could calculate more as needed.)

can you see that
$f(0,0) = f_x(0,0) = f_y(0,0) = f_{xx}(0,0) = f_{xy}(0,0) = 0$?

Problem 141 Expand $f(x,y,z) = xyz + x^2$ about the center $(1,0,3)$.

$$f(x,y,z) = \left[(x-1)+1\right] y \left[(z-3)+3\right] + (x-1+1)^2$$
$$= y\left[(x-1)(z-3) + 3(x-1) + (z-3) + 3\right] + (x-1)^2 + 2(x-1) + 1$$
$$= \boxed{1 + 2(x-1) + 3y + (x-1)^2 + 3y(x-1) + y(z-3) + 3y(x-1) + y(x-1)(z-3)}$$

Problem 142 Given that $f(x,y) = 3 + 2x^2 + 3y^2 - 2xy + \cdots$ determine if $(0,0)$ is a critical point and is $f(0,0)$ a local extremum.

YES, by examining the series above we identify that
$f_x(0,0) = f_y(0,0) = 0 \quad \therefore \quad (0,0)$ is critical pt.
Moreover, read that $\frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right)$
must match with $2x^2 + 3y^2 - 2xy$ hence we deduce,

$$f_{xx}(0,0) = 4, \quad f_{yy}(0,0) = 6, \quad f_{xy}(0,0) = -2$$

Apply 2nd derivative test as derived in my notes (or Stewart)

$$D(0,0) = 24 - (-2)^2 = 20 > 0 \quad \text{and} \quad f_{xx}(0,0) > 0$$

$\therefore$ $\boxed{f(0,0) \text{ is local minimum for } z = f(x,y)}$

Problem 143 Use Clairaut's Theorem to show it is impossible for $\vec{F} = \langle y^3 + x, x^2 + y \rangle = \nabla f$.

Assume $\vec{F} = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ we must have that

$\frac{\partial f}{\partial x} = y^3 + x$ and $\frac{\partial f}{\partial y} = x^2 + y$. These are continuous

functions f_x, f_y and clearly f_{xx}, f_{xy}, f_{yy} are also continuous

so Clairaut's $Th^{\underline{m}}$ applies and $f_{xy} = f_{yx}$. Consider that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^3 + x\right) = 3y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + y\right) = 2x \quad \therefore f_{xy} \neq f_{yx} \text{ which}$$

is a <u>contradiction</u>. It follows our assumption $\vec{F} = \nabla f$ is false.

Problem 144 Suppose $\vec{F} = \langle P, Q \rangle$ and suppose $P_y = Q_x$ for all points in some subset $U \subseteq \mathbb{R}^2$. Does it follow that $\vec{F} = \nabla f$ on U for some scalar function f? Discuss.

Hint: the polar angle θ has total differential $d\theta = d(\tan^{-1}(y/x)) = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dx$, think about the example $\vec{F} = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \rangle$. This function has domain $U = \mathbb{R}^2 - \{(0,0)\}$, can you find f such that $\vec{F} = \nabla f$ on all of U?

$\vec{F} = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right\rangle$ has $\text{dom}(\vec{F}) = U = \underbrace{\mathbb{R}^2 - \{(0,0)\}}$.

this is called the <u>punctured</u> plane.

Now try to find f such that $\vec{F} = \nabla f$.

Need to solve both

$$\frac{\partial f}{\partial x} = \frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-x}{x^2+y^2}$$

One obvious $sol^{\underline{n}}$ is that $f(x,y) = \tan^{-1}(y/x)$.

(it's obvious because of my hint, or if you wish trig subst.)

But, observe $\text{dom}(f) = \mathbb{R}^2 - \{(0,y) \mid y \in \mathbb{R}\}$, this

is <u>not</u> all of U. Ok, is this an artifact of the trig. subst. or will this happen no matter how we integrate?

Let's make a $\cot(\theta) = x/y$ calculation to investigate.

