Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy! This quiz has 3pts of bonus credit.

Problem 1 Convert $f(x,y,z) = \frac{z}{x^2+y^2+z^2}$ to spherical coordinates and calculate ∇f in terms of the spherical coordinate frame. (leave answer in terms of $\hat{\rho}, \hat{\phi}, \hat{\theta}$ and the spherical coordinates ρ, ϕ, θ)

$$f = \frac{\rho\cos\phi}{\rho^2} = \frac{\cos\phi}{\rho}$$

$$\frac{\partial f}{\partial \rho} = -\frac{\cos\phi}{\rho^2}$$

$$\frac{\partial f}{\partial \phi} = -\frac{\sin\phi}{\rho}$$

$$\frac{\partial f}{\partial \theta} = 0$$

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\hat{\phi}\frac{\partial f}{\partial \phi} + \frac{\hat{\theta}}{\rho\sin\phi}\frac{\partial f}{\partial \theta}$$

$$= -\frac{\cos\phi}{\rho^2}\hat{\rho} - \frac{\sin\phi}{\rho^2}\hat{\phi} = \boxed{\frac{-1}{\rho^2}(\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi})}$$

Problem 2 Find the multivariate power series expansion of $f(x,y) = e^{x+y^2}$ at $(0,0)$ to second order. Is $(0,0)$ a critical point? If so, classify the nature of the point.

$$f(x,y) = e^x e^{y^2}$$

$$= \left(1 + x + \tfrac{1}{2}x^2 + \cdots\right)\left(1 + y^2 + \cdots\right)$$

$$= 1 + x + \tfrac{1}{2}x^2 + y^2 + \cdots$$

$f_x(0,0) = 1$ $\quad\therefore\quad \nabla f(0,0) = \langle 1,0\rangle \neq \langle 0,0\rangle$

$f_y(0,0) = 0$ $\quad\therefore\quad (0,0)$ not a critical pt.

Problem 3 Find all critical points of $f(x,y,z) = \cos(x+y+z)$.

$$\nabla f = -\sin(x+y+z)\,\nabla(x+y+z) = -\sin(x+y+z)\langle 1,1,1\rangle$$

$$\nabla f = 0 \text{ iff } \sin(x+y+z) = 0 \Rightarrow \boxed{x+y+z = n\pi,\ n \in \mathbb{Z}.}$$

Problem 4 Find the extreme values of $f(x,y) = x^2 + y^2 + x^2 e^y$ on the disk $D = \{(x,y) \mid x^2+y^2 \le 1\}$.

$$\nabla f = \langle 2x + 2xe^y,\ 2y + x^2 e^y \rangle$$

$$2x(1+e^y) = 0 \implies \underline{x = 0} \text{ necessary.}$$

$$2y + x^2 e^y = 0 \implies 2y = 0 \therefore \underline{y = 0}.$$

Hence $(0,0)$ only critical pt. interior of disk. $\underline{f(0,0) = 0}$. for later reference.

Consider the boundary, $g = x^2+y^2$, $\nabla g = \langle 2x, 2y \rangle$

$$\nabla f = \lambda \nabla g \quad \begin{cases} 2x + 2xe^y = 2\lambda x \\ 2y + x^2 e^y = 2\lambda y \end{cases}$$

$$2(x-y) + (2x - x^2)e^y = 2\lambda(x-y)$$

$$(2x - x^2)e^y = 2(\lambda - 1)(x-y)$$

If $\lambda = 1$ then $2x - x^2 = 0 \to 2x = x^2 \to \underline{x = 0, x = 2}$. not on disk.

Hence $x = 0 \implies y = \pm 1$ thus $f(0, \pm 1) = 1 + 0 \cdot e^{\pm 1} = 1$.

Adam's Method, ∂D : $x = \cos t$, $y = \sin t$ hence

$$g(t) = f(\cos t, \sin t) = 1 + \cos^2 t\, e^{\sin t}$$

$$\frac{dg}{dt} = (-2\cos t \sin t + \cos^3 t)e^{\sin t} = \cos t\,[-2\sin t + 1 - \sin^2 t]\,e^{\sin t}$$

$$= -\cos t\,[\sin^2 t + 2\sin t - 1]$$

$$= -\cos t\,[(\sin t + 1)^2 - 2]$$

Problem 5 Suppose $f(x,y) = 6 + 3(x-1)^2 - 4(y+2)^2 - 10(x-1)(y+2) + \cdots$. Determine if $(1,-2)$ is a critical point and if it is use the second derivative test to classify the nature of the point as min/max/saddle if possible.

Observe by inspection $f_{xx}(1,-2) = 6$, $f_{yy}(1,-2) = -8$ and $f_{xy}(1,-2) = -10$ therefore, as $f_x = f_y = 0$ at $(1,-2)$,

$D(1,-2) = f_{xx}f_{yy} - f_{xy}^2 = -48 - 100 < 0$

$\therefore$ it is a saddle point.

$-\cos t = 0$ or $\sin t + 1 = \pm\sqrt{2}$

$t = \pi/2, 3\pi/2$ $\quad$ $\sin t = -1 + \sqrt{2}$

$\to (0,1)$, $(0,-1)$ $\quad$ $\underline{t = \sin^{-1}(-1+\sqrt{2})}$

didn't find this yet in my Lagrange algebra.

