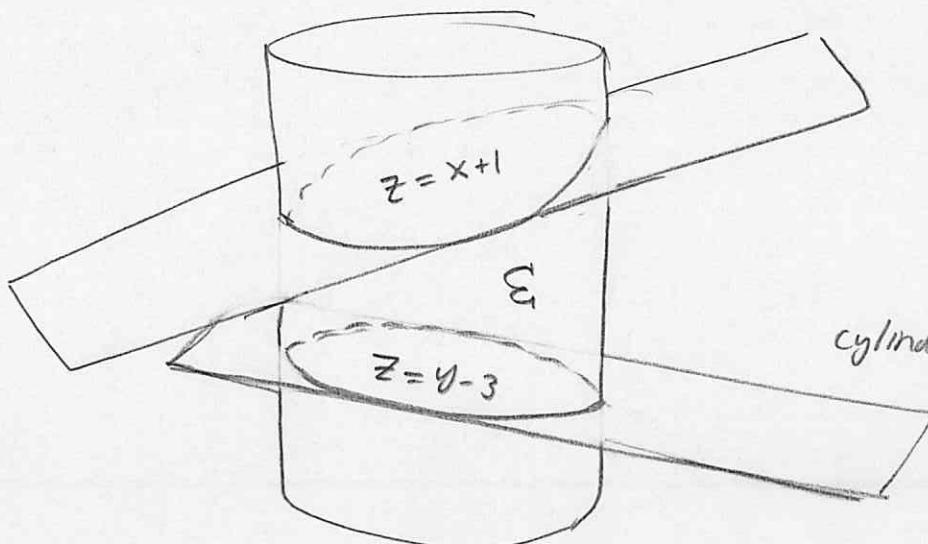


Feel free to use Mathematica or some other CAS to illustrate as needed.

**Problem 165** Find the volume bounded inside the cylinder  $x^2 + y^2 = 1$  and the planes  $z = x + 1$  and  $z = y - 3$ .



$$y - 3 \leq z \leq x + 1$$

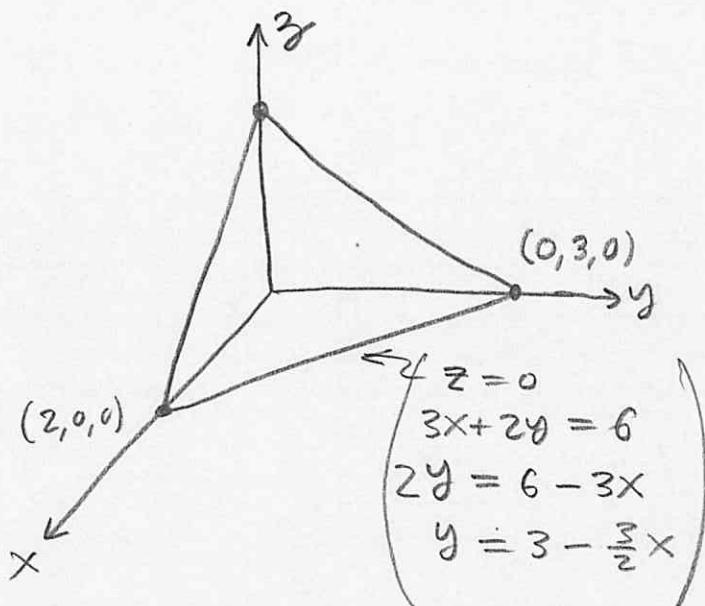
$$0 \leq x^2 + y^2 \leq 1$$

cylindrical

$$\left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ r\sin\theta - 3 \leq z \leq r\cos\theta + 1 \end{array} \right.$$

$$\begin{aligned}
 V &= \iiint_{\Sigma} dV = \int_0^{2\pi} \int_0^1 \int_{r\sin\theta - 3}^{r\cos\theta + 1} r dz dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 r [r\cos\theta + 1 - (r\sin\theta - 3)] dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 4r dr d\theta \\
 &= \int_0^{2\pi} 2r^2 \Big|_0^1 d\theta \\
 &= 2\theta \Big|_0^{2\pi} \\
 &= \boxed{4\pi}
 \end{aligned}$$

**Problem 166** Find the volume bounded by the coordinate planes and the plane  $3x + 2y + z = 6$ .



$$\begin{aligned} y=0, z=0 &\Rightarrow 3x=6 \therefore x=2 \\ x=0, z=0 &\Rightarrow 2y=6 \therefore y=3 \end{aligned}$$

