Lecture Notes for Linear Algebra

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Fall 2009

introduction and motivations for these notes

These notes should cover what is said in lecture at a minimum. However, I'm still learning so I may have a new thought or two in the middle of the semester, especially if there are interesting questions. We will have a number of short quizzes and the solutions and/or commentary about those is not in these notes for the most part. It is thus important that you never miss class. The text is good, but it does not quite fit the lecture plan this semester.

I probably would have changed it if I realized earlier but alas it is too late. Fortunately, these notes are for all intents and purposes a text for the course. But, these notes lack exercises, hence the required text. The text does have a good assortment of exercises but please bear in mind that the ordering of the exercises assigned matches my lecture plan for the course and not the text's. Finally, there are a couple topics missing from the text which we will cover and I will write up some standard homework problems for those sections.

As usual there are many things in lecture which you will not really understand until later. I will regularly give quizzes on the material we covered in previous lectures. I expect you to keep up with the course. Procrastinating until the test to study will not work in this course. The difficulty and predictability of upcoming quizzes will be a function of how I percieve the class is following along.

Doing the homework is doing the course. I cannot overemphasize the importance of thinking through the homework. I would be happy if you left this course with a working knowledge of:

- $\checkmark\,$ how to solve a system of linear equations
- \checkmark Gaussian Elimination and how to interpret the rref(A)
- $\checkmark\,$ concrete and abstract matrix calculations
- \checkmark determinants
- $\checkmark\,$ real vector spaces both abstract and concrete
- $\checkmark~$ subspaces of vector space
- $\checkmark\,$ how to test for linear independence
- $\checkmark\,$ how to prove a set spans a space
- $\checkmark\,$ coordinates and bases
- $\checkmark\,$ column, row and null spaces for a matrix
- $\checkmark~$ basis of an abstract vector space
- $\checkmark\,$ linear transformations

- \checkmark matrix of linear transformation
- $\checkmark\,$ change of basis on vector space
- $\checkmark\,$ geometry of Euclidean Space
- $\checkmark\,$ orthogonal bases and the Gram-Schmidt algorithm
- $\checkmark\,$ least squares fitting of experimental data
- \checkmark best fit trigonmetric polynomials (Fourier Analysis)
- \checkmark Eigenvalues and Eigenvectors
- \checkmark Diagonalization
- \checkmark how to solve a system of linear differential equations
- $\checkmark\,$ principle axis theorems for conic sections and quadric surfaces

I hope that I have struck a fair balance between pure theory and application. The treatment of systems of differential equations is somewhat unusual for a first course in linear algebra. No apologies though, I love the example because it shows nontrivial applications of a large swath of the theory in the course while at the same time treating problems that are simply impossible to solve without theory. Generally speaking, I tried to spread out the applications so that if you hate the theoretical part then there is still something fun in every chapter. If you don't like the applications then you'll just have to deal (as my little sister says)

Before we begin, I should warn you that I assume quite a few things from the reader. These notes are intended for someone who has already grappled with the problem of constructing proofs. I assume you know the difference between \Rightarrow and \Leftrightarrow . I assume the phrase "iff" is known to you. I assume you are ready and willing to do a proof by induction, strong or weak. I assume you know what \mathbb{R} , \mathbb{C} , \mathbb{Q} , \mathbb{N} and \mathbb{Z} denote. I assume you know what a subset of a set is. I assume you know how to prove two sets are equal. I assume you are familar with basic set operations such as union and intersection (although we don't use those much). More importantly, I assume you have started to appreciate that mathematics is more than just calculations. Calculations without context, without theory, are doomed to failure. At a minimum theory and proper mathematics allows you to communicate analytical concepts to other like-educated individuals.

Some of the most seemingly basic objects in mathematics are insidiously complex. We've been taught they're simple since our childhood, but as adults, mathematical adults, we find the actual definitions of such objects as \mathbb{R} or \mathbb{C} are rather involved. I will not attempt to provide foundational arguments to build numbers from basic set theory. I believe it is possible, I think it's well-thought-out mathematics, but we take the existence of the real numbers as an axiom for these notes. We assume that \mathbb{R} exists and that the real numbers possess all their usual properties. In fact, I assume \mathbb{R} , \mathbb{C} , \mathbb{Q} , \mathbb{N} and \mathbb{Z} all exist complete with their standard properties. In short, I assume we have numbers to work with. We leave the rigorization of numbers to a different course.

