Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy! This quiz has 3pts of bonus credit.

Problem 1 (9pts) Let $f(x, y) = x + x^2y^2$. Find the following:

(a.) $\frac{\partial f}{\partial x} = \boxed{1 + 2xy^2}$ $\rightarrow$ $f_x(1,2) = 1 + 2(1)(4) = 9$

(b.) $\frac{\partial f}{\partial y} = \boxed{2x^2y}$ $\rightarrow$ $f_y(1,2) = 2(1)(2) = 4$

(c.) the equation of the tangent plane at $(1, 2)$. Note $f(1,2) = 1 + 1^2 \cdot 2^2 = 1 + 4 = 5$.

$$\underbrace{\boxed{z = 5 + 9(x-1) + 4(y-2)}}_{z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)}$$

$\sqrt{64+36} = \sqrt{100} = 10.$

Problem 2 (6pts) Let $f(x, y) = \dfrac{1}{1 + x^2 + y^3}$. Find the rate of change in f at $(1, 0)$ in the $\langle 8, -6\rangle$ direction.

$\hat{u} = \frac{1}{10}\langle 8, -6\rangle$

$$\nabla f(x,y) = \left\langle \frac{-2x}{(1+x^2+y^3)^2}, \frac{-3y^2}{(1+x^2+y^3)^2} \right\rangle$$

$$\nabla f(1,0) = \left\langle \frac{-2}{4}, 0 \right\rangle = \langle -1/2, 0 \rangle$$

$$(D_{\hat{u}} f)(1,0) = \nabla f(1,0) \cdot \hat{u} = \langle -1/2, 0\rangle \cdot \langle 8/10, -6/10\rangle = \frac{-8}{20} = \boxed{\frac{-2}{5}}$$

Problem 3 (6pts) Let $f(x, y) = x^2 + y^2$. Plot level curves $f(x, y) = k$ for $k = 1, 2, 9$ on the graph template given below. Furthermore, find (with calculus) the direction in which f is locally constant at $(1, 1)$ and indicate why it is reasonable in view of your plotted level curves.

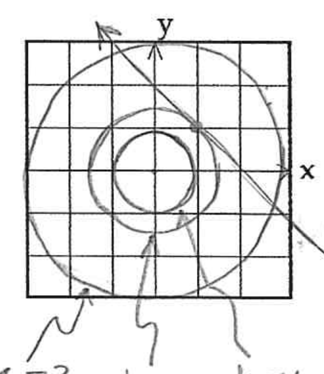
$$\nabla f(x,y) = \langle 2x, 2y \rangle$$

$$(\nabla f)(1,1) = \langle 2, 2 \rangle$$

$$(D_{\hat{u}} f)(1,1) = (\nabla f)(1,1) \cdot \langle a, b \rangle = 0$$

$$2a + 2b = 0 \Rightarrow b = -a$$

choose $a = 1 \Rightarrow \underbrace{\langle 1, -1 \rangle}_{\text{locally constant direction}}$



has $\langle 1, -1 \rangle$ direction

$k=3$ $k=2$ $k=1$

reasonable as line points in direction in which $f(x,y)$ stays at 2. (locally)

$$dy = (2\beta\sin\beta + \beta^2\cos\beta)\,d\beta$$

Problem 4 (10pts) Let $z = \sin(xy^2)$. Furthermore, $x = \alpha + \beta^2$ and $y = \beta^2\sin(\beta)$. Use the multivariate chain rule to calculate $\dfrac{\partial z}{\partial \alpha}$ and $\dfrac{\partial z}{\partial \beta}$.

