These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed. If I have derived a normal vector field in the notes or lecture then you may use it, but please mention the source of the fact.

Problem 205 Show that for a simply connected region R with consistently oriented boundary ∂R if f, g are differentiable on some open set containing R then

$$\iint_R (f\nabla^2 g + \nabla f \bullet \nabla g)\,dA = \int_{\partial R} f\nabla g \bullet \widehat{n}\,ds.$$

Problem 206 Show that for a simply connected region R with consistently oriented boundary ∂R if f, g are differentiable on some open set containing R then

$$\iint_R (f\nabla^2 g - g\nabla^2 f)\,dA = \int_{\partial R} \left[f\nabla g \bullet \widehat{n} - g\nabla f \bullet \widehat{n}\right] ds.$$

Problem 207 Suppose $\nabla^2 f = 0$ on a simply connected region R. If $f|_{\partial R} = 0$ then what can you say about f throughout R?
(here $|_{\partial R}$ denotes restriction of f to the subset ∂R. In particular this means you are given that $f(x,y) = 0$ for all $(x,y) \in \partial R$.)

Problem 208 Suppose $b : \mathbb{R}^2 \to \mathbb{R}$ is a particular function and $\nabla^2 f = b$ on a simply connected region R. If g is another such solution ($\nabla^2 g = b$) on R then show that $f = g$ on R.
The equation $\nabla^2 f = b$ is called Poisson's Equation. When $b = 0$ then it's called Laplace's Equation. You are showing the solution to Poisson's Equation is unique on a simply connected region. **Hint:** *use the last problem's result on $f - g$... hmmm... I guess this is a retroactive hint for Problem 208 if you think about it.*

Problem 209 Find the surface area of $z = xy$ for $x^2 + y^2 \leq 1$.

Problem 210 Find the surface area of the plane $y + 2z = 2$ bounded by the cylinder $x^2 + y^2 = 1$.

Problem 211 Find the surface area of the cone frustrum $z = \frac{1}{3}\sqrt{x^2 + y^2}$ with $1 \leq z \leq 4/3$

Problem 212 Find the surface area of torus with radii $A, R > 0$ and $R \geq A$ parametrized by

$$\vec{X}(\alpha, \beta) = \Big\langle \big[R + A\cos(\alpha)\big]\cos(\beta), \big[R + A\cos(\alpha)\big]\sin(\beta), A\sin(\alpha) \Big\rangle$$

for $0 \leq \alpha \leq 2\pi$ and $0 \leq \beta \leq 2\pi$.

Problem 213 Find an explicit double integral which gives the surface area of the graph $x = g(y, z)$ for $(y, z) \in D$.

Problem 214 Consider a napkin ring which is formed by taking a sphere of radius R and drilling out a circular cylinder of radius B through the center of the sphere. Find the surface area of the napkin ring (include the inner as well as outer surfaces).

Problem 215 Integrate $H(x, y, z) = xyz$ over the surface of the solid $[0, a] \times [0, b] \times [0, c]$ where $a, b, c > 0$.

Problem 216 Integrate $G(x, y, z) = x^2$ over the surface of the unit-sphere.

Problem 217 Integrate $H(x, y, z) = z - x$ on the graph $z = x + y^2$ over the triangular region with vertices $(0, 0, 0), (1, 0, 0)$ and $(0, 1, 0)$.

Problem 218 Consider a thin-shell of constant density δ. Let the shell be cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$. Find **(a.)** the center of mass and **(b.)** the moment of intertia with respect to the z-axis.

Problem 219 Find the flux of $\vec{F}(x, y, z) = \langle z^2, x, -3z \rangle$ through the parabolic cylinder $z = 4 - y^2$ bounded by the planes $x = 0$, $x = 1$ and $z = 0$. Assume the orientation of the surface is outward, away from the x-axis.

Problem 220 Find the flux of $\vec{F}(x, y, z) = z\widehat{\mathbf{z}}$ through the portion the sphere of radius R in the first octant. Give the sphere an orientation which points away from the origin. In other words, assume the sphere is outwardly oriented.

Problem 221 Find the flux of $\vec{F}(x, y, z) = \langle -x, -y, z^2 \rangle$ through the conical frustrum $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$ with outward orientation.

Problem 222 Let S be the outward oriented paraboloid $z = 6 - x^2 - y^2$ for $x \geq 0$. Calculate the flux of $\vec{F} = (x^2 + y^2)\widehat{\mathbf{z}}$ for $z \geq 0$

Problem 223 Find the flux of $\vec{F}(x, y, z) = \langle 2xy, 2yz, 2xz \rangle$ upward through the subset of the plane $x + y + z = 2c$ where $(x, y) \in [0, c] \times [0, c]$.

Problem 224 Suppose $\vec{C}$ is a constant vector. Let $\vec{F}(x, y, z) = \vec{C}$ find the flux of $\vec{F}$ through a surface S on plane with nonzero vectors $\vec{A}, \vec{B}$. In particular, the surface S is parametrized by $\vec{r}(u, v) = \vec{r}_o + u\vec{A} + v\vec{B}$ for $(u, v) \in \Omega$.

Problem 225 Let $\vec{F}(x, y, z) = \langle a, b, c \rangle$ for some constants a, b, c. Calculate the flux of $\vec{F}$ through the upper-half of the outward oriented sphere $\rho = R$.

Problem 226 Once more consider the constant vector field $\vec{F}(x, y, z) = \langle a, b, c \rangle$. Calculate the flux of $\vec{F}$ through the downward oriented disk $z = 0$ for $\phi = \pi/2$.

Problem 227 Let $\vec{F} = \langle x^2, y^2, z^2 \rangle$. Calculate the flux of $\vec{F}$ through $z = 4 - x^2 - y^2$ for $z \geq 0$.

Problem 228 Let $\vec{F} = \langle x^2, y^2, z^2 \rangle$. Calculate the flux of $\vec{F}$ through the downward oriented disk $x^2 + y^2 \leq 4$ with $\phi = \pi/2$.

Problem 229 Let $\phi = \pi/4$ define a closed surface S with $0 \leq \rho \leq 2$. Find the flux of

$$\vec{F}(\rho, \phi, \theta) = \phi^2\widehat{\rho} + \rho\widehat{\phi} + \widehat{\theta}$$

through the outward oriented S.

Problem 230 Consider the closed cylinder $x^2 + y^2 = R^2$ for $0 \leq z \leq L$. Find the flux of

$$\vec{F}(r, \theta, z) = \theta \widehat{\mathbf{z}} + z\widehat{\theta} + r^2\widehat{\mathbf{r}}$$

out of the cylinder.

Problem 231 Let S be the pseudo-tetrahedra with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Three of the faces of S_1 are subsets of the coordinate planes call these S_{xy}, S_{zx}, S_{yz} with the obvious meanings and call S_T the top face. Let $\vec{F} = \langle y, -x, y \rangle$ and define $\vec{G} = \nabla \times \vec{F}$.

(a.) Calculate the circulation of $\vec{F}$ around each face .