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Chapter 1

Gauss-Jordan elimination

Gauss-Jordan elimination is an optimal method for solving a system of linear equations. Logically it may be equivalent to methods you are already familar with but the matrix notation is by far the most efficient method. This is important since throughout this course we will be faced with the problem of solving linear equations. Additionally, the Gauss-Jordan produces the *reduced row echelon form*(rref) of the matrix. Given a particular matrix the rref is unique. This is of particular use in theoretical applications.

1.1 systems of linear equations

Let me begin with a few examples before I state the general definition.

Example 1.1.1. Consider the following system of 2 equations and 2 unknowns,

$$x + y = 2$$
$$x - y = 0$$

Adding equations reveals 2x = 2 hence x = 1. Then substitute that into either equation to deduce y = 1. Hence the solution (1,1) is unique

Example 1.1.2. Consider the following system of 2 equations and 2 unknowns,

$$x + y = 2$$
$$3x + 3y = 6$$

We can multiply the second equation by 1/3 to see that it is equivalent to x + y = 2 thus our two equations are in fact the same equation. There are infinitely many equations of the form (x, y) where x + y = 2. In other words, the solutions are (x, 2 - x) for all $x \in \mathbb{R}$.

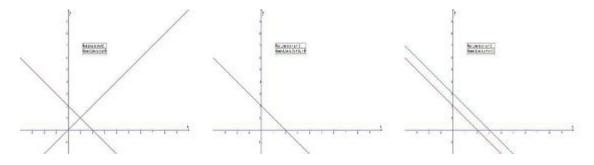
Both of the examples thus far were **consistent**.

Example 1.1.3. Consider the following system of 2 equations and 2 unknowns,

$$x + y = 2$$
$$x + y = 3$$

These equations are inconsistent. Notice substracting the second equation yields that 0 = 1. This system has no solutions, it is inconsistent

It is remarkable that these three simple examples reveal the general structure of solutions to linear systems. Either we get a unique solution, infinitely many solutions or no solution at all. For our examples above, these cases correspond to the possible graphs for a pair of lines in the plane. A pair of lines may intersect at a point (unique solution), be the same line (infinitely many solutions) or be paralell (inconsistent).¹



Remark 1.1.4.

It is understood in this course that i, j, k, l, m, n, p, q, r, s are in N. I will not belabor this point. Please ask if in doubt.

Definition 1.1.5. system of *m*-linear equations in *n*-unknowns

Let x_1, x_2, \ldots, x_m be m variables and suppose $b_i, A_{ij} \in \mathbb{R}$ for $1 \le i \le m$ and $1 \le j \le n$ then $A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1$ $A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2$ $\vdots \quad \vdots \quad \vdots \quad \vdots$ $A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m$

is called a system of linear equations. If $b_i = 0$ for $1 \le i \le m$ then we say the system is homogeneous. The solution set is the set of all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^m$ which satisfy all the equations in the system simultaneously.

¹I used the *Graph* program to generate these graphs. It makes nice pictures, these are ugly due to user error.

Remark 1.1.6.

We use variables x_1, x_2, \ldots, x_n mainly for general theoretical statements. In particular problems and especially for applications we tend to defer to the notation x, y, z etc...

Definition 1.1.7.

The augmented coefficient matrix is an array of numbers which provides an abbreviated notation for a system of linear equations.

$\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = b_2 \end{bmatrix}$			$\begin{array}{c} A_{12} \\ A_{22} \end{array}$		$\begin{array}{c} A_{1n} \\ A_{2n} \end{array}$	b_1 b_2	
$\begin{bmatrix} \vdots \vdots \vdots \vdots \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = b_m \end{bmatrix}$	abbreviated by	\vdots A_{m1}	•	•	\vdots A_{mn}	$\vdots \\ b_m$	

The vertical bar is optional, I include it to draw attention to the distinction between the matrix of coefficients A_{ij} and the nonhomogeneous terms b_i . Let's revisit my three simple examples in this new notation. I illustrate the Gauss-Jordan method for each.

