Please show your work. Enjoy! There are at least 150pts to earn here.

(25pts) Solve 2 of the problems on this page. Do not work more than 2, you need the time remaining for the rest of the test. Thanks.

Problem 1 How many positive integers between 1 and 221 are relatively prime to 221?
Hint: $221 = (13)(17)$.

Problem 2 Find the Egyptian fraction presentation of $\frac{20}{13}$.

Problem 3 Find the continued fraction form of $\frac{99}{26}$.

(25pts) Solve 2 of the problems on this page. Do not work more than 2, you need the time remaining for the rest of the test. Thanks.

Problem 4 Express 101 in its base-two representation. Then, for $m \in \mathbb{Z}$, show how we can calculate m^{101} from the value 1 by successive multiplications by m and squaring. Loosely, your exponentiation should not have more than about 10 steps.

Problem 5 Calculate the least positive residue of 50^{201} mod 33.

Problem 6 Find the last two digits of $5 \cdot 33^{7,000,321} \cdot 3^{7,000,322}$. Hint: $99 = 3 \cdot 33$.

(25pts) Solve 2 of the problems on this page. Do not work more than 2, you need the time remaining for the rest of the test. Thanks.

Problem 7 Find $m, n \in \mathbb{Z}$ such that $gcd(131, 30) = m(131) + n(30)$. Also, calculate $[30]^{-1}$ in $\mathbb{Z}_{131}^{\times}$.

Problem 8 Find all integer solutions of $2x + 3y = 20$.

Problem 9 Simultaneously solve the system of congruences:

$$x \equiv 3 \pmod{5} \qquad \& \qquad x \equiv 4 \pmod{17}.$$

(25pts) Solve 2 of the problems on this page. Do not work more than 2, you need the time remaining for the rest of the test. Thanks.

Problem 10 Find the order of 10 modulo 41. What is the signifcance of your result as you compare the order to the decimal expansion of $1/41$? (use your calculator to see the expansion)

Problem 11 Let $b \in \mathbb{N}$ and suppose that $\gcd(b+18, b)$ is even and divisible by an odd prime. List the possible values for $\gcd(b+18, b)$.

Problem 12 Recall the base-eight representation of a number:

$$(c_d \dots c_2 c_1 c_0)_8 = c_d \times 8^d + \cdots c_2 \times 8^2 + c_1 \times 8 + c_0 \times 1$$

where $0 \leq c_0, c_1, \dots, c_d \leq 7$. Is $(7654321)_8$ divisible by 9 ?

To be clear: from here on out, attempt all the problems you can. Thanks!

Problem 13 (10pts) Let $a, b, c, d \in \mathbb{Z}$ and $a \mid b$ and $c \mid d$. Prove that $ac \mid bd$.

Problem 14 (10pts) Prove that the product of successive integers is even. (I expect a proof referencing integers and precise definitions)

Problem 15 (10pts) Suppose $[a] \in \mathbb{Z}_n$ has order $k > 1$. Furthermore, suppose $b \in \mathbb{Z}$ has $ab \equiv 1 \text{ mod}(n)$. Prove $[b]$ also has order k.

Problem 16 (5pts) Arrange the following list of mathematicians in chronological order from ancient to modern: Gauss, Dirichlet, Diophantus, Euclid, Euler, Sprano :

Problem 17 (10pts) Consider $G = (\mathbb{Z}/5\mathbb{Z})^\times = \{1, 2, 3, 4\}$ where the multiplication is defined modulo 5. Fill out the following multiplication table:

$(\mathbb{Z}_5)^\times$	1	2	3	4
1				
2				
3				
4				

Define $H = \{1, 4\}$. Show how G is partitioned by the cosets of H.

Problem 18 (15pts) State and prove Lagrange's Theorem.