We find for  $(x, y, z)$  in the volume,

$$0 \leq x \leq 2$$

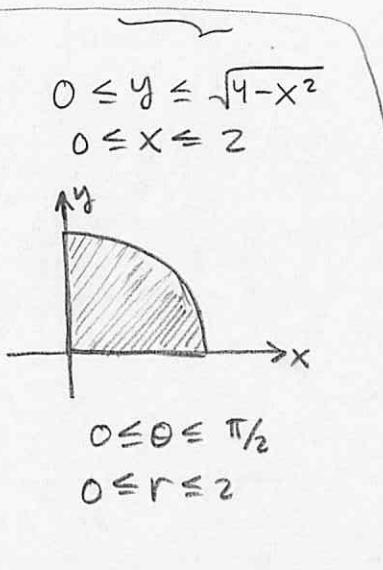
$$0 \leq y \leq 3 - \frac{3}{2}x$$

$$0 \leq z \leq 6 - 3x - 2y$$

Thus,

$$\begin{aligned} V &= \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} dz dy dx \\ &= \int_0^2 \int_0^{3-\frac{3}{2}x} (6 - 3x - 2y) dy dx \\ &= \int_0^2 \left[ 6(3 - \frac{3}{2}x) - 3x(3 - \frac{3}{2}x) - (3 - \frac{3}{2}x)^2 \right] dx \\ &= \int_0^2 \left[ 18 - 9x - 9x - \frac{9}{2}x^2 - (9 - 9x + \frac{9}{4}x^2) \right] dx \\ &= \int_0^2 \left[ 9 - 9x + \frac{9}{4}x^2 \right] dx \\ &= \left. \left( 9x - \frac{9}{2}x^2 + \frac{9}{12}x^3 \right) \right|_0^2 \\ &= 18 - \frac{36}{2} + \frac{9(8)}{12} = \boxed{6}. \end{aligned}$$

Problem 167 Calculate the integral (use polars):



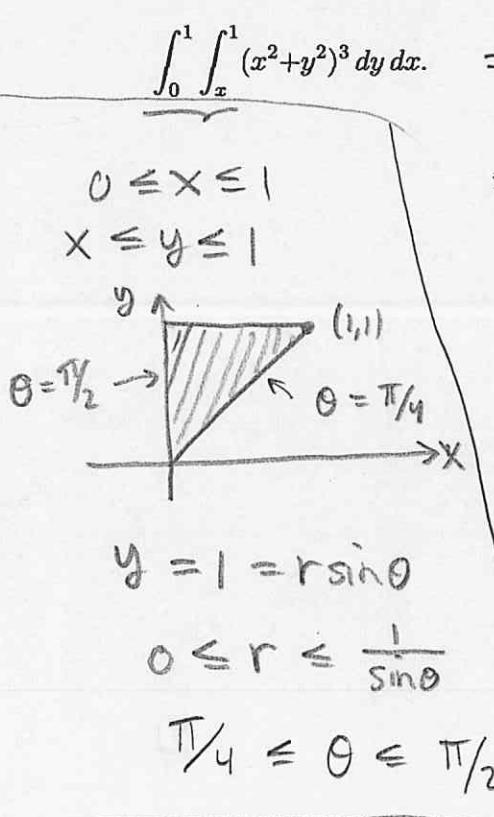
$$\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)^{3/2} dy dx = \int_0^2 \int_0^{\pi/2} r^3 r dr d\theta$$

$$= \left[ \frac{r^5}{5} \right]_0^{\pi/2}$$

$$= \left( \frac{\pi}{2} \right) \left( \frac{32}{5} \right)$$

$$= \boxed{\frac{16\pi}{5}}$$

Problem 168 Calculate the integral (use polars):



$$\int_0^1 \int_x^1 (x^2+y^2)^3 dy dx = \int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} (r^2)^3 r dr d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} r^7 dr d\theta$$