Example 1.1.8. The system x + y = 2 and x - y = 0 has augmented coefficient matrix:

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 1 & -1 & | & 0 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & | & 2 \\ 0 & -2 & | & -2 \end{bmatrix}$$
$$\underbrace{r_2/-2 \to r_2} \begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$$

The last augmented matrix represents the equations x = 1 and y = 1. Rather than adding and subtracting equations we added and subtracted rows in the matrix. Incidentally, the last step is called the **backward pass** whereas the first couple steps are called the **forward pass**. Gauss is credited with figuring out the forward pass then Jordan added the backward pass. Calculators can accomplish these via the commands ref (Gauss' row echelon form) and rref (Jordan's reduced row echelon form). In particular,

$$ref\begin{bmatrix} 1 & 1 & | & 2\\ 1 & -1 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & | & 2\\ 0 & 1 & | & 1 \end{bmatrix} \qquad rref\begin{bmatrix} 1 & 1 & | & 2\\ 1 & -1 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 1\\ 0 & 1 & | & 1 \end{bmatrix}$$

Example 1.1.9. The system x + y = 2 and 3x + 3y = 6 has augmented coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row in the last augmented matrix represents the equation x + y = 2. In this case we cannot make a backwards pass so the ref and rref are the same.

Example 1.1.10. The system x + y = 3 and x + y = 2 has augmented coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The last row indicates that 0x+0y = 1 which means that there is no solution since $0 \neq 1$. Generally, when the bottom row of the rref(A|b) is zeros with a 1 in the far right column then the system Ax = b is inconsistent because there is no solution to the equation.

1.2 Gauss-Jordan algorithm

To begin we need to identify three basic operations we do when solving systems of equations. I'll define them for system of 3 equations and 3 unknowns, but it should be obvious this generalizes to m equations and n unknowns without much thought. The following operations are called **Elementary Row Operations**.

(1.) scaling row 1 by nonzero constant c

$\begin{bmatrix} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{bmatrix} \xrightarrow{cr_1 \to r_1}$	$\begin{bmatrix} cA_{11} & cA_{12} & cA_{13} & cb_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{bmatrix}$
---	--

(2.) replace row 1 with the sum of row 1 and row 2

$\begin{bmatrix} A_{11} & A_{12} & A_{13} & b_1 \end{bmatrix}$		$A_{11} + A_{21}$	$A_{12} + A_{22}$	$A_{13} + A_{23}$	$b_1 + b_2$	1
$A_{21} A_{22} A_{23} b_2$	$r_1 + r_2 \rightarrow r_1$	A_{21}	A_{22}		b_2	
$\left[\begin{array}{cccc} A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$,	A_{31}	A_{32}	A_{33}	b_3]

(3.) swap rows 1 and 2

$\begin{bmatrix} A_{11} & A \end{bmatrix}$	$A_{12} A_{13} \mid b_{13}$	1	A_{21}	A_{22}	A_{23}	b_2
A_{21} A	$A_{22} A_{23} b_2$	$\left] \xrightarrow{r_1 \longleftrightarrow r_2} \right.$	A_{11}	A_{12}	A_{13}	b_1
$\begin{bmatrix} A_{31} & A \end{bmatrix}$	$A_{32} A_{33} \mid b_3$		$\begin{bmatrix} A_{31} \end{bmatrix}$	A_{32}	A_{33}	b_3

Each of the operations above corresponds to an allowed operation on a system of linear equations. When we make these operations we will not change the solution set. Notice the notation tells us what we did and also where it is going. I do expect you to use the same notation. I also expect you can figure out what is meant by $cr_2 \rightarrow r_2$ or $r_1 - 3r_2 \rightarrow r_1$. We are only allowed to make a finite number of the operations (1.),(2.) and (3.). The Gauss-Jordan algorithm tells us which order to make these operations in order to reduce the matrix to a particularly simple format called the "reduced row echelon form" (I abbreviate this rref most places). The following definition is borrowed from the text *Elementary Linear Algebra: A Matrix Approach*, 2nd ed. by Spence, Insel and Friedberg, however you can probably find nearly the same algorithm in dozens of other texts.

