Feel free to use Mathematica or some other CAS to illustrate as needed.

Problem 151 Calculate

$$\int_0^2\int_0^4 (3x+4y)\,dx\,dy = \int_0^2 \left(\tfrac{3}{2}x^2+4xy\right)\Big|_0^4 dy$$

$$= \int_0^2 (24+16y)\,dy$$

$$= (24y+8y^2)\Big|_0^2$$

$$= 48+8(4)$$

$$= \boxed{80.}$$

Problem 152

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x)\cos(y)\,dx\,dy = \int_0^{\pi/2}\left(-\cos(x)\cos(y)\Big|_{x=0}^{x=\frac{\pi}{2}}\right)dy$$

$$= \int_0^{\pi/2} \cos(y)\,dy$$

$$= \sin(y)\Big|_0^{\pi/2}$$

$$= \sin(\pi/2)$$

$$= \boxed{1.}$$

Problem 153

$$\int_{-1}^1\int_0^1 \sin^3(x)\cos^{42}(y)\,dy\,dx = \int_0^1\int_{-1}^1 \sin^3(x)\cos^{42}(y)\,dx\,dy$$

$$= \int_0^1 \cos^{42}(y)\underbrace{\int_{-1}^1 \sin^3(x)\,dx}_{\text{zero}} = \boxed{0.}$$

zero as $\sin^3(x)$ is odd function of x.

Problem 154 Calculate the average of $f(x, y) = x^2 + y^2$ on the unit-square.

$$f_{avg} = \frac{1}{\text{area (unit-square)}} \int_0^1 \int_0^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{1}{3}x^3 + xy^2\right)\Big|_{0=x}^{1=x} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2\right) dy$$

$$= \frac{1}{3} + \frac{y^3}{3}\Big|_0^1$$

$$= \boxed{2/3.}$$

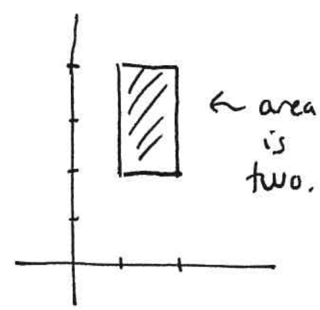
Problem 155 Calculate the average of $f(x, y) = x^2 + y^2$ on the region bounded by $x^2 + y^2 = R^2$.

$$f_{avg} = \frac{1}{\text{area}(x^2+y^2 \leq R^2)} \int_0^{2\pi} \int_0^R r^2 r\,dr\,d\theta$$

$$= \frac{1}{\pi R^2} \int_0^{2\pi} d\theta \int_0^R r^3\,dr$$

$$= \frac{2\pi}{\pi R^2} \frac{r^4}{4}\Big|_0^R$$

$$= \boxed{\frac{R^2}{2}}$$



Problem 156 Calculate the average of $f(x, y) = xy$ on $[1, 2] \times [3, 4]$.

$$f_{avg} = \frac{1}{2} \int_3^4 \int_1^2 xy\,dx\,dy$$

$$= \frac{1}{2} \int_3^4 \left(\frac{1}{2}x^2\Big|_1^2\right) y\,dy$$

$$= \frac{1}{4} \int_3^4 3y\,dy$$

$$= \frac{3}{8} y^2\Big|_3^4$$

$$= \frac{3}{8}\left[16 - 9\right]$$

$$= \boxed{21/8}$$

Problem 157 Show that

$$\lim_{n\to\infty}\int_0^1\int_0^1 x^n y^n\,dx\,dy = 0.$$

$$\int_0^1\int_0^1 x^n y^n dx\,dy = \int_0^1 x^n dx \int_0^1 y^n dy$$

$$= \left(\int_0^1 x^n dx\right)^2$$

$$= \left(\frac{x^{n+1}}{n+1}\Big|_0^1\right)^2$$

$$= \left(\frac{1}{n+1}\right)^2$$

$$\Rightarrow \lim_{n\to\infty}\int_0^1\int_0^1 x^n y^n dxdy = \lim_{n\to\infty}\left(\frac{1}{n+1}\right)^2 = \boxed{0.}$$