(we're going to find that the new $\tilde{f}$ also suffers the fate of our $f = \tan^{-1}(y/x)$, however, the trouble with $\tilde{f} = \cot^{-1}(x/y)$ is it misses the x-axis... you can see any f will be arctangent for some angle and they all miss some line in U.)

PROBLEM 144 continued

Try to solve $\frac{\partial f}{\partial x} = \frac{y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{-x}{x^2+y^2}$

with an cotangent substitution.

$$\cot\theta = \frac{x}{y} \qquad \sin^2\theta + \cos^2\theta = 1$$

Divide by $\sin^2\theta \longrightarrow 1 + \cot^2\theta = \csc^2\theta$

I forgot $\frac{d}{du}\left[\cot^{-1}(u)\right]$ so I'll derive it here,

$$z = \cot^{-1}(u) \Rightarrow \cot(z) = u \Rightarrow -\csc^2(z)\frac{dz}{du} = 1$$

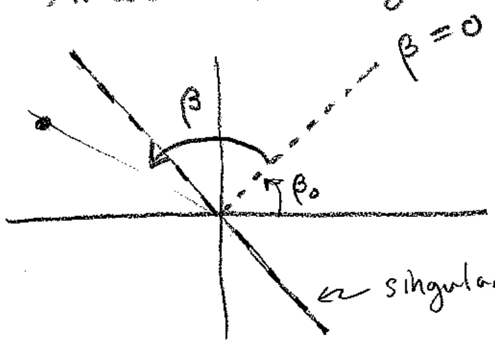
$$\Rightarrow \frac{d}{du}\left[\cot^{-1}(u)\right] = \frac{-1}{\csc^2 z} = \frac{-1}{1+u^2}.$$

Very well, now guess $\tilde{f}(x,y) = -\cot^{-1}(x/y)$ and check,

$$\frac{\partial \tilde{f}}{\partial x} = \frac{1}{1+(x/y)^2}\frac{\partial}{\partial x}\left[\frac{x}{y}\right] = \frac{1}{y + x^2/y} = \frac{y}{y^2+x^2}$$

$$\frac{\partial \tilde{f}}{\partial y} = \frac{1}{1+(x/y)^2}\frac{\partial}{\partial y}\left[\frac{x}{y}\right] = \frac{1}{1+x^2/y^2}\left(\frac{-x}{y^2}\right) = \frac{-x}{x^2+y^2}.$$

Thus $\tilde{f}(x,y) = \cot^{-1}(x/y)$ also provides a "potential" for the force field $\vec{F}$. However, this time the $\operatorname{dom}\tilde{f} = \mathbb{R} - \{(x,0) \mid x \in \mathbb{R}\}$. We lost the x-axis this time. More generally, could measure angle relative to $y = (\tan\beta_0)x$ axis



$$x = r\cos(\beta_0 + \beta)$$
$$y = r\sin(\beta_0 + \beta)$$

← singularity in $\tan^{-1}(\beta)$.

I conjecture that $d\beta = \omega_{\vec{F}} = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$

but, $\operatorname{dom}(\beta) = \mathbb{R}^2 - \{(x,y) \mid \theta = \beta_0 + \pi/2 \text{ or } \theta = \beta_0 + \frac{3\pi}{2}\}$

(I did not expect you all to find the ideas on this page, I mainly want you to see my hint provides a warning about reversing the $\vec{F} = \nabla f$ fail test of 143.)

Problem 145 We say $U \subseteq \mathbb{R}^n$ is path-connected iff any pair of points in U can be connected by a polygonal-path (this is a path made from stringing together finitely many line-segments one after the other) . Show that if $\nabla f = 0$ on a path-connected set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = c$ for each $\vec{x} \in U$. You may use the theorem from calculus I which states that if $f'(t) = 0$ for all t in a connected domain then $f = c$ on that domain.