$$= \frac{1}{8} \int_{\pi/4}^{\pi/2} \csc^8 \theta d\theta$$

$$= -\frac{1}{8} \int_{\pi/4}^{\pi/2} (1 + \cot^2 \theta)^3 d(\cot \theta)$$

$$= -\frac{1}{8} \int_1^0 (1+u^2)^3 du$$

$$= \frac{1}{8} \int_0^1 (1+3u^2+3u^4+u^6) du$$

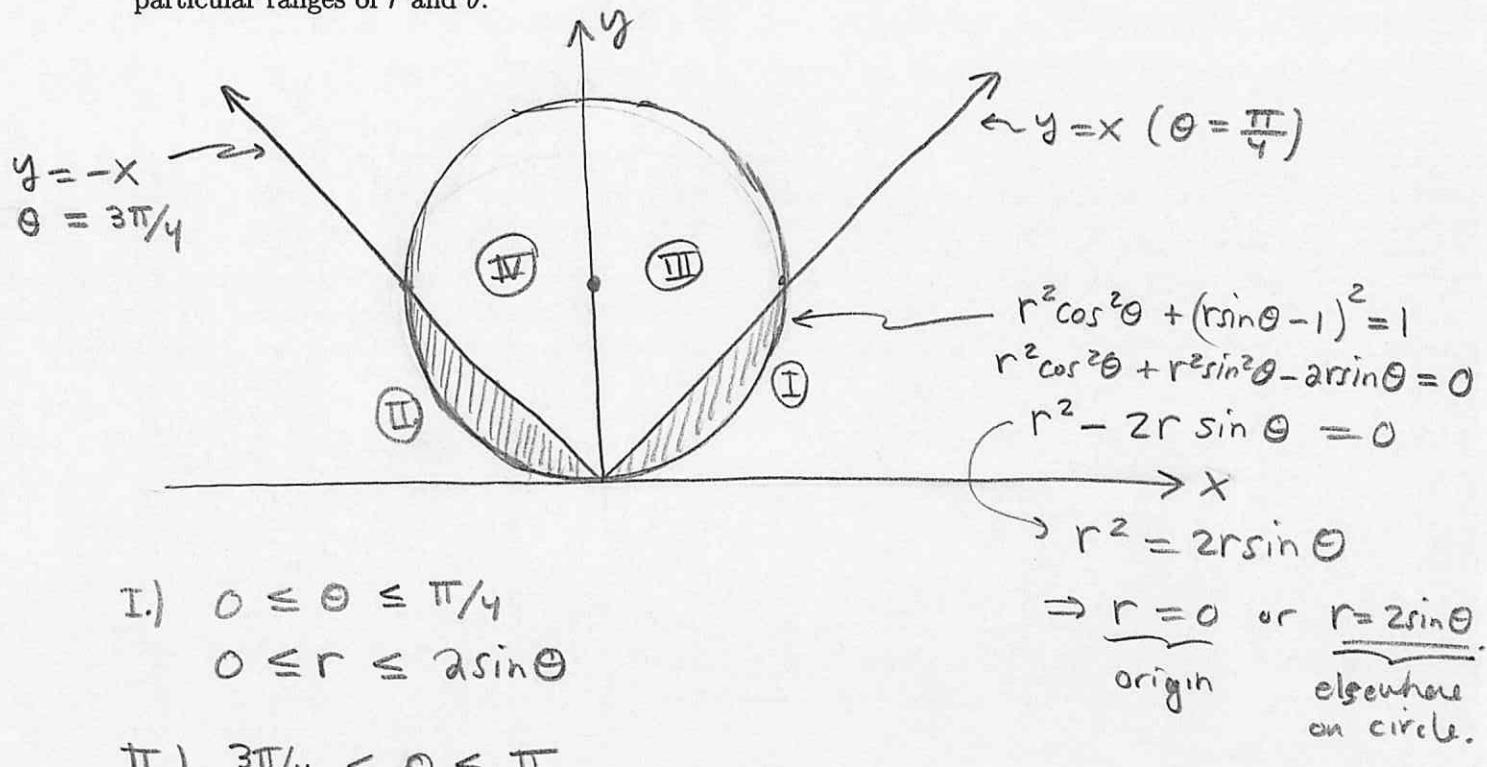
$$= \frac{1}{8} \left[ 1 + \frac{3}{3} + \frac{3}{5} + \frac{1}{7} \right]$$

$$= \boxed{\frac{12}{35}}$$

$\begin{cases} \sin^2 \theta + \cos^2 \theta = 1 \\ 1 + \cot^2 \theta = \csc^2 \theta \\ d(\cot \theta) = -\csc^2 \theta d\theta \end{cases}$

$u = \cot \theta$

**Problem 169** Suppose  $R$  is the region bounded by  $y + |x|$  and  $x^2 + (y - 1)^2 = 1$ . Express  $R$  in polar coordinates. In other words, draw a picture and indicate how the points in  $R$  are reached by particular ranges of  $r$  and  $\theta$ .



It would also be reasonable to suppose the bounded region was (or includes) III & IV,