Definition 1.2.1. Gauss-Jordan Algorithm.

Given an m by n matrix A the following sequence of steps is called the Gauss-Jordan algorithm or Gaussian elimination. I define terms such as **pivot column** and **pivot position** as they arise in the algorithm below.

- **Step 1:** Determine the leftmost nonzero column. This is a **pivot column** and the topmost position in this column is a **pivot position**.
- **Step 2:** Perform a row swap to bring a nonzero entry of the pivot column below the pivot row to the top position in the pivot column (in the first step there are no rows above the pivot position, however in future iterations there may be rows above the pivot position, see 4).
- **Step 3:** Add multiples of the pivot row to create zeros **below** the pivot position. This is called "clearing out the entries below the pivot position".
- **Step 4:** If there is a nonzero row below the pivot row from (3.) then find the next pivot postion by looking for the next nonzero column to the right of the previous pivot column. Then perform steps 1-3 on the new pivot column. When no more nonzero rows below the pivot row are found then go on to step 5.
- **Step 5:** the leftmost entry in each nonzero row is called the **leading entry**. Scale the bottommost nonzero row to make the leading entry 1 and use row additions to clear out any remaining nonzero entries **above** the leading entries.
- **Step 6:** If step 5 was performed on the top row then stop, otherwise apply Step 5 to the next row up the matrix.

Steps (1.)-(4.) are called the **forward pass**. A matrix produced by a foward pass is called the reduced echelon form of the matrix and it is denoted ref(A). Steps (5.) and (6.) are called the **backwards pass**. The matrix produced by completing Steps (1.)-(6.) is called the reduced row echelon form of A and it is denoted rref(A).

The ref(A) is not unique because there may be multiple choices for how Step 2 is executed. On the other hand, it turns out that rref(A) is unique. The proof of uniqueness can be found in Appendix E of your text. The backwards pass takes the ambiguity out of the algorithm. Notice the forward pass goes down the matrix while the backwards pass goes up the matrix.

Example 1.2.2. Given $A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{bmatrix}$ calculate rref(A). $A = \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 2 & 4 & 0 & | & 7 \\ -1 & 3 & 2 & | & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 0 & 6 & | & 5 \\ -1 & 3 & 2 & | & 0 \end{bmatrix} \xrightarrow{r_1 + r_3 \to r_3}$ $\begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 0 & 6 & | & 5 \\ 0 & 5 & -1 & | & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 5 & -1 & | & 1 \\ 0 & 0 & 6 & | & 5 \end{bmatrix} = ref(A)$

that completes the forward pass. We begin the backwards pass,

$$ref(A) = \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 5 & -1 & | & 1 \\ 0 & 0 & 6 & | & 5 \end{bmatrix} \xrightarrow{r_3 \leftarrow \frac{1}{6}r_3} \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 5 & -1 & | & 1 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} \xrightarrow{r_2 + r_3 \leftarrow r_2} \\ \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 0 & 5 & 0 & | & 11/6 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} \xrightarrow{r_1 + 3r_3 \leftarrow r_1} \begin{bmatrix} 1 & 2 & 0 & | & 21/6 \\ 0 & 5 & 0 & | & 11/6 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} \xrightarrow{r_1 - 2r_2 \leftarrow r_1} \begin{bmatrix} 1 & 0 & 0 & | & 83/30 \\ 0 & 1 & 0 & | & 11/30 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} \xrightarrow{r_1 - 2r_2 \leftarrow r_1} \begin{bmatrix} 1 & 0 & 0 & | & 83/30 \\ 0 & 1 & 0 & | & 11/30 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} = rref(A)$$

Example 1.2.3. Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix}$ calculate rref(A).