Let $\vec{P}_1, \vec{P}_2$ be a pair of points in U. Hence, there exists paths $l_1, \dots, l_N$

where $\vec{\gamma}([0,N]) = l_1[0,1] \cup l_2[1,2] \cup \dots \cup l_N[N-1,N]$ and we have $\vec{\gamma}(0) = l_1(0) = \vec{P}_1$ and $\vec{\gamma}(N) = l_N(N) = \vec{P}_2$. I chose the path so that $\mathrm{dom}(l_j) = [j-1, j]$ for each $j = 1, 2, \dots, N$. These line segments meet at the connecting points $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_{N-1}$. Given this set-up, assume $\nabla f = 0$ on U and consider,

$$\frac{d}{dt}(f \circ l_j)(t) = \underbrace{(\nabla f)(l_j(t))}_{} \cdot \frac{dl_j}{dt} = 0 \quad \text{for } j = 1, 2, \dots, N.$$

(since $l_j(t) \in U$ $\forall j = 1, 2, \dots, N$ we have this is zero by the assumption $\nabla f = 0$ on U.)

Thus $f(l_j(t)) = k_j$ for each $j = 1, 2, \dots, N$ since we have constant function on closed interval. But $f(l_j(j)) = f(l_{j+1}(j))$ hence $k_j = k_{j+1}$ and this holds for $j = 1, 2, \dots, N-1$ $\therefore$ $k_1 = k_2 = \dots = k_N$

but notice $k_1 = f(\vec{P}_1)$ and $f(\vec{P}_2) = k_N$ hence $f(\vec{P}_1) = f(\vec{P}_2)$. But, $\vec{P}_1$ & $\vec{P}_2$ we arbitrary so we find f has same value on <u>all</u> of U.

Problem 146 Show that if $\nabla f = \nabla g$ on a path-connected set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = g(\vec{x}) + c$ for each $\vec{x} \in U$. *Hint: you can use Problem 145.*

Consider $h = f - g$.
Suppose $\nabla f = \nabla g$ on path-connected set $U \subseteq \mathbb{R}^n$. Notice that $\nabla h = \nabla f - \nabla g = 0$ and apply Problem 145 to obtain $h(\vec{x}) = c$ for all $\vec{x} \in U$.
But, this is precisely our goal

$$h(\vec{x}) = f(\vec{x}) - g(\vec{x}) = c \implies \boxed{f(\vec{x}) = g(\vec{x}) + c \text{ for all } \vec{x} \in U.}$$

Problem 147 Prove the mean-value theorem for functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. In particular, show that if f is differentiable at each point of the line-segment connecting $\vec{P}$ and $\vec{Q}$ then there exists a point $\vec{C}$ on the line-segment $\overline{PQ}$ such that $\nabla f(\vec{C}) \bullet (\vec{Q} - \vec{P}) = f(\vec{Q}) - f(\vec{P})$.
Hint: parametrize the line-segment and construct a function on $\mathbb{R}$ to which you can apply the ordinary mean value theorem, use the multivariate chain-rule and win.

Suppose f is differentiable on each point of $\overline{PQ}$.
Parametrize $\overline{PQ}$ via $\vec{\gamma}(t) = \vec{P} + t(\vec{Q} - \vec{P})$ for $0 \leq t \leq 1$ and consider $g = f \circ \vec{\gamma} : [0,1] \xrightarrow{\vec{\gamma}} \overline{PQ} \xrightarrow{f} \mathbb{R}$.
Calculate, by chain rule

$$
\begin{aligned}
g'(t) = \frac{dg}{dt} &= \frac{d}{dt}\left(f(\vec{\gamma}(t))\right) \\
&= (\nabla f)(\vec{\gamma}(t)) \bullet \frac{d\vec{\gamma}}{dt} \\
&= (\nabla f)(\vec{\gamma}(t)) \bullet (\vec{Q} - \vec{P}). \quad (\star)
\end{aligned}
$$

Observe that g is continuous on $[0,1]$ and differentiable on $(0,1)$ apply the mean value theorem to obtain $c \in (0,1)$ with

$$g'(c) = \frac{g(1) - g(0)}{1 - 0} = \frac{f(\vec{Q}) - f(\vec{P})}{1} \quad (\star\star)$$