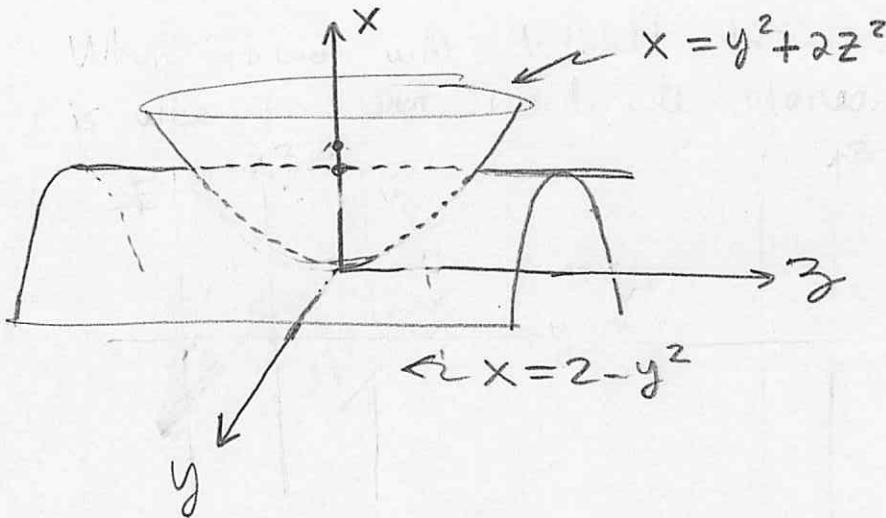
$$\left. \begin{array}{l} \text{III.) } \pi/4 \leq \theta \leq \pi/2 \\ \quad 0 \leq r \leq 2 \sin \theta \end{array} \right\} \text{a.k.a. } \left. \begin{array}{l} \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ \quad 0 \leq r \leq 2 \sin \theta. \end{array} \right\}$$

IV.)  $\pi/2 \leq \theta \leq 3\pi/4$   
 $0 \leq r \leq 2 \sin \theta$

Observe, the region  $x^2 + (y - 1)^2 \leq 1$  is nicely written as  $0 \leq \theta \leq \pi$  with  $0 \leq r \leq 2 \sin \theta$  in polar coordinates.

(Some logical subset of these comments  
earns good credit)

Problem 170 Find volume bounded by the paraboloid  $x = y^2 + 2z^2$  and the parabolic cylinder  $x = 2 - y^2$ .



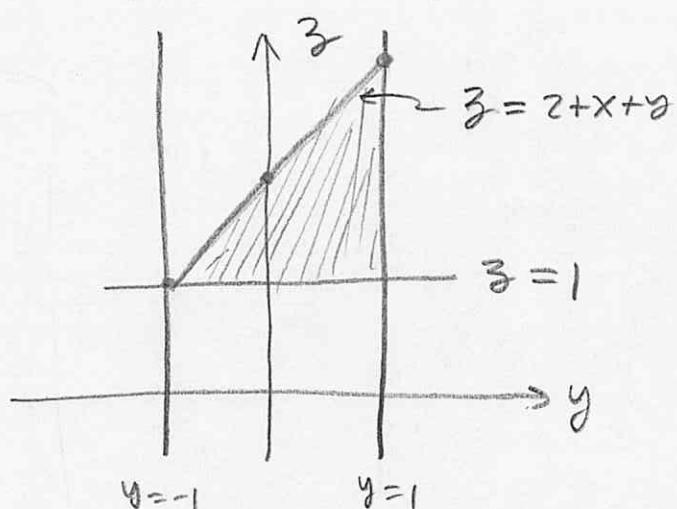
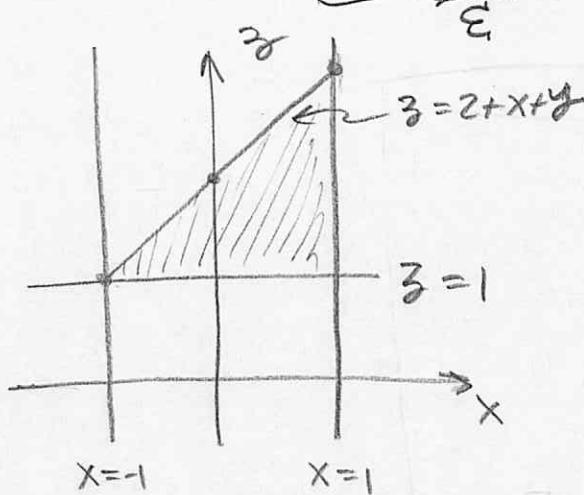
It is clear that  $y^2 + 2z^2 \leq x \leq 2 - y^2$  for a certain region of space between  $x = 2 - y^2$  and  $x = y^2 + 2z^2$ . These intersect at  $(x, y, z)$  for which  $2 - y^2 = y^2 + 2z^2 \Rightarrow z = \sqrt{y^2 + 2z^2} \Rightarrow 1 = y^2 + z^2$  ah. nice.