(b.) Calculate the flux of $\vec{G}$ through each face.

(c.) do you see any pattern?

there will be more problems... coming soon

Problem 232 Let $\vec{F}(x, y, z) = \langle x^2, 2x, z^2 \rangle$. Calculate $\int_E \vec{F} \cdot d\vec{r}$ where E is the CCW oriented ellipse $4x^2 + y^2 = 4$ with $z = 0$. (use Stoke's Theorem)

Problem 233 Let $\vec{F}(x, y, z) = \langle y^2 + z^2, x^2 + z^2, x^2 + y^2 \rangle$. Find the work done by $\vec{F}$ around the CCW (as viewed from above) triangle formed from the intersection of the plane $x + y + z = 1$ and the coordinate planes. (use Stoke's Theorem)

Problem 234 Let $\vec{F}(x, y, z) = \langle y^2 + z^2, x^2 + y^2, x^2 + y^2 \rangle$. Find the work done by $\vec{F}$ around the CCW-oriented square bounded by $x = \pm 1$ and $y = \pm 1$ in the $z = 0$ plane (use Stoke's Theorem).

Problem 235 Consider the elliptical shell $4x^2 + 9y^2 + 36z^2 = 36$ with $z \geq 0$ and let

$$\vec{F}(x, y, z) = \left\langle y, x^2, (x^2 + y^4)^{\frac{3}{2}} \sin(exp(\sqrt{xyz})) \right\rangle.$$

Find the flux of $\nabla \times \vec{F}$ through the outwards oriented shell.

Problem 236 Let $\vec{F} = \langle 2x, 2y, 2z \rangle$ and suppose S is any simply connected surface with boundary ∂S. Show by Stoke's theorem that $\int_{\partial S} \vec{F} \cdot d\vec{r} = 0$

Problem 237

Problem 238

Problem 239

Problem 240

Problem 205 Show for simply connected R and differentiable f, g

$$\iint_R \left(f\,\nabla^2 g + \nabla f \cdot \nabla g\right) dA = \int_{\partial R} f\,(\nabla g)\cdot \hat{n}\, ds$$

Let $\vec{F} = f\nabla g$ be a vector field apply Green's Th$^{\underline{m}}$ to $\vec{F}$ on R, use divergence form $\oint_{\partial R}(\vec{F}\cdot\hat{n})\,ds = \iint_R (\nabla\cdot\vec{F})\,dA$

$$\begin{aligned}\int_{\partial R}(f\nabla g)\cdot\hat{n}\,ds &= \iint_R \nabla\cdot(f\nabla g)\,dA \\ &= \iint_R \left[\nabla f\cdot\nabla g + f\,\nabla\cdot\nabla g\right] dA \\ &= \underline{\iint_R \left(f\,\nabla^2 g + \nabla f\cdot\nabla g\right) dA} \,.\,//\end{aligned}$$

Problem 206 Show that for simply connected R with consistently oriented boundary ∂R and diff. f, g on a set containing R,

$$\iint_R \left(f\,\nabla^2 g - g\,\nabla^2 f\right) dA = \int_{\partial R}\left(f\,\nabla g\cdot\hat{n} - g\,\nabla f\cdot\hat{n}\right) ds$$

Apply Problem 205 twice,

$$\int_{\partial R} f\,\nabla g\cdot\hat{n}\,ds = \iint_R \left(f\,\nabla^2 g + \nabla f\cdot\nabla g\right) dA$$

$$\int_{\partial R} g\,\nabla f\cdot\hat{n}\,ds = \iint_R \left(g\,\nabla^2 f + \nabla g\cdot\nabla f\right) dA$$

Take the difference of these identities, use linearity of the double integral and commutivity of $\nabla f\cdot\nabla g = \nabla g\cdot\nabla f$ to derive,

$$\underline{\int_{\partial R}\left(f\,\nabla g\cdot\hat{n} - g\,\nabla f\cdot\hat{n}\right) ds = \iint_R \left(f\,\nabla^2 g - g\,\nabla^2 f\right) dA} \,.\,//$$

PROBLEM 207 | Suppose $\nabla^2 f = 0$ on R. Suppose $f|_{\partial R} = 0$ what can we say about f through R ← simply connected

Apply divergence form of Green's Thm,

$$\int_{\partial R} (\nabla f \cdot \hat{n})\, ds = \iint_R (\nabla \cdot \nabla f)\, dA = \iint_R (\nabla^2 f)\, dA = \iint_R 0\, dA = 0$$

The flux of ∇f through ∂R is zero. Now suppose $f|_{\partial R}(x,y) = 0 \quad \forall (x,y) \in \partial R$. Apply $\iint_R (f \nabla^2 f + \nabla f \cdot \nabla f)\, dA = \int_{\partial R} f (\nabla f) \cdot \hat{n}\, ds$ (from Problem 205). note that $\nabla^2 f = 0$ hence we obtain $\iint_R (\nabla f) \cdot (\nabla f)\, dA = \int_{\partial R} (f \nabla f) \cdot \hat{n}\, ds$. Moreover, since $f|_{\partial R} = 0$ it follows immediately that $\int_{\partial R} (f \nabla f) \cdot \hat{n}\, ds = 0$. We find $\iint_R (\nabla f) \cdot (\nabla f)\, dA = 0$ consequently $\iint_R \|\nabla f\|^2\, dA = 0$ and thus $\nabla f = 0$ on R. Therefore, f is constant on R and this means $f|_R = f|_{\partial R} = 0$. To conclude, we find f is zero throughout R.

PROBLEM 208 | Suppose $b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function such that $\nabla^2 f = b$ on simply connected R. Suppose g is another function s.t. $\nabla^2 g = b$ show that $f = g$ on R

Let $h = f - g$ note that $h|_{\partial R} = 0$ thus as $\nabla^2 h = \nabla^2 f - \nabla^2 g = b - b = 0$ we may apply PROBLEM 207 to find $h = 0$ on R

$\therefore\ f = g$ on R.

PROBLEM 209 Find surface area $z = xy$ for $x^2+y^2 \le 1$

You can show $dS = \sqrt{1+f_x^2+f_y^2}\,dx\,dy$ for $z = f(x,y)$

applying that here $dS = \sqrt{1+y^2+x^2}\,dx\,dy$

$$\text{AREA} = \iint_{x^2+y^2\le 1} \sqrt{1+x^2+y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\,dr\,d\theta$$

$$\boxed{\begin{array}{l} u = 1+r^2 \\ u(0) = 1 \\ u(1) = 1+1 = 2 \end{array}}$$

$$= 2\pi \int_1^2 \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{2\pi}{3}(u)^{3/2}\Big|_1^2$$

$$= \boxed{\frac{2\pi}{3}\left[2^{3/2}-1\right].}$$

PROBLEM 210 Find surface area of $y+2z=2$ bounded by $x^2+y^2=1$.