Problem 158 Calculate

$$\int_0^{\ln(2)}\int_0^{\ln(3)} e^{x+y}dx\,dy = \left(\int_0^{\ln(2)} e^y dy\right)\left(\int_0^{\ln(3)} e^x dx\right) \qquad e^{x+y} = e^x e^y$$

$$= \left(e^y\Big|_0^{\ln(2)}\right)\left(e^x\Big|_0^{\ln(3)}\right)$$

$$= \left(e^{\ln(2)} - 1\right)\left(e^{\ln(3)} - 1\right)$$

$$= (2-1)(3-1)$$

$$= \boxed{2.}$$

Problem 159 Suppose $\int\int_R f\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (1+x)dy\, dx$. Calculate the given integral.

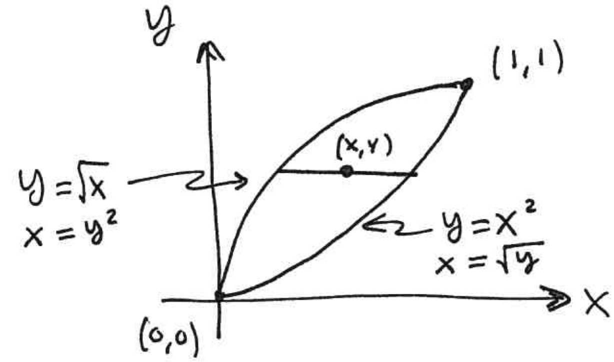
$$\int_0^1 \int_{x^2}^{\sqrt{x}} (1+x)\,dy\,dx = \int_0^1 \left(y(1+x)\Big|_{x^2=y}^{\sqrt{x}=y} \right) dx$$

$$= \int_0^1 (\sqrt{x} - x^2)(1+x)\,dx$$

$$= \int_0^1 \left(\sqrt{x} + x - x^2 - x^{3/2}\right) dx$$

$$= \frac{2}{3} + \frac{1}{2} - \frac{1}{3} - \frac{2}{5} = \boxed{\frac{29}{60}}$$

Problem 160 For the integral given in the previous problem, explicitly write R as a subset of $\mathbb{R}^2$ using set-builder notation. In addition, calculate the integral once more with the interation of the integrals beginning with dx. Draw a picture to explain the inequalities which form the basis for your new set-up to the integral.



$$0 \le y \le 1$$
$$y^2 \le x \le \sqrt{y}$$

$$\iint_R f\,dA = \int_0^1 \int_{y^2}^{\sqrt{y}} (1+x)\,dx\,dy$$

$$= \int_0^1 \left(x + \tfrac{1}{2}x^2\right)\Big|_{x=y^2}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \left(\sqrt{y} - y^2 + \tfrac{1}{2}(\sqrt{y})^2 - \tfrac{1}{2}(y^2)^2\right) dy$$

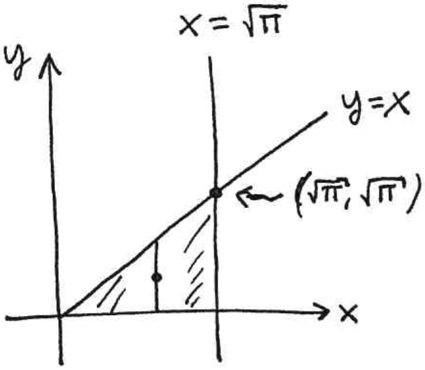
$$= \int_0^1 \left(\sqrt{y} - y^2 + \tfrac{1}{2}y - \tfrac{1}{2}y^4\right) dy$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{10}$$

$$= \frac{40 - 20 + 15 - 6}{60} = \boxed{\frac{29}{60}.}$$

Problem 161 Reverse the order of integration in order to calculate the following integral:

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dx\, dy. = \int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2)\, dy\, dx$$



$$\left.\begin{aligned} 0 &\le y \le \sqrt{\pi} \\ y &\le x \le \sqrt{\pi} \end{aligned}\right\} \text{given}$$

by picture.