They intersect on the cylinder  $y^2 + z^2 = 1$ . We should use modified polars to solve this.

$$\begin{aligned} y &= r \cos \theta \\ z &= r \sin \theta \\ x &= x \end{aligned} \Rightarrow \begin{cases} y^2 + 2z^2 = r^2(\cos^2 \theta + 2\sin^2 \theta) \\ 2 - y^2 = 2 - r^2 \sin^2 \theta \end{cases}$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_{r^2 \cos^2 \theta + 2r^2 \sin^2 \theta}^{2 - r^2 \sin^2 \theta} r dx dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r(2 - r^2 \sin^2 \theta - r^2 \cos^2 \theta - 2r^2 \sin^2 \theta) dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r - r^2 - 2r^3 \sin^2 \theta) dr d\theta \\ &= \int_0^{2\pi} \left(1 - \frac{1}{3} - \frac{1}{2} \sin^2 \theta\right) d\theta \\ &= \frac{4\pi}{3} - \frac{1}{4} \int_0^{2\pi} (1 - \cos 2\theta) d\theta = \frac{4\pi}{3} - \frac{\pi}{2} = \boxed{\frac{5\pi}{6}} \end{aligned}$$

Problem 171 Find the volume bounded by the cylinder  $x^2 + y^2 = 1$  and  $z = 2 + x + y$  and  $z = 1$ .



$$y=0 \Rightarrow z = 2+x$$

$$x=0 \Rightarrow z = 2+y$$

$$(x, y, z) \in E \Rightarrow \begin{cases} 0 \leq x^2 + y^2 \leq 1 \\ 1 \leq z \leq 2+x+y \end{cases} \quad \left. \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ 1 \leq z \leq 2+r\cos\theta+r\sin\theta \end{array} \right\}$$

$$\begin{aligned} V_E &= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_1^{2+r\cos\theta+r\sin\theta} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r(2+r\cos\theta+r\sin\theta - 1) dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (r + r^2(\cos\theta + \sin\theta)) dr d\theta \\ &= (2\pi) \left( \frac{1}{2} \right) \\ &= \boxed{\pi}. \end{aligned}$$

**Problem 171** Find the volume bounded by the cones  $\underbrace{z = \sqrt{x^2 + y^2}}$  and  $\underbrace{z = 2\sqrt{x^2 + y^2}}$  and the sphere  $\rho = 3$ .

$$\begin{aligned} \rho \cos \phi &= \rho \sin \phi & \rho \cos \phi &= 2\rho \sin \phi \\ \Rightarrow \phi &= \pi/4 & \phi &= \tan^{-1}(1/2) \end{aligned}$$

$$\begin{aligned} V &= \int_0^3 \int_0^{2\pi} \int_{\tan^{-1}(1/2)}^{\pi/4} \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \left( \frac{\rho^3}{3} \Big|_0^3 \right) \left( \Theta \Big|_0^{2\pi} \right) \left( -\cos \phi \Big|_{\tan^{-1}(1/2)}^{\pi/4} \right) \\ &= \left( \frac{27}{3} \right) (2\pi) \left[ -\frac{1}{\sqrt{2}} + \cos(\tan^{-1}(1/2)) \right] \end{aligned}$$

**Problem 172** Let  $B$  be a ball of radius  $R$  centered at the origin. Calculate  $\iiint_B e^{-\rho^3} dV$

$$\begin{aligned} \iiint_B e^{-\rho^3} dV &= \int_0^{2\pi} \int_0^\pi \int_0^R e^{-\rho^3} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= (2\pi)(2) \left( -\frac{1}{3} e^{-\rho^3} \Big|_0^R \right) \\ &= -\frac{4\pi}{3} (e^{-R^3} - 1) \\ &= \boxed{\frac{4\pi}{3} [1 - \exp(-R^3)]} \end{aligned}$$

**Problem 173** Let  $u = \frac{2x}{x^2+y^2}$  and  $v = \frac{-2y}{x^2+y^2}$  calculate  $\frac{\partial(x,y)}{\partial(u,v)}$ .

Solving for  $x, y$  is a bit involved, so... instead,

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \\ &= \left[ \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} \right] \left[ \frac{-2(x^2+y^2) + 2y(2y)}{(x^2+y^2)^2} \right] \\ &\quad - \left[ \frac{-2xy}{(x^2+y^2)^2} \right] \left[ \frac{+2xy}{(x^2+y^2)^2} \right] \\ &= \frac{2(y^2-x^2)2(y^2-x^2) + 4x^2y^2}{(x^2+y^2)^2} \\ &= 4 \left[ \frac{x^4 - 2x^2y^2 + y^4 + x^2y^2}{(x^2+y^2)^2} \right] = 4 \left( \frac{x^4 - x^2y^2 + y^4}{(x^2+y^2)^2} \right)\end{aligned}$$

**Problem 174** Suppose  $\delta(x,y,z) = 1 = dM/dV$  for  $x,y,z > 0$ . Find center of mass for a sphere with this density  $\delta$  centered at  $(1,2,3)$ . Told class to use