We have $z = 1-\frac{1}{2}y$ for $x^2+y^2\le 1$

$$\text{AREA} = \iint_{x^2+y^2\le 1}\sqrt{1+\left(\tfrac{1}{2}\right)^2}\,dx\,dy = \sqrt{\tfrac{5}{4}}\iint_{x^2+y^2\le 1} dA = \boxed{\frac{\pi\sqrt{5}}{2}.}$$

PROBLEM 211 Find area of surface $z = \frac{1}{3}\sqrt{x^2+y^2}$ for $1 \le z \le 4/3$

Note: $1 \le \frac{1}{3}\sqrt{x^2+y^2} \le 4/3 \Rightarrow \underline{3 \le r \le 4} : \Omega$

$$\text{AREA} = \iint_\Omega \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_\Omega \sqrt{1+\left(\frac{x}{3r}\right)^2+\left(\frac{y}{3r}\right)^2}\,dA$$

$$= \iint_\Omega \sqrt{1+\frac{1}{9}\left(\frac{x^2+y^2}{r^2}\right)}\,dA$$

$$= \iint_\Omega \sqrt{\frac{10}{9}}\,dA = \boxed{\frac{\sqrt{10}}{3}\pi(16-9).}$$

$$= \boxed{\frac{7\pi\sqrt{10}}{3}}$$

PROBLEM 212) Find surface area of torus $A, R > 0$ and $R \geq A$

$$\vec{X}(\alpha,\beta) = \left\langle [R + A\cos\alpha]\cos\beta,\ [R + A\cos\alpha]\sin\beta,\ A\sin\alpha \right\rangle$$

for $(\alpha,\beta) \in [0, 2\pi]^2$

This time dS is not known from more general result. We must <u>derive</u> it; $dS = \| \partial_\alpha \vec{X} \times \partial_\beta \vec{X} \| \, d\alpha \, d\beta$.

$$\partial_\alpha \vec{X} = \langle -A\sin\alpha\cos\beta,\ -A\sin\alpha\sin\beta,\ A\cos\alpha \rangle$$

$$\partial_\beta \vec{X} = \langle -[R + A\cos\alpha]\sin\beta,\ [R + A\cos\alpha]\cos\beta,\ 0 \rangle$$

$$\vec{N}(\alpha,\beta) = \partial_\alpha \vec{X} \times \partial_\beta \vec{X} = \Big\langle -A\cos\alpha\cos\beta[R + A\cos\alpha],\ A\cos\alpha\sin\beta[R + A\cos\alpha],$$

$$-A\sin\alpha[R + A\cos\alpha](\cos^2\beta + \sin^2\beta) \Big\rangle$$

$$\therefore\ \vec{N}(\alpha,\beta) = A[R + A\cos\alpha]\underbrace{\langle -\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ -\sin\alpha \rangle}_{\text{unit-vector.}}$$

$$N = |\vec{N}| = |A(R + A\cos\alpha)|$$

Thus $dS = |AR + A^2\cos\alpha| \, d\alpha \, d\beta$

$$\text{AREA}_S = \int_0^{2\pi}\int_0^{2\pi} |AR + A^2\cos\alpha| \, d\alpha \, d\beta$$

$$= 2\pi A \int_0^{2\pi} |R + A\cos\alpha| \, d\alpha$$

note $R \geq A \Rightarrow |R + A\cos\alpha| = R + A\cos\alpha$.

$$= 2\pi A \int_0^{2\pi} (R + \underbrace{A\cos\alpha}_{0}) \, d\alpha$$

$$= (2\pi A)(2\pi R)$$

$$= \boxed{4\pi^2 AR.}$$

PROBLEM 213 Find double integral to give surface area of graph $x = g(y,z)$ for $(y,z) \in D$

Let $\vec{r}(y,z) = \langle g(y,z), y, z \rangle$ for $(y,z) \in D$

$$\frac{\partial \vec{r}}{\partial y} = \langle \partial_y g, 1, 0 \rangle$$

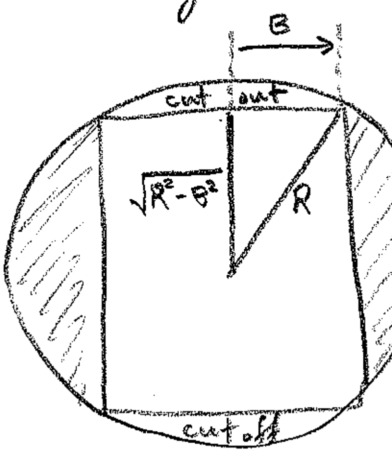
$$\frac{\partial \vec{r}}{\partial z} = \langle \partial_z g, 0, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial y} \times \frac{\partial \vec{r}}{\partial z} = \langle 1, -\partial_y g, -\partial_z g \rangle$$

$$dS = \left\| \frac{\partial \vec{r}}{\partial y} \times \frac{\partial \vec{r}}{\partial z} \right\| dy\,dz = \sqrt{1 + (\partial_y g)^2 + (\partial_z g)^2}\; dy\,dz$$

$$\boxed{\text{AREA} = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}\; dy\,dz}$$

PROBLEM 214 Partly left to reader ☺,



$$-\sqrt{R^2 - B^2} \leq z \leq \sqrt{R^2 - B^2}$$

$$-\sqrt{R^2 - B^2} \leq R\cos\phi \leq \sqrt{R^2 - B^2}$$

$$-\frac{\sqrt{R^2 - B^2}}{R} \leq \cos\phi \leq \frac{\sqrt{R^2 - B^2}}{R}$$

$$\Rightarrow \phi_0 = \cos^{-1}\left(\sqrt{\frac{R^2 - B^2}{R^2}}\right) \leq \phi \leq \pi - \phi_0$$

$$\text{Surface Area} = \int_0^{2\pi} \int_{\phi_0}^{\pi - \phi_0} R^2 \sin\phi \, d\phi \, d\theta + (2\pi B)\left(2\sqrt{R^2 - B^2}\right)$$

PROBLEM 215 Integrate $H(x,y,z) = xyz$ over the surface of the solid $[0,a]\times[0,b]\times[0,c]$; $a,b,c>0$

There are 6 faces to S but only $\underbrace{x=a}_{S_a}$, $\underbrace{y=b}_{S_b}$, $\underbrace{z=c}_{S_c}$ give nontrivial results.

$$\iint_S H\,dS = \iint_{S_a} xyz\,dS + \iiint_{S_b} xyz\,dS + \iiint_{S_c} xyz\,dS$$

$$= \int_0^b\int_0^c ayz\,dz\,dy + \int_0^a\int_0^c bxz\,dz\,dx + \int_0^a\int_0^b cxy\,dy\,dx$$

$$= a\frac{b^2}{2}\frac{c^2}{2} + b\frac{a^2}{2}\frac{c^2}{2} + c\frac{a^2}{2}\frac{b^2}{2}$$

$$= \boxed{\frac{1}{4}\left(ab^2c^2 + a^2bc^2 + a^2b^2c\right)}$$

PROBLEM 216 $\iint_S x^2\,dS$ where $S: x^2+y^2+z^2=1$

Recall $dS = \sin\phi\,d\phi\,d\theta$ since $R=1$ here.