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_3 - 2r_1 \to r_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{\frac{3r_3 \to r_3}{5r_2 \to r_2}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -15 \\ 0 & 0 & -15 \end{bmatrix} \xrightarrow{\frac{r_3 - r_2 \to r_3}{\frac{-1}{15}r_2 \to r_2}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -15 \\ \frac{-1}{15}r_2 \to r_2} \xrightarrow{\frac{-1}{15}r_2 \to r_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -15 \end{bmatrix} \xrightarrow{\frac{r_3 - r_2 \to r_3}{\frac{-1}{15}r_2 \to r_2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -15 \end{bmatrix} \xrightarrow{r_3 - r_2 \to r_3} \xrightarrow{\frac{r_3 - r_2 \to r_3}{15}r_2 \to r_2} \xrightarrow{r_3 - r_2 \to r_3} \xrightarrow{r_3 - r_3 - r_2 \to r_3} \xrightarrow{r_3 - r_3 - r_2 \to r_3} \xrightarrow{r_3 - r_3 - r_3} \xrightarrow{r_3 - r_3 \to r_3} \xrightarrow{r_3 - r_3 - r_3} \xrightarrow{r_3 - r_3}$$

Note it is customary to read multiple row operations from top to bottom if more than one is listed between two of the matrices. The multiple arrow notation should be used with caution as it has great potential to confuse. Also, you might notice that I did not strictly-speaking follow Gauss-Jordan in the operations $3r_3 \rightarrow r_3$ and $5r_2 \rightarrow r_2$. It is sometimes convenient to modify the algorithm slightly in order to avoid fractions.

Example 1.2.4. *easy examples are sometimes disquieting, let* $r \in \mathbb{R}$ *,*

$$v = \begin{bmatrix} 2 & -4 & 2r \end{bmatrix} \xrightarrow{\frac{1}{2}r_1 \to r_1} \boxed{\begin{bmatrix} 1 & -2 & r \end{bmatrix} = rref(v)}$$

Example 1.2.5. here's another next to useless example,

$$v = \begin{bmatrix} 0\\1\\3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1\\0\\3 \end{bmatrix} \xrightarrow{r_3 - 3r_1 \to r_3} \begin{bmatrix} 1\\0\\0 \end{bmatrix} = rref(v)$$

Example 1.2.6. Find the rref of the matrix A given below:

$$\begin{split} A &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \underbrace{r_2 - r_1 \to r_2}_{1 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \underbrace{r_3 + r_1 \to r_3}_{1 \to r_1 \to r_3} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 & 2 \end{bmatrix} \underbrace{r_2 \leftrightarrow r_3}_{2 \to r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & -2 & 0 & -1 & 0 \end{bmatrix} \underbrace{r_3 + 2r_2 \to r_3}_{1 \to r_1} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{4r_1 \to r_1}{2r_2 \to r_2}}_{2r_2 \to r_2} \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{r_2 - r_3 \to r_2}{r_1 - r_3 \to r_1}}_{1 \to r_1 \to r_1} \\ \begin{bmatrix} 4 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{r_1 - 2r_2 \to r_1}_{0 \to 1 \to 1} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{r_1 - 2r_2 \to r_1}{r_2 \to r_2}}_{1 \to r_1} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{r_1 - 2r_2 \to r_1}{r_2 \to r_1}}_{1 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{r_1 - 2r_2 \to r_1}{r_2 \to r_2}}_{1 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 / 2 \to r_2}}_{1 \to r_2 \to r_1} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_1 \to r_1} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_1 \to r_1} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r_1}{r_2 \to r_2}}_{1 \to r_2 \to r_2} \underbrace{\frac{r_1 / 4 \to r$$

Example 1.2.7.

$$\begin{bmatrix} A|I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 2 & 0 & | & 0 & 1 & 0 \\ 4 & 4 & 4 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2}_{r_3 - 4r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 2 & 0 & | & -2 & 1 & 0 \\ 0 & 4 & 4 & | & -4 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - 2r_2 \to r_3}_{r_3 - 2r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 2 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 4 & | & 0 & -2 & 1 \end{bmatrix} \xrightarrow{r_3/4 \to r_3}_{r_3/4 \to r_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1/2 & 0 \\ 0 & 0 & 1 & | & 0 & -1/2 & 1/4 \end{bmatrix} = rref[A|I]$$

Example 1.2.8.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{r_4 - 3r_1 \to r_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & -3 & 0 \end{bmatrix} \xrightarrow{r_4 - r_2 \to r_4} \xrightarrow{r_4 \to r_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} \xrightarrow{r_4 - r_2 \to r_4} \xrightarrow{r_4 \to r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \xrightarrow{r_4 + r_3 \to r_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_4 \to r_3} \xrightarrow{r_4 \to r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_4 \to r_3} \xrightarrow{r_1 \to r_3 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = rref(A)$$

Proposition 1.2.9.