Define $\vec{\gamma}(c) = \vec{c}$ and observe by $\star$

$$g'(c) = \nabla f(\vec{c}) \bullet (\vec{Q} - \vec{P}).$$

Therefore, for $\vec{c} \in \overline{PQ}$

$$\boxed{f(\vec{Q}) - f(\vec{P}) = \nabla f(\vec{c}) \bullet (\vec{Q} - \vec{P})}\ .$$

Problem 148 The method of characteristics is one of the many calculational techniques suggested by the total differential. The idea is simply this: given $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ we can solve both of these for dt to eliminate time. This leaves a differential equation in just the cartesian coordinates x, y and we can usually use a separation of variables argument to solve for the level curves which the solutions to $dx/dt = f(x,y)$ and $dy/dt = g(x,y)$ parametrize. Use the technique just described to solve

$$\boxed{\frac{dx}{dt} = -y \quad \& \quad \frac{dy}{dt} = x.} \quad (*)$$

$$dt = \frac{dx}{-y} = \frac{dy}{x}$$

obtain $x\,dx = -y\,dy \Rightarrow \int x\,dx = -\int y\,dy$

$$\Rightarrow \frac{1}{2}x^2 = -\frac{1}{2}y^2 + \frac{1}{2}R^2$$

$$\Rightarrow \boxed{x^2 + y^2 = R^2}$$ sol$^{\underline{n}}$'s parametrize circles.

Thus $x = R\cos t$, $y = R\sin t$ ought to solve the given system $(*)$.

(you can easily see these sol$^{\underline{n}}$'s are valid)

Problem 149 Suppose that the force $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$ is the net-force on a mass m. Furthermore, suppose $\vec{B} = B\,\widehat{\mathbf{z}}$ and $\vec{E} = E\,\widehat{\mathbf{z}}$ where E and B are constants. Find the equations of motion in terms of the initial position $\vec{r}_o = \langle x_o, y_o, z_o \rangle$ and velocity $\vec{v}_o = \langle v_{ox}, v_{oy}, v_{oz} \rangle$ by solving the differential equations given by $\vec{F} = m\frac{d\vec{v}}{dt}$. If $E = 0$ and $v_{oz} = 0$ then find the radius of the circle in which the charge q orbits.

Hint: first solve for the velocity components via the technique from Problem 148 then integrate to get the components of the position vector. not partial derivatives!

$$\vec{F} = q\left[(\vec{V} \times B\hat{z}) + E\hat{z}\right] = q\left[\langle V_x, V_y, V_z\rangle \times \langle 0,0,B\rangle + E\hat{z}\right]$$

$$= q\left[\langle BV_y, -BV_x, 0\rangle + E\hat{z}\right]$$

$$= \langle qBV_y, -qBV_x, E\rangle$$

Sir Isaac tells us,

$$m\vec{a} = \vec{F} \Rightarrow \underline{m\left\langle \frac{dV_x}{dt}, \frac{dV_y}{dt}, \frac{dV_z}{dt}\right\rangle = \langle qBV_y, -qBV_x, E\rangle}$$

need to solve these!

Problem 149 continued

We must solve the system,

$$\left.\begin{aligned}\frac{dV_x}{dt} &= \frac{qB}{m}V_y \\ \frac{dV_y}{dt} &= \frac{-qB}{m}V_x\end{aligned}\right\}$$ coupled. But, can use 148 idea to rip-apart.

$$\underbrace{\frac{dV_z}{dt} = \frac{E}{m}}_{} \longrightarrow V_z(t) = \frac{Et}{m} + V_{z0} = \frac{dz}{dt}$$

uncoupled, easy to integrate to get z-motion

$$\boxed{z(t) = z_0 + V_{z0}t + \frac{E}{2m}t^2}$$

Eliminate time at differential level,

$$dt = \frac{m\,dV_x}{qBV_y} = -\frac{m\,dV_y}{qB\,V_x} \Rightarrow V_x dV_x = -V_y dV_y$$

$$\Rightarrow \frac{1}{2}V_x^2 = \frac{-1}{2}V_y^2 + \frac{E_0^2}{2m}$$

$$\Rightarrow \boxed{\frac{1}{2}V_x^2 + \frac{1}{2}V_y^2 = \frac{\mathcal{E}}{m}}$$

Sorry folks, need to capture qB/m factor by different technique. (as far as I think of at present ...)