Let  $\bar{x} = x-1$   
 $\bar{y} = y-2$   
 $\bar{z} = z-3$

radius 1 for sphere.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{(x^2+y^2)^2}{4(x^4 - x^2y^2 + y^4)}$$

then the sphere has  $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = 1$ . We can calculate,

$$M = \iiint_{\text{sphere}} \delta dV = \iiint_{\text{sphere}} dV = \frac{4\pi}{3}. \quad \text{Let sphere } = B$$

$$MX_{cm} = \iiint_B x \delta dV = \iint_0^1 \iint_B (x-1) dV + \iint_0^1 \iint_B dV = \frac{4\pi}{3} \therefore \underline{x_{cm} = 1}.$$

(symmetry)

$$MY_{cm} = \iiint_B y \delta dV = \iint_0^1 \iint_B (\bar{y}-2) dV + 2 \iint_0^1 \iint_B dV = \frac{8\pi}{3} \therefore \underline{y_{cm} = 2}$$

Likewise,  $z_{cm} = 3$  hence  $\boxed{(1,2,3) \text{ is centroid.}}$

(this could reasonably be claimed w/o work here, note to grader.)

I use  $\bar{x}, \bar{y}, \bar{z}$  introduced in prob 174 sol<sup>1/2</sup> in what follows,

**Problem 175** Suppose  $\delta(x, y, z) = xyz = dM/dV$  for  $x, y, z > 0$ . Find center of mass for a sphere with this density centered at  $(1, 2, 3)$ . Assume radius 1.

Introduce modified sphericals,

$$\bar{x} = 1 + \rho \cos \theta \sin \phi$$

$$y = 2 + \rho \sin \theta \sin \phi$$

$$z = 3 + \rho \cos \phi$$

Then  $0 \leq \rho \leq 1$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \phi \leq \pi$  models  $B$ .

$$\begin{aligned} M &= \int_0^1 \int_0^{2\pi} \int_0^\pi (1 + \rho \cos \theta \sin \phi)(2 + \rho \sin \theta \sin \phi)(3 + \rho \cos \phi) \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi (2 + \rho \sin \theta \sin \phi + 2\rho \cos \theta \sin \phi + \rho^2 \cos \theta \sin \phi \cos \phi)(3 + \rho \cos \phi) dV \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi (6 + 2\rho \cos \phi) \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi 6 \rho^2 \sin \phi d\phi d\theta d\rho \quad (2\cos \phi \sin \phi = \sin(2\phi) \text{ so } 0 \leq \phi \leq \pi \text{ is full cycle which averages to zero.}) \\ &= 2(2\pi)(-\cos \phi)|_0^\pi \\ &= \boxed{8\pi}. \end{aligned}$$

$$\begin{aligned} x_{cm} &= \frac{1}{8\pi} \iiint_B x \delta dV \rightarrow x\delta = x^2 y z = (x-1+1)^2 (y-2+2) (z-3+3) \\ &\quad = (\bar{x}^2 - 2\bar{x} + 1)(\bar{y} + 2)(\bar{z} + 3) \\ &\quad = (\bar{x}^2 - 2\bar{x} + 1)(\bar{y}\bar{z} + 2\bar{z} + 3\bar{y} + 6) \\ &\quad = 6 + 6\bar{x}^2 + \underbrace{\text{terms odd in either } \bar{y}, \bar{z} \text{ or } \bar{x}}_{\text{integrate to zero over } B.} \\ &= \frac{1}{8\pi} \iiint_B (6 + 6\bar{x}^2) dV \\ &= \frac{3}{4\pi} \int_0^1 \int_0^\pi \int_0^{2\pi} (1 + \rho^2 \cos^2 \theta \sin^2 \phi) \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \frac{3}{4\pi} \left[ \frac{4\pi}{3} + \left(\frac{1}{5}\right) \left( \int_0^\pi \sin^3 \phi d\phi \right) \left( \int_0^{2\pi} \cos^2 \theta d\theta \right) \right] \\ &= \frac{3}{4\pi} \left[ \frac{4\pi}{3} + \left(\frac{1}{5}\right) \left(\frac{4}{3}\right) (\pi) \right] \\ &= \boxed{6/5}. \end{aligned}$$

175 continued

$$y_{cm} = \frac{1}{8\pi} \iiint_B y \delta dV$$

$$yS = y^2 \times 3$$

$$= (y-2+2)^2 (x-1+1) (z-3+3)$$

$$= (\bar{y}+2)^2 (\bar{x}+1) (\bar{z}+3)$$

$$= (\bar{y}^2 + 4\bar{y} + 4)(\bar{x}+1)(\bar{z}+3)$$

$$= 12 + 3\bar{y}^2 + \text{odd terms which vanish once integrated}$$

$$y_{cm} = \frac{1}{8\pi} \iiint_B (12 + 3\bar{y}^2) dV$$

$$= \frac{1}{8\pi} \left[ 12 \left( \frac{4\pi}{3} \right) + 3 \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin^2 \theta \sin^2 \phi \rho^2 \sin \phi d\rho d\phi d\theta \right]$$