Now $x = \cos\theta\sin\phi$ hence $x^2 = \cos^2\theta\sin^2\phi$. Calculate

$$\iint_S x^2\,dS = \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}(1+\underbrace{\cos(2\theta)}_{0})\,d\theta\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$= \left(\frac{2\pi}{2}\right)\int_1^{-1}(u^2-1)\,du$$

$u = \cos\phi$

$du = -\sin\phi\,d\phi$

$u(0) = \cos(0) = 1$

$u(\pi) = \cos(\pi) = -1$

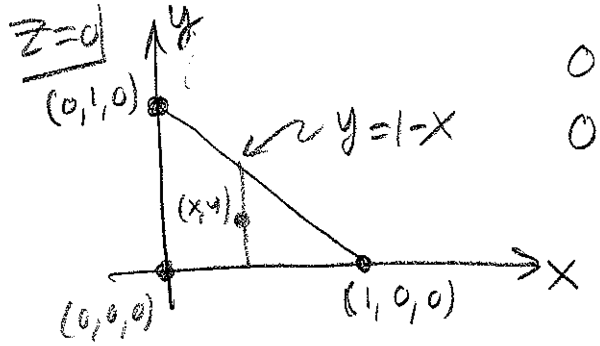
$$= \pi\left(\frac{1}{3}u^3 - u\right)\Big|_1^{-1}$$

$$= \pi\left(\frac{1}{3}(-1) + 1 - \frac{1}{3} + 1\right)$$

$$= \pi\left(2 - \frac{2}{3}\right)$$

$$= \boxed{\frac{4\pi}{3}}.$$

PROBLEM 217 $H(x,y,z) = z - x$. Calculate $\iint_S H\,dS$ on the graph $z = x + y^2$ over the triangular region with vertices $(0,0,0), (1,0,0), (0,1,0)$



$0 \le x \le 1$

$0 \le y \le 1-x$

$\underbrace{\vec{r}(x,y) = \langle x, y, x+y^2 \rangle}_{\text{graph parametrization}}$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$
$$= \sqrt{1 + 1 + (2y)^2}\,dx\,dy$$
$$= \sqrt{2 + 4y^2}\,dx\,dy$$

$$\iint_S H\,dS = \int_0^1 \int_0^{1-x} (x + y^2 - x)\sqrt{2+4y^2}\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} y^2\sqrt{2+4y^2}\,dy\,dx$$

ouch. Let's change order of integration.

$$= \int_0^1 \int_0^{1-y} y^2\sqrt{2+4y^2}\,dx\,dy$$
$$= \int_0^1 y^2\sqrt{2+4y^2}\,(1-y)\,dy$$
$$= \sqrt{2}\int_0^1 \left(y^2\sqrt{1+2y^2} - y^3\sqrt{1+2y^2}\right)dy =$$

Let us appreciate $\cosh^2\theta - \sinh^2\theta = 1 \Rightarrow 1 + \sinh^2\theta = \cosh^2\theta$

Let $2y^2 = \sinh^2\theta \leftrightarrow \sinh\theta = \sqrt{2}\,y \therefore \cosh\theta\,d\theta = \sqrt{2}\,dy$

Moreover, $y = 1 \Rightarrow \sinh\theta = \sqrt{2} \Rightarrow \theta = \sinh^{-1}(\sqrt{2})$ and $y = 0 \to \theta = 0$.

Note $dy = \frac{1}{\sqrt{2}}\cosh\theta\,d\theta$ and $\sqrt{1+2y^2} = \cosh\theta$ and $y = \frac{\sinh\theta}{\sqrt{2}}$

Hence $$\iint_S H\,dS = \int_0^{\sinh^{-1}\sqrt{2}} \left(\frac{\sinh^2\theta}{2}\cosh\theta\frac{\cosh\theta}{\sqrt{2}} - \frac{\sinh^3\theta}{2\sqrt{2}}\frac{\cosh^2\theta}{\sqrt{2}}\right)d\theta$$

$\vdots$

$$= \boxed{\frac{1}{240}\left(-8\sqrt{2} + 27\sqrt{6} - 15\sinh^{-1}(\sqrt{2})\right) \approx 0.1568}$$

PROBLEM 218 Cone $x^2+y^2=z^2$ has density δ and is bounded by $z=1$ and $z=2$
(a.) find center of mass
(b.) $I_z = \iint_{cone} (x^2+y^2)\delta\, dS$

(a.) By symmetry $\bar{x} = \bar{y} = 0$. We need to calculate $\bar{z} = \frac{1}{M}\iint_{cone} z\delta\, dS$. We need a nice parametrization of the cone, $\vec{r}(\theta, z) = \langle z\cos\theta, z\sin\theta, z\rangle$ for $0 \le \theta \le 2\pi$ and $1 \le z \le 2$ will do nicely.

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = \langle -z\sin\theta, z\cos\theta, 0\rangle \times \langle \cos\theta, \sin\theta, 1\rangle$$

$$= \langle z\cos\theta, z\sin\theta, -z\rangle$$

Thus, $dS = \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} \right\| d\theta\, dz = \sqrt{2}\, z\, d\theta\, dz$ (since $z > 0$)

$$M = \int_1^2\int_0^{2\pi} \sqrt{2}\,\delta z\, d\theta\, dz = 2\pi\sqrt{2}\,\delta \left.\frac{z^2}{2}\right|_1^2 = \underline{3\pi\delta\sqrt{2}}.$$

$$\bar{z} = \frac{1}{3\pi\cancel{\delta\sqrt{2}}}\int_1^2\int_0^{2\pi} \cancel{\delta\sqrt{2}}\, z^2\, d\theta\, dz$$

$$= \frac{1}{3\pi}(2\pi)\left(\left.\frac{z^3}{3}\right|_1^2\right)$$

$$= \frac{2}{3}\left(\frac{8}{3} - \frac{1}{3}\right)$$

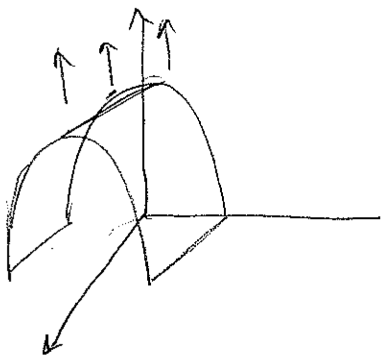
$$= \boxed{\frac{14}{9}} \hookrightarrow \boxed{(0, 0, 14/9) \text{ center of mass}}$$

(b.) $I_z = \iint_S (x^2+y^2)\, dm = \int_1^2\int_0^{2\pi} \delta z^2 \sqrt{2}\, z\, d\theta\, dz$

$$= \delta\sqrt{2}(2\pi)\left.\frac{z^4}{4}\right|_1^2 = \frac{\pi\delta\sqrt{2}}{2}(16-1) = \boxed{\frac{15\pi\delta\sqrt{2}}{2}}$$

PROBLEM 219 $\vec{F}(x,y,z) = \langle z^2, x, -3z \rangle$ find flux of $\vec{F}$ through parabolic cylinder $z = 4-y^2$ bounded by planes $x=0$, $x=1$, $z=0$ oriented away from x-axis.