For $\sin t = -1 + \sqrt{2}$ have $\cos t = \sqrt{1 - (\sqrt{2}-1)^2}$ hence

$$f\left(\sqrt{1-(\sqrt{2}-1)^2}, -1+\sqrt{2}\right) = 1 + (1 - (\sqrt{2}-1)^2)e^{-1+\sqrt{2}}$$

$$= 1 + (1 - (2 - \sqrt{2} + 1))e^{-1+\sqrt{2}}$$

$$= 1 + (\sqrt{2} - 2)e^{-1+\sqrt{2}} < 1$$

Thus $f(0,0) = 0$ is minimum and $f(0,\pm 1) = 1$ gives max.

Problem 6 Calculate $\int_0^1 \int_0^{x^2} e^{x^3}\,dy\,dx$

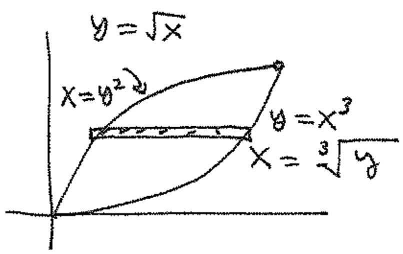
$$\int_0^1 y e^{x^3}\Big|_{y=0}^{y=x^2} dx = \int_0^1 x^2 e^{x^3}\,dx$$

$$u = x^3$$
$$du = 3x^2 dx$$

$$= \frac{1}{3}\int_0^1 e^u\,du$$

$$= \boxed{\frac{1}{3}(e-1)}$$

Problem 7 Express the integral $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dy\,dx$ as an iterated integral based on a Type II region.



$$0 \le y \le 1$$

$$\boxed{\int_0^1 \int_{y^2}^{\sqrt[3]{y}} f(x,y)\,dx\,dy}$$

Problem 8 Calculate $\iiint_C (x^2+y^2)z\,dV$ where C is the solid cylinder of radius $r = 3$ for $0 \le z \le 1$.

$$\iiint_C (x^2+y^2)z\,dV = \int_0^{2\pi}\int_0^3\int_0^1 r^2 z\, r\,dz\,dr\,d\theta$$

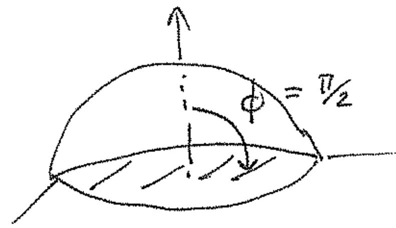
$$= \int_0^{2\pi} d\theta \int_0^3 z\,dz \int_0^1 r^3\,dr$$

$$= (2\pi)\left(\frac{9}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{18\pi}{8} = \boxed{\frac{9\pi}{4}}$$

Problem 9 Let $B = \{(x,y,z) \mid \underbrace{x^2+y^2+z^2 \leq 4,\ z \geq 0}\}$. Calculate $\iiint_B (x^2+y^2)dV$.

$$0 \leq \rho \leq 2$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \frac{\pi}{2}$$



Note, $r = \rho \sin\phi$ hence $x^2+y^2 = \rho^2 \sin^2\phi$

$$\iiint_B (x^2+y^2)dV = \int_0^2 \int_0^{2\pi} \int_0^{\pi/2} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\, d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3\phi\, d\phi$$

$$= \left(\frac{2^4}{4}\right)(2\pi) \int_0^{\pi/2} (1-\cos^2\phi)\sin\phi\, d\phi$$

$$= 8\pi \int_1^0 (1-u^2)(-du)$$

$u = \cos\phi$
$du = -\sin\phi\, d\phi$
$u(0) = 1$
$u(\pi/2) = 0$

$$= 8\pi \int_0^1 (u^2-1)du$$

$$= -8\pi\left(\frac{1}{3} - 1\right)$$

$$= -8\pi\left(-\frac{2}{3}\right)$$

$$= \boxed{\frac{16\pi}{3}}$$

Problem 10 Suppose the mass per unit area is given by $\sigma(x, y) = x$. Find the center of mass for the rectangular region $R = [0, 1] \times [0, 1]$.

$$M = \int_0^1\int_0^1 x\,dx\,dy = \int_0^1 dy \int_0^1 x\,dx = \underline{\frac{1}{2}}.$$

$$X_{cm} = \frac{1}{M}\iint_R \sigma x\,dA = 2\int_0^1\int_0^1 x^2\,dx\,dy = 2\left(\frac{1}{3}\right) = \underline{\frac{2}{3}}$$

$$y_{cm} = \frac{1}{M}\iint_R \sigma y\,dA = 2\int_0^1\int_0^1 xy\,dx\,dy$$

$$= 2\int_0^1 x\,dx \int_0^1 y\,dy$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \underline{1/2}.$$

Thus $(2/3, 1/2)$ is the c.o.m.

Problem 11 Find the area of the ellipitical region $x^2/a^2 + y^2/b^2 \leq 1$.

$\mathcal{E}$

~~u = x a, v = y b~~

$$\underbrace{\frac{x}{a} = u, \quad \frac{y}{b} = v}$$

$u^2+v^2 \leq 1 : \tilde{\mathcal{E}}$

$x = au, \quad y = bv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\iint_{\mathcal{E}} dA = \iint_{\tilde{\mathcal{E}}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{u^2+v^2\leq 1} ab\,du\,dv$$

$$= ab \iint_{u^2+v^2\leq 1} du\,dv$$

$$= \boxed{\pi ab}$$