($\mathcal{E}$ becomes energy in certain context.)

$$V_x = \frac{m}{qB}\frac{dV_y}{dt} \Rightarrow \frac{dV_x}{dt} = \frac{d}{dt}\left[\frac{m}{qB}\frac{dV_y}{dt}\right] = -\frac{qB}{m}V_y$$

$$\Rightarrow V_y'' + \left(\frac{qB}{m}\right)^2 V_y = 0$$

$$\Rightarrow V_y = A\sin\left(\frac{qBt}{m} + \phi\right)$$

my calc II course, or your DEq^ns course.

$$\Rightarrow V_x = A\cos\left(\frac{qBt}{m} + \phi\right)$$

But, by previous part we know $A = \sqrt{\frac{2\mathcal{E}}{m}}$ (save for later)

$$V_x = \frac{dx}{dt} = A\cos\left(\frac{qBt}{m} + \phi\right)$$

$$V_y = \frac{dy}{dt} = A\sin\left(\frac{qBt}{m} + \phi\right)$$

Continuing 149

Integrate V_x and V_y to obtain,

$$x(t) = x_o + \left(\frac{mA}{qB}\right)\sin\left(\frac{qBt}{m} + \phi\right)$$

$$y(t) = y_o - \left(\frac{mA}{qB}\right)\cos\left(\frac{qBt}{m} + \phi\right)$$

Thus, $mA/qB = \frac{m}{qB}\sqrt{\frac{2\mathcal{E}}{m}} = \sqrt{\frac{2\mathcal{E}m}{q^2B^2}}$

$$\boxed{\vec{r}(t) = \left\langle x_o + \sqrt{\frac{2\mathcal{E}m}{q^2B^2}}\sin\left(\frac{qBt}{m}+\phi\right),\ y_o - \sqrt{\frac{2\mathcal{E}m}{q^2B^2}}\cos\left(\frac{qBt}{m}+\phi\right),\ z_o + V_{oz}t + \frac{E}{2m}t^2 \right\rangle}$$

nonlinear-helix of radius $\sqrt{\frac{2\mathcal{E}m}{q^2B^2}}$

where $\mathcal{E} = \frac{1}{2}mV_{ox}^2 + \frac{1}{2}mV_{oy}^2$ is conserved KE in xy-degrees of freedom.

Suppose $E = 0$ and $V_{oz} = 0$ in this case $\mathcal{E} = \frac{1}{2}mV^2$ since $V_z(t) = 0$, all KE is in xy-degrees of freedom and $\mathcal{E}$ = total energy.

$$R = \sqrt{\frac{2\mathcal{E}m}{q^2B^2}} = \sqrt{\frac{2\left(\frac{1}{2}mv^2\right)m}{q^2B^2}} = \sqrt{\frac{m^2v^2}{q^2B^2}} = \frac{mv}{qB}.$$

In 231/232 you might derive the same by using the centripetal force eq$^{\underline{n}}$;

$$\frac{mv^2}{R} = qvB \implies R = \frac{mv}{qB}$$

Problem 150 Suppose objective function $f(x,y)$ has an extremum on $g(x,y) = 0$. Show that F defined by $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ recovers the extremum as a critical point. From this viewpoint, the adjoining of the multiplier converts the constrained problem in n-dimensions to an unconstrained problem in $(n+1)$-dimensions (you can easily generalize your argument to $n > 2$).

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial \lambda} \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x}[f - \lambda g], \frac{\partial}{\partial y}[f - \lambda g], \frac{\partial}{\partial \lambda}[f - \lambda g] \right\rangle$$

$$= \left\langle \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y}, -g \right\rangle$$

Observe that $\nabla F = \langle 0, 0, 0 \rangle$ yields

$\nabla f = \lambda \nabla g$ from the x, y components.

$-g = 0$ from the λ-component

(these are the Lagrange Multiplier Eq's!)