$\vec{r}(x,y) = \langle x, y, \underbrace{4-y^2} \rangle$ have $\underline{0 \le x \le 1}$



$0 \le 4-y^2 \Rightarrow y^2 \le 4 \Rightarrow \underline{-2 \le y \le 2}$.

$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \langle 0, 2y, 1 \rangle$

$\vec{F}(x, y, 4-y^2) = \langle (4-y^2)^2, x, -3(4-y^2) \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_{-2}^{2} \langle (4-y^2)^2, x, -3(4-y^2) \rangle \cdot \langle 0, 2y, 1 \rangle \, dy\, dx$$

$$= \int_0^1 \int_{-2}^{2} [\underbrace{2xy}_{0} - 3(4-y^2)] \, dy\, dx$$

$$= \int_0^1 [-12y + y^3 \Big|_{-2}^{2} \, dx$$

$$= \int_0^1 [-12(4) + (8+8)] \, dx$$

$$= -48 + 16$$

$$= \boxed{-32}$$

PROBLEM 220 Let $\vec{F}(x,y,z) = z\hat{z}$ find flux through S_R with $x,y,z \ge 0$

$$\Phi = \int_0^{\pi/2} \int_0^{\pi/2} (R^2 \sin\phi)(R\cos\phi)(\cos\phi) \, d\phi\, d\theta$$

$$= R^3 \, \theta \Big|_0^{\pi/2} \left(\frac{-1}{3} \cos^3\phi \Big|_0^{\pi/2} \right)$$

$$= R^3 \left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)$$

$$= \boxed{\frac{1}{6}\pi R^3} \leftarrow \frac{1}{8} \text{ of total flux through } S_R.$$

$x = R\cos\theta\sin\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\phi$

$d\vec{S} = R^2 \sin\phi \, d\phi\, d\theta \, \hat{\rho}$

$\hat{\rho} = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$

$\underline{\hat{\rho} \cdot \hat{z} = \cos\phi}$

<u>Note</u>: Gauss' Th^{m}; $\nabla \cdot \vec{F} = 1$ $\quad \iint_{S_R} \vec{F} \cdot d\vec{S} = \iiint \nabla \cdot \vec{F} \, dV = \frac{4}{3}\pi R^3 = 8\left(\frac{1}{6}\pi R^3\right)$

PROBLEM 235 Consider the elliptical shell $4x^2+9y^2+36z^2=36$ ($\mathcal{E}$) with $z \geq 0$ and $\vec{F}(x,y,z) = \left\langle y, x^2, (x^2+y^4)^{3/2}\sin\left(\exp\sqrt{xyz}\right)\right\rangle$

Find $\iint_{\mathcal{E}} (\nabla\times\vec{F})\cdot d\vec{S}$

Note $\partial\mathcal{E} \Rightarrow 4x^2+9y^2=36 \hookrightarrow \underline{x = 3\cos\theta}$, $\underline{y = 4\sin\theta}$, $z=0$

$dx = -3\sin\theta\, d\theta$, $dy = 4\cos\theta\, d\theta$, $dz = 0$

Apply Stoke's Th$^{\underline{m}}$

$$\iint_{\mathcal{E}} (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial\mathcal{E}} \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} (4\sin\theta)(-3\sin\theta\, d\theta) + (9\cos^2\theta)(4\cos\theta\, d\theta) + \underline{\underline{0}}$$

$$= \int_0^{2\pi} \left[-12\sin^2\theta + 36\cos^3\theta\right] d\theta$$

$$= \int_0^{2\pi} \left[-6(1-\underbrace{\cos(2\theta)}_{0}) + \underbrace{36(1-\sin^2\theta)\cos\theta}_{0}\right] d\theta$$

$$= \boxed{-12\pi.}$$

PROBLEM 236 Let $\vec{F}(x,y,z) = \langle 2x, 2y, 2z\rangle$ and let S be simply connected with boundary ∂S. Show $\oint_{\partial S} \vec{F}\cdot d\vec{r} = 0$

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = \boxed{0.}$$

PROBLEM 237 Let S be the cylinder  $z=1$, $0 \leq x^2+y^2 \leq 1$ with open base $x^2+y^2=1$, $0 \leq z \leq 1$

Suppose $\nabla\times\vec{F} = \left\langle (\sinh z)(x^2+y^2), z e^{xy+\cos(x+y)}, (xz+y)\tan^{-1}(z)\right\rangle$.

Find $\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$

shrink the cylinder to the xy-plane and notice that $(\nabla\times\vec{F})\big|_{z=0} = \langle 0,0,0\rangle$ thus by Stoke's Th$^{\underline{m}}$

Cylinder with cap & disk share same boundary $\partial S = \partial S_2$

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r} = \int_{\partial S_2} \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S} = \boxed{0.}$$

PROBLEM 238 Suppose $\vec{F}(x,y,z) = \langle y-x,\ z-y,\ y-x \rangle$ and let $E = [-1,1]^3$ find flux of $\vec{F}$ through ∂E

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F})\, dV \qquad (\text{divergence Th}^{m})$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\, dV$$

$$= -2 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dx\,dy\,dz$$

$$= \boxed{-16}$$

PROBLEM 239 Let $\vec{F}(x,y,z) = \langle y, xy, -z \rangle$ find flux of $\vec{F}$ across ∂E for $E:\ x^2+y^2 \le 4,\ 0 \le z \le x^2+y^2$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (x-1)\, dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} (r\cos\theta - 1)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (r^4 \underbrace{\cos\theta}_{0} - r^3)\, dr\, d\theta$$

$$= 2\pi \left(-\frac{1}{4} r^4 \Big|_0^2 \right)$$

$$= 2\pi \left(\frac{-16}{4} \right)$$

$$= \boxed{-8\pi.}$$

PROBLEM 240 $E:\ R_1 \le \rho \le R_2$ and let $\vec{F} = \nabla \times \vec{A}$ for smooth $\vec{A}$. Show $\Phi_{R_1} = \Phi_{R_2}$ by applying div. Th^{m}

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot (\nabla \times \vec{A})\, dV = 0 \qquad \left(\begin{array}{c}\text{div. Th}^{m} \text{ for solid} \\ \text{with hole}\end{array}\right)$$

$$\parallel$$

Consistent orientation for ∂E

$$\iint_{-S_{R_1}} \vec{F} \cdot d\vec{S} + \iint_{S_{R_2}} \vec{F} \cdot d\vec{S} \implies \boxed{\iint_{S_{R_1}} \vec{F} \cdot d\vec{S} = \iint_{S_{R_2}} \vec{F} \cdot d\vec{S}}$$

