Math 307	Test 3

Please show your work. Enjoy! There are at least 150pts to earn here.

Problem 1 If I is an ideal of $\mathbb{Z}$ then $I = (n)$ for some $n \in \mathbb{Z}$. This proves $\mathbb{Z}$ is a principal ideal domain.

Problem 2 Consider $\alpha = 7/13$. Is α an algebraic number? Is α an algebraic integer. Briefly explain.

Problem 3 prove, if $1 \in I$ then $I = R$.

Problem 4 Show $\mathbb{Z}[\sqrt{-14}\,] = (2,\, 1 + \sqrt{-14}\,)$

Problem 5 find the units in $\mathbb{Z}[\sqrt{-14}\,]$

Problem 6 Use quadratic reciprocity to answer the following question: Can you solve $x^2 \equiv 24 \ (\mathrm{mod}(31))$ for any $x \in \mathbb{Z}$?

Problem 7 Let R be a ring and I a nonzero proper ideal (proper meaning $I \neq R$). Define multiplications of cosets $I + a \in R/I$ by

$$(I + a)(I + b) = I + ab.$$

Show that this multiplication is well-defined. In particular, **show that:** if $I + a = I + a'$ and $I + b = I + b'$ then $I + ab = I + a'b'$.

Problem 8 Let R be integers of $\mathbb{Q}(\sqrt{d})$ where d is squarefree. If $z = a+b\sqrt{d}$ then we define $\bar{z} = a-b\sqrt{d}$. Likewise, if A is an ideal of R then the **conjugate ideal** $\bar{A} = \{\bar{z} \mid z \in A\}$. Prove:

(a) $\overline{zw} = \overline{z} \cdot \overline{w}$ for all $z, w \in R$. (here $\cdot$ is the ring multiplication)

(b) for ideals A, B show $\overline{AB} = \overline{A} \cdot \overline{B}$ (here $\cdot$ is the multiplication of ideals)

Problem 9 A new idea which is fun to study in a relaxed setting like a test is that of the **ideal norm**. In particular, for a nonzero ideal A we know $\bar{A}A = (k)$ for some $k \in \mathbb{Z}$ where without loss of generality we may suppose $k > 0$. Define the **ideal norm of** A by $\mathrm{N}(A) = k$.

(a) If $A = (\alpha)$ then show $\mathrm{N}(A) = \mathrm{norm}(\alpha)$.

(b) If A, B are nonzero ideals then $\mathrm{N}(AB) = \mathrm{N}(A)\mathrm{N}(B)$

(c) If $A \mid B$ then $\mathrm{N}(A) \mid \mathrm{N}(B)$

(d) an ideal whose norm is prime in $\mathbb{Z}$ is a prime ideal.

Problem 10 Prove $(3-\sqrt{-14})$ is prime in $\mathbb{Z}[\sqrt{-14}\,]$. Hint: 17 is prime.

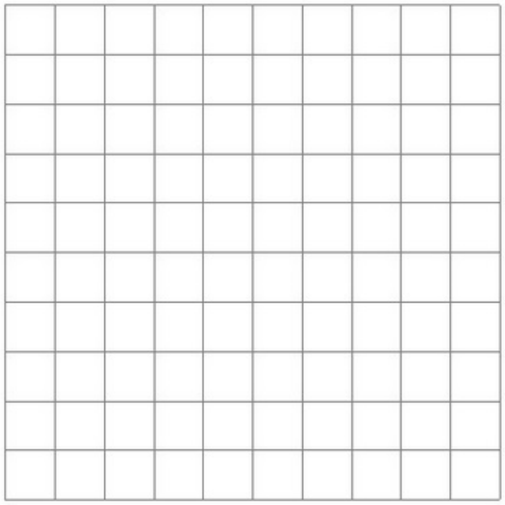
Problem 11 Show that $(3-\sqrt{-14})$ is maximal in $\mathbb{Z}[\sqrt{-14}\,]$.

Problem 12 Mordell's Equation: to find integer solutions of $y^2+26=x^3$ we can guess the simple solutions $x=3$ and $y=\pm 1$. However, less trivial solutions may be found from factoring in $\mathbb{Z}[\sqrt{-26}]$. Conjecture:

$$x^3 = y^2+26 = (y-\sqrt{-26})(y+\sqrt{-26}) \quad \Rightarrow \quad y+\sqrt{-26} = (a+b\sqrt{-26})^3$$

for some $a,b \in \mathbb{Z}$. Derive two integer solutions from the above conjecture.

Problem 13 Let $R = \mathbb{Z}[i]$ and consider $J = (1 + 2i)$. Show J is a lattice by finding $\alpha, \beta \in R$ for which $J = \{m\alpha + n\beta \mid m, n \in \mathbb{Z}\}$. Picture the ideal J and, with the help of your picture, describe the structure of R/J. In particular, find a representative for each distinctl coset in R/J. (there are 5).



Challenge: are there zero divisors in R/J *? If not, can you find a finite field which is ring isomorphic to* R/J *?*

Problem 14 Recall the equivalence of the following properties of an ideal P:

$$(1.)\ AB \subseteq P \ \Rightarrow\ A \subseteq P \text{ or } B \subseteq P, \qquad\qquad (2.)\ ab \in P \ \Rightarrow\ a \in P \text{ or } b \in P$$

Please prove just one direction of (1.) $\Leftrightarrow$ (2.). (your choice)

Problem 15 Prove: every maximal ideal is prime.

Problem 16 Explain why every prime ideal is maximal in the integers of an imaginary quadratic field.