$$dx = d\alpha + 2\beta\,d\beta$$

$$dz = \cos(xy^2)\left[y^2dx + 2xy\,dy\right]$$

$$= \cos(xy^2)\left[y^2(d\alpha + 2\beta d\beta) + 2xy(2\beta\sin\beta + \beta^2\cos\beta)d\beta\right]$$

$$= \underbrace{y^2\cos(xy^2)}_{\frac{\partial z}{\partial \alpha}}d\alpha + \underbrace{\cos(xy^2)\left[2\beta y^2 + 2xy\left[2\beta\sin\beta + \beta^2\cos\beta\right]\right]}_{\frac{\partial z}{\partial \beta}}d\beta$$

$$\boxed{\frac{\partial z}{\partial \alpha} = y^2\cos(xy^2)} \qquad \boxed{\frac{\partial z}{\partial \beta} = \cos(xy^2)\left[2y^2\beta + 2xy\beta\left[2\sin\beta + \beta\cos\beta\right]\right]}$$

Problem 5 (6pts) Suppose $h_x(1,2) = 3$ and $h_y(1,2) = 7$. In addition, $x(u,v) = uv^2$ and $y(u,v) = u^2 - v$. Let $g(u,v) = h(x(u,v), y(u,v))$ and calculate $\dfrac{\partial g}{\partial u}(1,-1)$.

$$x(1,-1) = 1(-1)^2 = 1$$
$$y(1,-1) = 1^2 - (-1) = 2$$

$$\frac{\partial g}{\partial u} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial u}(1,-1) = h_x(x(1,-1), y(1,-1))\frac{\partial x}{\partial u}(1,-1) + h_y(x(1,-1), y(1,-1))\frac{\partial y}{\partial u}(1,-1)$$

$$= h_x(1,2)\,v^2\big|_{(1,-1)} + h_y(1,2)\,2u\big|_{(1,-1)}$$

$$= 3(-1)^2 + 7(2(1)) = 3 + 14 = \boxed{17}$$

Problem 6 (9 pts) Let $\vec{r}(s,t) = \langle t\cos 3s,\ t\sin 3s,\ 4s\rangle$ be the parametrization of a surface M. Find the normal vector field to M.

$$\frac{\partial \vec{r}}{\partial t} = \langle \overbrace{\cos 3s}^{C}, \overbrace{\sin 3s}^{S}, 0\rangle$$

$$\frac{\partial \vec{r}}{\partial s} = \langle -3t\sin 3s,\ 3t\cos 3s,\ 4\rangle$$

$$\vec{N}(s,t) = \frac{\partial \vec{r}}{\partial s}\times\frac{\partial \vec{r}}{\partial t} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3tS & 3tC & 4 \\ C & S & 0 \end{bmatrix}$$

$$S^2 + C^2 = 1.$$

$$= \hat{x}(-4S) - \hat{y}(-4C) + \hat{z}(-3tS^2 - 3tC^2)$$

$$= \boxed{\langle -4\sin(3s),\ 4\cos(3s),\ -3t\rangle}$$

$F \quad = \quad 5$

Problem 7 (6pts) A surface $\overbrace{Ax^2 + By^3 + Cz^4}^{F} = 5$ has a normal line at $(1, -1, 1)$ with direction vector $\langle 12, 12, 12\rangle$. Find the values of the constants A, B and C.

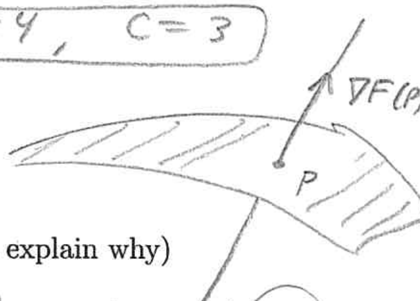
$$\nabla F(x,y,z) = \langle 2Ax,\ 3By^2,\ 4Cz^3 \rangle$$

$$\nabla F(1,-1,1) = \langle 2A,\ 3B,\ 4C \rangle = \langle 12, 12, 12 \rangle$$

$$\Rightarrow 2A = 12, \quad 3B = 12, \quad 4C = 12$$

$$\Rightarrow \boxed{A = 6, \quad B = 4, \quad C = 3}$$

direction of normal line matches the normal (upto rescaling)



Problem 8 (6 pts) Calculate the limit below (if it doesn't exist then explain why)

$$\lim_{(x,y)\to(0,0)} \frac{y - x^2}{x^2 + y^2} = \lim_{r \to 0} \left(\frac{r\sin\theta - r^2\cos^2\theta}{r^2} \right) = \lim_{r \to 0} \left(\frac{\sin\theta}{r} - \cos^2\theta \right)$$

blows-up.