PROBLEM 221 Find flux of $\vec{F}(x,y,z) = \langle -x, -y, z^2 \rangle$ through conical frustrum $z = \sqrt{x^2+y^2}$ for $1 \le z \le 2$, outward oriented

Parametrize S: $\vec{X}(\theta, z) = \langle z\cos\theta, z\sin\theta, z \rangle$

We found in Problem 218 that $\vec{N}(\theta, z) = \langle z\cos\theta, z\sin\theta, -z \rangle$ this points outward as desired. Calculate,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\text{dom}(\vec{X})} \vec{F}(z\cos\theta, z\sin\theta, z) \cdot \vec{N}(\theta, z)\, d\theta\, dz$$

$$= \int_1^2 \int_0^{2\pi} \langle -z\cos\theta, -z\sin\theta, z^2 \rangle \cdot \langle z\cos\theta, z\sin\theta, -z \rangle\, d\theta\, dz$$

$$= \int_1^2 \int_0^{2\pi} \left[-z^2 - z^3 \right] d\theta\, dz$$

$$= 2\pi \left[-\frac{1}{3} z^3 - \frac{1}{4} z^4 \right] \Big|_1^2$$

$$= 2\pi \left[\left(\frac{1}{3} + \frac{1}{4} \right) - \left(\frac{8}{3} + \frac{16}{4} \right) \right]$$

$$= 2\pi \left[\frac{7}{12} - \frac{32}{12} - \frac{48}{12} \right]$$

$$= \boxed{-\frac{73\pi}{6}}$$

PROBLEM 222 Let S: $z = 6 - x^2 - y^2$ for $x, z \ge 0$ and $\vec{F}(x,y,z) = (x^2+y^2)\hat{z}$. Find Φ_S

$$\Phi_S = \iint_{0 \le 6 - x^2 - y^2} (x^2+y^2)\hat{z} \cdot \langle 2x, 2y, 1 \rangle\, dx\, dy$$

$$= \iint_{r^2 \le 6} r^2\, dA = \int_0^{2\pi} \int_0^{\sqrt{6}} r^3\, dr\, d\theta = 2\pi \frac{r^4}{4} \Big|_0^{\sqrt{6}}$$

$$= \frac{\pi}{2}(36)$$

$$= \boxed{18\pi}$$

PROBLEM 223 | Let $\vec{F}(x,y,z) = \langle 2xy, 2yz, 2xz \rangle$
Find flux of $\vec{F}$ through plane $x+y+z = 2c$ where $(x,y) \in [0,c] \times [0,c]$.

Can use $\vec{r}(x,y) = \langle x, y, 2c - x - y \rangle$

Hence $\vec{N}(x,y) = \langle 1,1,1 \rangle$ (not surprising!)

$$\Phi_{\vec{F}} = \int_0^c \int_0^c \langle 2xy, 2y(2c-x-y), 2x(2c-x-y) \rangle \cdot \langle 1,1,1 \rangle \, dx\,dy$$

$$= \int_0^c \int_0^c \Big(2xy + 2y(2c-x-y) + 2x(2c-x-y) \Big) dx\,dy$$

$$= \int_0^c \int_0^c \left(\cancel{2xy} + 4cy - \cancel{2yx} - 2y^2 + 4xc - 2x^2 - 2xy \right) dx\,dy$$

$$= \int_0^c \left(4c^2 y - 2cy^2 + 2c^3 - \tfrac{2}{3}c^3 - c^2 y \right) dy$$

$$= 2c^2c^2 - \tfrac{2}{3}cc^3 + 2c^4 - \tfrac{2}{3}c^4 - \tfrac{1}{2}c^2c^2$$

$$= \left[2 - \tfrac{2}{3} + 2 - \tfrac{2}{3} - \tfrac{1}{2} \right] c^4 = \left[4 - \tfrac{4}{3} - \tfrac{1}{2} \right] c^4 = \left[\tfrac{24-8-3}{6} \right] c^4$$

$$= \boxed{\frac{13c^4}{6}}$$

PROBLEM 224 Suppose $\vec{F}(x,y,z) = \vec{C}$ and find the flux of this constant vector field through S: $\vec{r}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$ for $(u,v) \in \Omega$ where $\vec{A}, \vec{B} \neq 0$ and $\vec{A} \nparallel \vec{B}$

Calculate the normal vector field,

$$\vec{N}(u,v) = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \vec{A} \times \vec{B}$$

Thus,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_\Omega \vec{C} \cdot (\vec{A} \times \vec{B})\, du\,dv = \vec{C} \cdot (\vec{A} \times \vec{B}) \iint_\Omega du\,dv$$

$$= \boxed{\vec{C} \cdot (\vec{A} \times \vec{B})\ \text{AREA}(\Omega)}$$

PROBLEM 225 Let $\vec{F}(x,y,z) = \langle a, b, c \rangle$ for some constants a, b, c. Let $S_1 \subset S_R$ with $z \geq 0$ (or $0 \leq \phi \leq \pi/2$)

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \langle a, b, c \rangle \cdot R^2 \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \big[a R^2 \underbrace{\cos\theta}_{0} \sin^2\phi + b R^2 \underbrace{\sin\theta}_{0} \sin^2\phi + c R^2 \sin\phi \cos\phi \big]\, d\phi\, d\theta$$

$$= 2\pi c R^2 \left(\frac{1}{2} \sin^2\phi \right) \Big|_0^{\pi/2}$$

$$= \boxed{c \pi R^2.}$$

PROBLEM 226 $\vec{F}(x,y,z) = \langle a, b, c \rangle$, S_2 : $x^2 + y^2 \leq R^2$ and $z = 0$, $\hat{n} = -\hat{z}$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2 \leq R^2} \langle a, b, c \rangle \cdot (-\hat{z}\, dx\,dy)$$

$$= \iint_{x^2+y^2 \leq R^2} -c\, dx\,dy$$

$$= -c \iint_{x^2+y^2 \leq R^2} dA$$

$$= \boxed{-c \pi R^2}$$

Remark: $\nabla \cdot \vec{F} = 0$ hence

$$\iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{S} = \iiint_{\text{half-sphere}} (\nabla \cdot \vec{F})\, dV = 0$$

Problems 225 & 226 illustrate how this happens

PROBLEM 227 Let $\vec{F}(x,y,z) = \langle x^2, y^2, z^2 \rangle$ find flux through $z = 4 - x^2 - y^2$ for $z \geq 0$ (S_1)

Note, $z \geq 0 \Rightarrow 4 - x^2 - y^2 \geq 0 \Rightarrow 4 \geq x^2 + y^2$

Let $\vec{r}(x,y) = \langle x, y, 4 - x^2 - y^2 \rangle$ for $\underbrace{0 \leq x^2 + y^2 \leq 4}_{\Omega}$

then $\vec{N}(x,y) = \langle 2x, 2y, 1 \rangle$.