$$= 2 + 3 \left( \frac{1}{5} \right) \left( \int_0^\pi \sin^3 \phi d\phi \right) \left( \int_0^{2\pi} \sin^2 \theta d\theta \right)$$

$$= 2 + \frac{3}{8\pi} \left( \frac{1}{5} \right) \left( \frac{4}{3} \right) (\pi)$$

$$= 2 + \frac{1}{10}$$

$$= \boxed{21/10}$$

$$z_{cm} = \frac{1}{8\pi} \iiint_B z \delta dV, \quad zS = z^2 xy \quad z = \bar{z} + 3$$

$$= (\bar{z}^2 + 6\bar{z} + 9)(\bar{x}+1)(\bar{y}+2)$$

$$= \frac{1}{8\pi} \left[ 18 \left( \frac{4\pi}{3} \right) + 2 \iiint_B \bar{z}^2 dV \right] = 2\bar{z}^2 + 18 + \dots$$

$$= \frac{1}{8\pi} \left[ 18 \left( \frac{4\pi}{3} \right) + 2 \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 \cos^2 \phi \rho^2 \sin^2 \phi d\theta d\phi d\rho \right]$$

$$= \frac{1}{8\pi} \left[ 24\pi + 2 \left( \frac{1}{5} \right) (2\pi) \left( \int_0^\pi \cos^2 \phi \sin^2 \phi d\phi \right) \right]$$

$$= \frac{1}{8\pi} \left[ 24\pi + 2 \left( \frac{1}{5} \right) (2\pi) \left( \frac{\pi}{8} \right) \right]$$

$$= 3 + \frac{\pi^2}{10} - \frac{1}{8\pi}$$

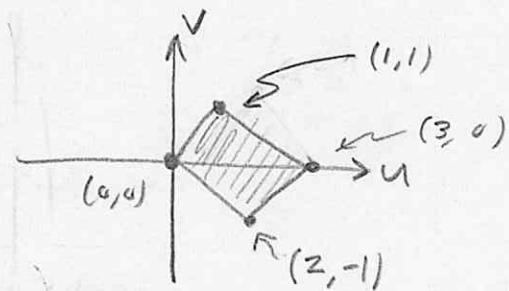
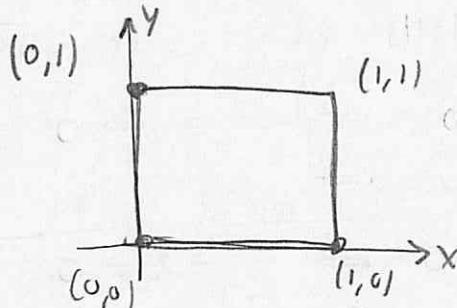
$$= 3 + \frac{\pi^2}{80}$$

$$\rightarrow \boxed{(6/5, 21/10, 3 + \pi^2/80)}$$

center of mass. (continued)

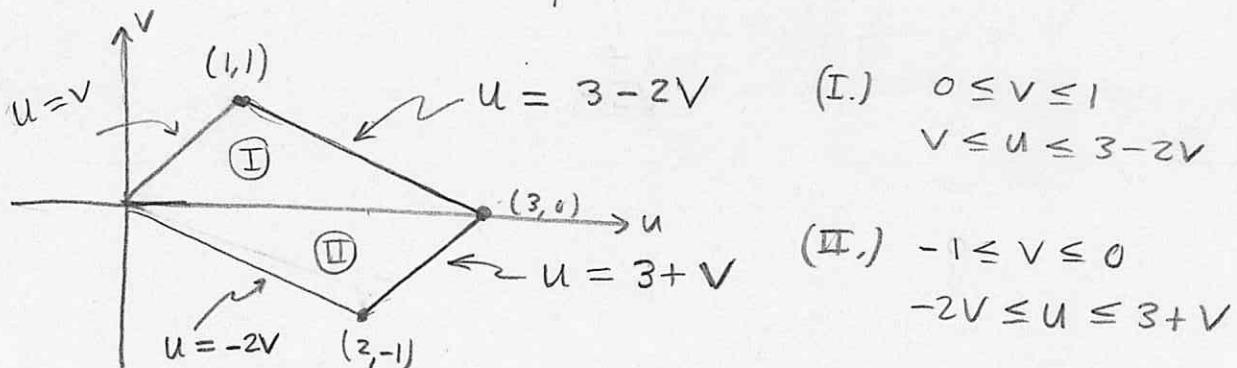
**Problem 176** Calculate  $\int_R \sqrt{x+2y} \sin(x-y) dA$  where  $R = [0, 1] \times [0, 1]$  by making an appropriate change of variables.