Notice,

$$\lim_{(0,y)\to(0,0)} \left(\frac{y - x^2}{x^2 + y^2} \right) = \lim_{y \to 0} \left(\frac{y - 0^2}{0^2 + y^2} \right) = \lim_{y \to 0} \left(\frac{1}{y} \right) = \text{d.n.e.}$$

Hence the given limit does not exist.

(if it did then all path limits converge to same value ... one diverging kills our hope.)

Problem 9 (6pts) You are stuck on the surface $xy^2z^3 = 1$. At the point $(1, 1, 1)$ you measure your x-velocity to be $2ft/s$ and your y-velocity is $-3ft/s$. Assuming units of ft and s find $\frac{dz}{dt}$.

$$\dot{x} + 2xyz^3\dot{y} + 3xy^2z^2\dot{z} = 0$$

$$\dot{z} = \frac{-\dot{x} - 2xyz^3\dot{y}}{3xy^2z^2} = \frac{-2 - 2(-3)}{3}\,\frac{ft}{s} = \boxed{\frac{4}{3}\,\frac{ft}{s}}$$

Problem 10 (6pts) Imagine yourself a pig. Ron Swanson is holding a barbeque of epic size in a heavily misty park. At the same time a bagpipe club is playing at an embarrassing volume. You can't see or hear in effect. The smell of meat is measured to roughly follow $M(x, y) = x^2 - 2xy$ at $(1, 1)$. In which direction should you travel to **get away** from the feast?

$$\nabla M = \langle 2x - 2y, \ -2x \rangle$$

$$\nabla M(1,1) = \langle 0, -2 \rangle$$ ← points directly toward the BBQ hence go opposite direction (N) $\boxed{\langle 0, 2 \rangle}$

Problem 11 (6pts) Suppose $w = x^2 + yz^3$ where $\cos(xyz) = y^2 + z^2$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$.

$$-\sin(xyz) \frac{\partial}{\partial x}\Big|_y (xyz) = \frac{\partial}{\partial x}\Big|_y (y^2 + z^2)$$

$$-\sin(xyz)\left[yz + xy \frac{\partial z}{\partial x}\Big|_y\right] = 2z \frac{\partial z}{\partial x}\Big|_y$$

$$\frac{\partial z}{\partial x}\Big|_y = \frac{-yz\sin(xyz)}{2z + xy\sin(xyz)}$$

$$\frac{\partial w}{\partial x}\Big|_y = \frac{\partial}{\partial x}\Big|_y (x^2 + yz^3)$$

$$= 2x + 3yz^2 \frac{\partial z}{\partial x}\Big|_y$$

$$= \boxed{2x - \frac{3y^2 z^3 \sin(xyz)}{2z + xy\sin(xyz)}}$$

Problem 12 (6pts) Calculate dF for $F(x, y, z) = \ln\left[e^x z^{z^2} \underbrace{[\cosh^2(xyz) - \sinh^2(xyz)]}_{1} \sqrt{\sec(y^2)}\right]$. Simplify your answer.

$$F = x + z^2 \ln(z) - \frac{1}{2}\ln(\cos(y^2))$$

$$dF = dx - \left[\frac{1}{2\cos(y^2)}(-\sin(y^2) \cdot 2y)\right] dy + \left[2z\ln(z) + \frac{z^2}{z}\right] dz$$

$$\boxed{dF = dx + y\tan(y^2)\, dy + z(2\ln(z) + 1)\, dz}$$

Problem 13 (10 pts) Solve $\underbrace{(2xy^2 + e^x)}_{\frac{\partial F}{\partial x}}dx + \underbrace{(2x^2y + y\cos(y^2))}_{\frac{\partial F}{\partial y}}dy = 0$.