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= \iint_{\Omega} \langle x^2, y^2, (4-x^2-y^2)^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx\,dy \\
&= \iint_{\Omega} \left[2x^3 + 2y^3 + (4 - x^2 - y^2)^2 \right] dx\,dy \\
&= \int_0^{2\pi} \int_0^2 \left(2r^3 \cos^3\theta + 2r^3 \sin^3\theta + (4 - r^2)^2 \right) r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 \left[2r^4 \cos^3\theta + 2r^4 \sin^3\theta + r(r^2 - 4)^2 \right] dr\,d\theta \\
&= \int_0^{2\pi} \left[\tfrac{2}{5}(32)\cos^3\theta + \tfrac{2}{5}(32)\sin^3\theta + \tfrac{1}{6}(0 + 64) \right] d\theta \\
&= \int_0^{2\pi} \left[\tfrac{64}{5}(1 - \underbrace{\sin^2\theta)\cos\theta}_{0} + \tfrac{64}{5}(1 - \underbrace{\cos^2\theta)\sin\theta}_{0} + \tfrac{32}{3} \right] d\theta \\
&= \boxed{64\pi/3}
\end{aligned}$$

PROBLEM 228 Let $\vec{F}(x,y,z) = \langle x^2, y^2, z^2 \rangle$ find flux through downward oriented disk $x^2 + y^2 \leq 4$ (S_2), $(z=0)$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{\Omega} \langle x^2, y^2, 0 \rangle \cdot (-\hat{z}\,dx\,dy) = \boxed{0.}$$

Remark: $\nabla \cdot \vec{F} = 2x + 2y + 2z$ hence,

$$\begin{aligned}
\iiint_E (\nabla \cdot \vec{F})\,dV &= \iint_{\Omega} \int_0^{4-x^2-y^2} 2(x + y + z)\,dz\,dy\,dx \\
&= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 2(\underbrace{r^2\cos\theta}_{0} + \underbrace{r^2\sin\theta}_{0} + zr)\,dz\,dr\,d\theta \\
&= 2\pi \int_0^2 \tfrac{1}{2} r (4 - r^2)^2\,dr \\
&= \pi \left(-\tfrac{1}{3}(4 - r^2)^3 \Big|_0^2 \right) = \frac{64\pi}{3} \quad \text{(checks 227 \& 228)}
\end{aligned}$$

PROBLEM 229 Let S: $\phi = \pi/4$ for $0 \le \rho \le 2$. Find flux of $\vec{F}(\rho, \phi, \theta) = \phi^2 \hat{\rho} + \rho \hat{\phi} + \hat{\theta}$

Let $\vec{r}(\theta, \rho) = \frac{1}{\sqrt{2}} \langle \rho \cos\theta, \rho \sin\theta, \rho \rangle \Rightarrow \underbrace{\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \rho} = \frac{\rho}{\sqrt{2}} \hat{\phi}}_{\text{short calculation. (in my notes)}}$

Thus $\vec{N}(\theta, \rho) = \frac{1}{\sqrt{2}} \rho \hat{\phi}$ and we calculate,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 \left(\phi^2 \hat{\rho} + \rho \hat{\phi} + \hat{\theta}\right) \cdot \frac{1}{\sqrt{2}} \rho \hat{\phi} \, d\rho \, d\theta$$

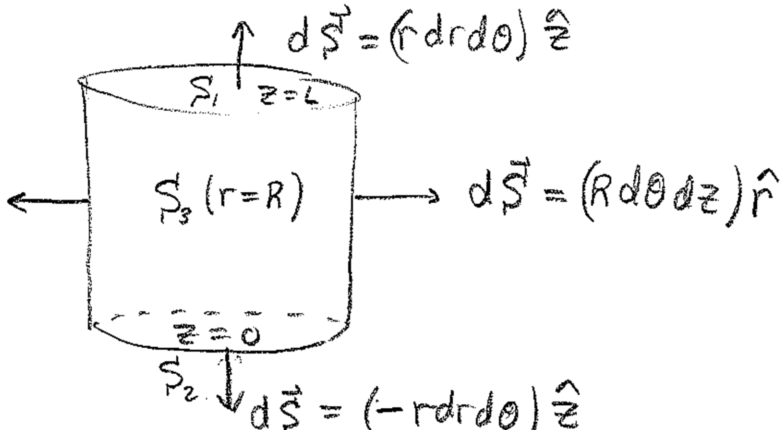
$$= \int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{2}} \rho^2 \, d\rho \, d\theta$$

$$= (2\pi)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{3} \rho^3 \Big|_0^2\right)$$

$$= \boxed{\frac{16\pi}{3\sqrt{2}}}$$

PROBLEM 230 Consider the surface $x^2 + y^2 = R^2$ for $0 \le z \le L$. (S) Find flux through S of $\vec{F}(r, \theta, z) = \theta \hat{z} + z \hat{\theta} + r^2 \hat{r}$

$S = S_1 \cup S_2 \cup S_3$



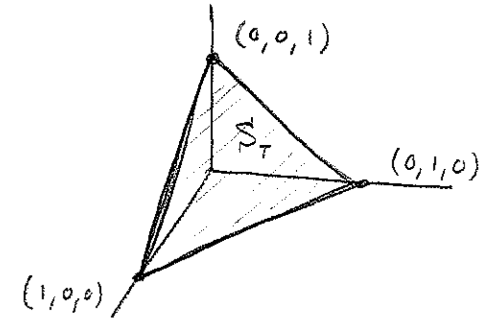
$$\iint_S \vec{F} \cdot d\vec{S} = \cancel{\int_0^{2\pi} \int_0^R \theta r \, dr \, d\theta} - \cancel{\int_0^{2\pi} \int_0^R \theta r \, dr \, d\theta} + \int_0^{2\pi} \int_0^L R^2 R \, d\theta \, dz$$

$$= \boxed{2\pi R^3 L}$$

Note: $\underbrace{\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r}\left[r r^2\right]}_{\text{divergence in cylindricals formula. see notes (around old 382)}} = \frac{3r^2}{r} = 3r \Rightarrow \iiint_{\text{cylind.}} (\nabla \cdot \vec{F}) \, dV = \iiint_{\text{cylinder}} 3r^2 \, dr \, d\theta \, dz$

$\overset{\checkmark}{=} R^3 (2\pi)(L)$

PROBLEM 231

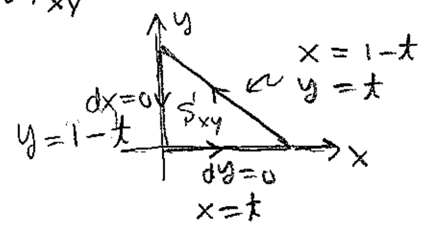


$S = S_T \cup S_{xy} \cup S_{yz} \cup S_{zx}$

Let $\vec{F}(x,y,z) = \langle y, -x, y \rangle$

$$\vec{G} = \nabla \times \vec{F} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y & -x & y \end{bmatrix} = \langle 1, 0, -2 \rangle$$