$$\begin{cases} u = x + 2y \\ v = x - y \end{cases} \Rightarrow \begin{cases} u - v = 3y \Rightarrow y = \frac{1}{3}(u-v) \\ u + 2v = 3x \Rightarrow x = \frac{1}{3}(u+2v) \end{cases}$$



$$\begin{aligned} (0,0) &\mapsto (0,0) \\ (1,0) &\mapsto (1+2(0), 1-0) = (1,1) \\ (1,1) &\mapsto (1+2, 1-1) = (3,0) \\ (0,1) &\mapsto (0+2, 0-1) = (2,-1) \end{aligned}$$

$$\begin{aligned} dA &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv \\ &= \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| du dv = \frac{1}{3} du dv. \end{aligned}$$



$$\iint_R f dA = \frac{1}{3} \left( \int_0^1 \int_{v}^{3-2v} \sqrt{u} \sin(v) du dv + \int_{-1}^0 \int_{-2v}^{3+v} \sqrt{u} \sin(v) du dv \right)$$

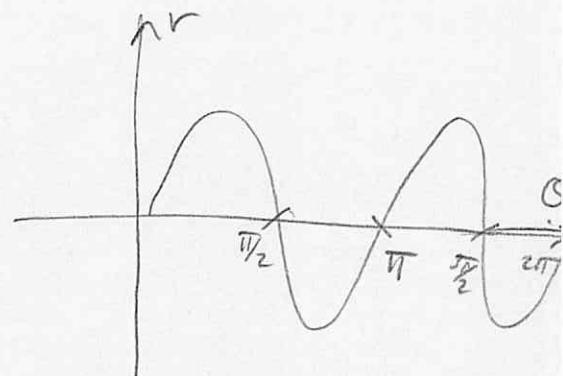
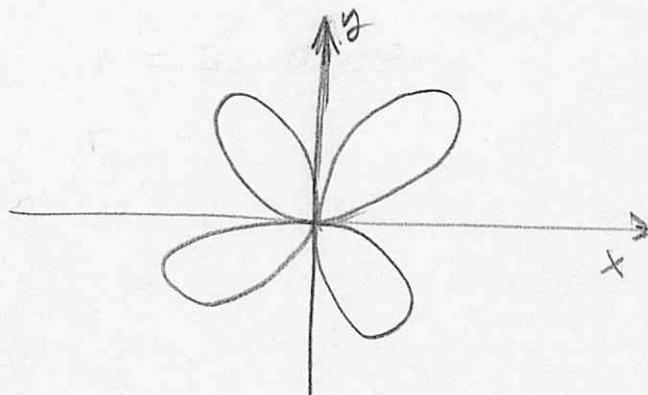
$$\approx \frac{1}{3} (0.521808 - 0.620951)$$

$$= \underline{-0.0330477}.$$

$$\begin{aligned} \iint_R \sqrt{x+2y} \sin(x-y) dA &= \int_0^1 \int_0^1 \sqrt{x+2y} \sin(x-y) dx dy \\ &= \underline{-0.0330474}. \end{aligned}$$

Remark: on a test, I'll find a way to either avoid numerics or to make it work out nicer...

**Problem 177** Find the center of mass for a laminate of variable density  $\delta(r, \theta) = r \sin^2(\theta)$  which is bounded by  $r = \sin(2\theta)$



$$\begin{aligned} 0 &\leq r \leq \sin(2\theta) \\ 0 &\leq \theta \leq 2\pi \end{aligned} \quad \left. \right\} \text{false}$$

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\sin(2\theta)} r \sin^2 \theta \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \sin^2 \theta \frac{1}{3} (\sin(2\theta))^3 \, d\theta \\ &= 0. \end{aligned}$$

See below

Well, this one is subtle. Should use

$$\begin{array}{ll} 0 \leq r \leq \sin(2\theta) & 0 \leq \theta \leq \pi/2 \\ 0 \leq r \leq -\sin(2\theta) & \pi/2 \leq \theta \leq \pi \\ 0 \leq r \leq \sin(2\theta) & \pi \leq \theta \leq 3\pi/2 \\ 0 \leq r \leq -\sin(2\theta) & 3\pi/2 \leq \theta \leq 2\pi \end{array}$$

So, alternatively, by symmetry,

$$\begin{aligned} M &= 4 \int_0^{\pi/2} \int_0^{\sin(2\theta)} r \sin^2 \theta \, r \, dr \, d\theta \\ &= \frac{4}{3} \int_0^{\pi/2} \sin^2 \theta \sin^3(2\theta) \, d\theta \\ &= \boxed{\frac{4}{9}} \quad \text{integrals to } \frac{1}{3}. \end{aligned}$$

By symmetry,  $(0, 0)$  is the center of mass.