$$\Rightarrow \quad F(x,y) = x^2y^2 + e^x + C_1(y) \quad \Rightarrow \quad \cancel{2x^2y} + \frac{dC_1}{dy} = \cancel{2x^2y} + y\cos y^2$$

$$\therefore \quad C_1 = \int y \cos y^2 \, dy = \frac{1}{2}\sin(y^2) \quad \therefore \quad \boxed{x^2y^2 + e^x + \frac{1}{2}\sin(y^2) = C}$$

Sol^{n} of given DEq^{n}.

Problem 14 (3pts) Let $\omega = -y\,dx + x\,dy$. Is this form closed? Is this form exact?

$$\frac{\partial}{\partial y}(-y) = -1\ ,\ \frac{\partial}{\partial x}(x) = 1\ \ \therefore\ \underline{\text{not closed}} \Rightarrow \underline{\text{not exact}}.$$

Problem 15 (5pts) Suppose f, g are smooth real-valued functions of n-variables. Show that

$$\nabla(fg) = (\nabla f)g + f(\nabla g).$$

$$\begin{aligned}
\nabla(fg) &= \sum_{i=1}^{n} \hat{x}_i \partial_i (fg) \\
&= \sum_{i=1}^{n} \hat{x}_i \left[(\partial_i f) g + f(\partial_i g)\right] \quad : \text{ordinary product rule} \\
&= \left(\sum_{i=1}^{n} \hat{x}_i \partial_i f\right) g + f\left(\sum_{i=1}^{n} \hat{x}_i \partial_i g\right) \\
&= (\nabla f) g + f(\nabla g) \ //
\end{aligned}$$

(I gave significant partial credit for $n=2$ or $n=3$ derivations)

Problem 16 (4pts) Let $\vec{A}(x,y,z) = \left\langle 2x,\ -2y,\ \dfrac{1}{z^2+1} \right\rangle$. If possible, find Φ such that $\nabla\Phi = \vec{A}$.

Need $\dfrac{\partial\Phi}{\partial x} = 2x\ ,\ \dfrac{\partial\Phi}{\partial y} = -2y\ ,\ \dfrac{\partial\Phi}{\partial z} = \dfrac{1}{z^2+1}$

$$\Rightarrow \boxed{\Phi(x,y,z) = x^2 - y^2 + \tan^{-1}(z)}$$

(integrate each, then check that $\nabla\Phi = \vec{A}$ as desired, the combine method worked here)

Problem 17 (4pts) Let $\vec{B}(x,y,z) = \langle 2x+z,\ x^3,\ y \rangle$. If possible, find Ψ such that $\nabla\Psi = \vec{B}$.

$$\nabla\Psi = \left\langle \frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial\Psi}{\partial z} \right\rangle = \langle \underbrace{2x+z}_{P}, \underbrace{x^3}_{Q}, \underbrace{y}_{R} \rangle$$

Notice we have
$$\left.\begin{aligned}
\Psi_{xy} &= \Psi_{yx} \longrightarrow P_y = Q_x \\
\Psi_{xz} &= \Psi_{zx} \longrightarrow P_z = R_x \\
\Psi_{yz} &= \Psi_{zy} \longrightarrow Q_z = R_y
\end{aligned}\right\} \quad
\begin{aligned}
&P_y = 0,\ Q_x = 3x^2 \\
&P_z = 1,\ R_x = 0 \\
&Q_z = 0,\ R_y = 1
\end{aligned}$$

this extrapolates our $n=2$ work in class

The criteria are not met $\therefore\ \Psi$ d.n.e.