1.) $$\oint_{\partial S_{xy}} y\,dx - x\,dy + y\,dz = \int_0^1 0\,dt + \int_0^1 t(-dt) - (1-t)\,dt + \int_0^1 0\,dt$$



$$= -\int_0^1 dt$$

$$= \boxed{-1}$$

$$\iint_{S_{xy}} \vec{G} \cdot d\vec{S} = \int_0^1 \int_0^{1-x} \langle 1, 0, -2 \rangle \cdot \langle 0, 0, 1 \rangle \, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} -2\,dy\,dx$$

$$= \int_0^1 -2(1-x)\,dx$$

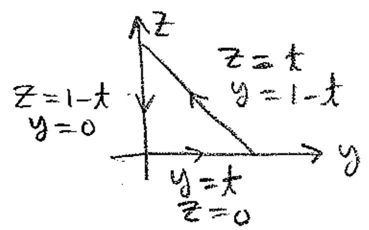
$$= -2\left(x - \tfrac{1}{2}x^2\right)\Big|_0^1$$

$$= -2\left(1 - \tfrac{1}{2}\right)$$

$$= -2\left(\tfrac{1}{2}\right)$$

$$= \boxed{-1.}$$

2.) $$\oint_{\partial S_{yz}} y\,\underbrace{dx}_{0} - \underbrace{x}_{0}\,dy + y\,dz = \int_{\partial S_{yz}} y\,dz = \underbrace{\int_0^1 (1-t)\,dt}_{\substack{\text{only} \\ z=t,\ y=1-t \\ \text{gives nonzero result.}}} = 1 - \tfrac{1}{2} = \boxed{\tfrac{1}{2}.}$$



$X = 0$, $dx = 0$

$$\iint_{S_{yz}} \vec{G} \cdot d\vec{S} = \int_0^1 \int_0^{1-z} \langle 1, 0, -2 \rangle \cdot \hat{x} \, dy\,dz$$

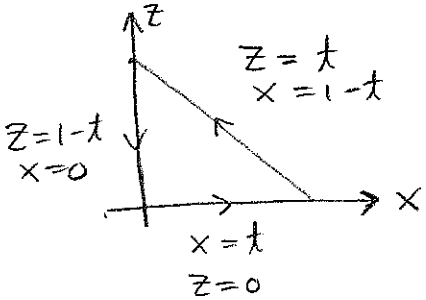
$$= \int_0^1 \int_0^{1-z} dy\,dz$$

$$= \int_0^1 (1-z)\,dz$$

$$= \boxed{\tfrac{1}{2}.}$$

PROBLEM 231 continued

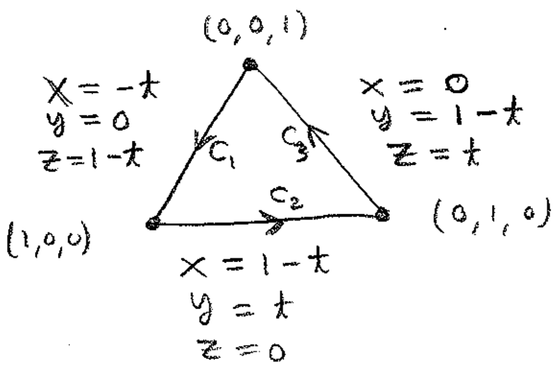
3.) $$\int_{\partial S_{xz}} \underset{0}{y}\,dx - x\,\underset{0}{dy} + \underset{0}{y}\,dz = \boxed{0.}$$



$\underline{y = 0,\ dy = 0}$

$$\iint_{S_{xz}} \vec{G}\cdot d\vec{S} = \int_0^1\int_0^{1-z} \underbrace{\langle 1, 0, -2\rangle \cdot \langle 0, 1, 0\rangle}_{\text{ZERO.}}\, dx\,dz = \boxed{0}$$

4.) $$\oint_{\partial S_{TOP}} y\,dx - x\,dy + y\,dz = \underbrace{\int_0^1 t(-dt) - \cancel{(1-t)dt}}_{C_2} + \underbrace{\int_0^1 \cancel{(1-t)dt}}_{C_3} = \boxed{\frac{-1}{2}}$$



$S_T :\ \underbrace{x + y + z = 1}$

$$\iint_{S_{TOP}} \vec{G}\cdot d\vec{S} = \int_0^1\int_0^{1-x} \langle 1, 0, -2\rangle \cdot \langle 1, 1, 1\rangle\, dy\,dx$$

$$= \int_0^1 \left(-y\Big|_0^{1-x}\right)dx$$

$$= \int_0^1 (x-1)\,dx$$

$$= \frac{1}{2} - 1$$

$$= \boxed{\frac{-1}{2}}$$

We find

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_{S} (\nabla\times\vec{F})\cdot d\vec{S}$$

in each case. Stokes' Th$^{\underline{m}}$ is verified.

PROBLEM 232 | Let $\vec{F}(x,y,z) = \langle x^2, 2x, z^2 \rangle$ calculate $\int_E \vec{F}\cdot d\vec{r}$ where E is CCW ellipse $4x^2+y^2=4$ with $z=0$

$$\int_E \vec{F}\cdot d\vec{r} = \iint_{\substack{4x^2+y^2\le 4 \\ z=0}} (\nabla\times\vec{F})\cdot d\vec{S} = \iint_{x^2+y^2/4\le 1} \langle 0,0,2\rangle\cdot\hat{z}\,dx\,dy$$

$$= 2\iint_{x^2+\frac{y^2}{4}\le 1} dA$$

$$= 2\pi ab$$

$$= \boxed{4\pi}$$

PROBLEM 233 | Consider $x+y+z=1$ intersect the coord. planes with CCW orientation. Find $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = \langle y^2+z^2, x^2+z^2, x^2+y^2\rangle$

$$\nabla\times\vec{F} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{bmatrix} = \langle 2y-2z, 2z-2x, 2x-2y\rangle$$

$$\oint_{\partial S=C} \vec{F}\cdot d\vec{r} = 2\iint_S \langle y-z, z-x, x-y\rangle\cdot\langle 1,1,1\rangle\,dx\,dy$$

$$= 2\int_0^1\int_0^{1-x} (\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y})\,dx\,dy$$

$$= \boxed{0.}$$

PROBLEM 234 | Let $\vec{F}(x,y,z) = \langle y^2+z^2, x^2+y^2, x^2+y^2\rangle$. Let $S: [-1,1]^2$

Find $\oint_{\partial S}\vec{F}\cdot d\vec{r}$

$$\nabla\times\vec{F} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{bmatrix} = \langle 2y, 2z-2x, 2x-2y\rangle$$

$$\oint_{\partial S}\vec{F}\cdot d\vec{r} = \int_{-1}^1\int_{-1}^1 \langle 2y, 2z-2x, 2x-2y\rangle\cdot\hat{z}\,dx\,dy$$

$$= \int_{-1}^1\int_{-1}^1 2(x-y)\,dx\,dy = \int_{-1}^1 2\left(\cancelto{0}{\tfrac{1}{2}x^2\Big|_{-1}^1} - 2y\right)dy$$

$$= -4\int_{-1}^1 y\,dy = \boxed{0.}$$