$$0 \le x \le \sqrt{\pi}$$
$$0 \le y \le x$$

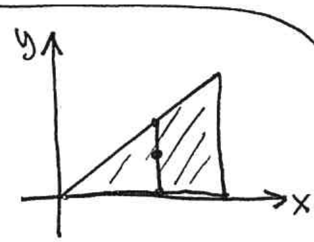
$$= \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$$
$$= \left. \frac{-1}{2} \cos(x^2) \right|_0^{\sqrt{\pi}}$$
$$= -\frac{1}{2} \cos \pi + \frac{1}{2} \cos(0)$$
$$= \boxed{1.}$$

Problem 162 Reverse the order of integration in order to calculate the following integral:

$$\int_0^1 \int_y^1 \frac{1}{1+x^4}\, dx\, dy. = \int_0^1 \int_0^x \frac{dy}{1+x^4}\, dx$$

$$= \int_0^1 \frac{x\, dx}{1+(x^2)^2}$$

$$\boxed{\begin{aligned} u &= x^2, & u(0) &= 0 \\ du &= 2x\, dx, & u(1) &= 1. \end{aligned}}$$



$$\left\{\begin{aligned} 0 &\le y \le 1 \\ y &\le x \le 1 \end{aligned}\right.$$

$$0 \le x \le 1$$
$$0 \le y \le x$$

$$= \frac{1}{2} \int_0^1 \frac{du}{1+u^2}$$
$$= \left. \frac{1}{2} \tan^{-1}(u) \right|_0^1$$
$$= \frac{1}{2} \left(\tan^{-1}(1) - \tan^{-1}(0) \right)$$
$$= \frac{1}{2} \left(\frac{\pi}{4} - 0 \right)$$
$$= \boxed{\pi/8}$$

Remark: it does not always have to happen that the x & y bounds match...

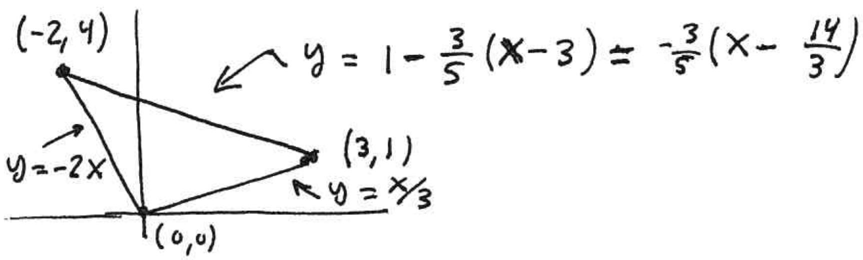
Problem 163 Find the average of $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(3,1)$ and $(-2,4)$.

$$\iint_T xy\,dy = \int_{-2}^{0}\int_{-2x}^{-\frac{3}{5}(x-14/3)} xy\,dy\,dx + \int_0^3 \int_{x/3}^{-\frac{3}{5}(x-14/3)} xy\,dy\,dx.$$

$$= \int_{-2}^{0} \frac{x}{2}\left(\frac{9}{25}\left(x-\frac{14}{3}\right)^2 - 4x^2\right)dx + \int_0^3 \frac{x}{2}\left(\frac{9}{25}\left(x-\frac{14}{3}\right)^2 - \frac{x^2}{9}\right)dx$$

$$= \frac{-126}{25} + \frac{126}{25}$$

$$= \boxed{0.}$$



Problem 164 Find volume bounded by $z = y + e^x$ and the xy-plane for $(x,y) \in [0,1] \times [0,2]$.

$$\int_0^2\int_0^1 (y+e^x)\,dx\,dy = \int_0^2 (y+e-1)\,dy$$

$$= \left[\frac{1}{2}y^2 + (e-1)y\right]\Big|_0^2$$

$$= \frac{4}{2} + (e-1)2$$

$$= 2 + 2e - 2$$

$$= \boxed{2e.}$$