If a particular column of a matrix is all zeros then it will be unchanged by the Gaussian elimination. Additionally, if we know rref(A) = B then rref[A|0] = [B|0] where 0 denotes one or more columns of zeros.

Proof: adding nonzero multiples of one row to another will result in adding zero to zero in the column. Likewise, if we multiply a row by a nonzero scalar then the zero column is uneffected. Finally, if we swap rows then this just interchanges two zeros. Gauss-Jordan elimination is just a finite sequence of these three basic row operations thus the column of zeros will remain zero as claimed. \Box

Example 1.2.10. Use Example 1.2.3 and Proposition 1.2.9 to calculate,

$$rref \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 2 & 0 & 0 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Similarly, use Example 1.2.5 and Proposition 1.2.9 to calculate:

	1	0	0	0		1	0	0	0]
rref	0	0	0	0	=	0	0	0	0
rref	3	0	0	0		0	0	0	0

I hope these examples suffice. One last advice, you should think of the Gauss-Jordan algorithm as a sort of road-map. It's ok to take detours to avoid fractions and such but the end goal should remain in sight. If you lose sight of that it's easy to go in circles. Incidentally, I would strongly recommend you find a way to check your calculations with technology. Mathematica will do any matrix calculation we learn. TI-85 and higher will do much of what we do with a few exceptions here and there. There are even websites which will do row operations, I provide a link on the course website. All of this said, I would remind you that I expect you be able perform Gaussian elimination correctly and quickly on the test and quizzes without the aid of a graphing calculator for the remainder of the course. The arithmetic matters. Unless I state otherwise it is expected you show the details of the Gauss-Jordan elimination in any system you solve in this course.

Theorem 1.2.11.

Let $A \in \mathbb{R}^{m \times n}$ then if R_1 and R_2 are both Gauss-Jordan eliminations of A then $R_1 = R_2$. In other words, the reduced row echelon form of a matrix of real numbers is unique.

Proof: see Appendix E in your text for details. This proof is the heart of most calculations we make in this course. \Box

1.3 classification of solutions

Surprisingly Examples 1.1.8,1.1.9 and 1.1.10 illustrate all the possible types of solutions for a linear system. In this section I interpret the calculations of the last section as they correspond to solving systems of equations.

Example 1.3.1. Solve the following system of linear equations if possible,

$$x + 2y - 3z = 1$$

$$2x + 4y = 7$$

$$-x + 3y + 2z = 0$$

We solve by doing Gaussian elimination on the augmented coefficient matrix (see Example 1.2.2 for details of the Gaussian elimination),

$$rref \begin{bmatrix} 1 & 2 & -3 & | & 1 \\ 2 & 4 & 0 & | & 7 \\ -1 & 3 & 2 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 83/30 \\ 0 & 1 & 0 & | & 11/30 \\ 0 & 0 & 1 & | & 5/6 \end{bmatrix} \Rightarrow \begin{vmatrix} x = 83/30 \\ y = 11/30 \\ z = 5/6 \end{vmatrix}$$

(We used the results of Example 1.2.2).

Remark 1.3.2.

The geometric interpretation of the last example is interesting. The equation of a plane with normal vector $\langle a, b, c \rangle$ is ax + by + cz = d. Each of the equations in the system of Example 1.2.2 has a solution set which is in one-one correspondence with a particular plane in \mathbb{R}^3 . The intersection of those three planes is the single point (83/30, 11/30, 5/6).

Example 1.3.3. Solve the following system of linear equations if possible,

$$x - y = 1$$

$$3x - 3y = 0$$

$$2x - 2y = -3$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.3 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & -1 & | & 1 \\ 3 & -3 & | & 0 \\ 2 & -2 & | & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which shows the system has no solutions. The given equations are inconsistent.

Remark 1.3.4.

The geometric interpretation of the last example is also interesting. The equation of a line in the xy-plane is is ax + by = c, hence the solution set of a particular equation corresponds to a line. To have a solution to all three equations at once that would mean that there is an intersection point which lies on all three lines. In the preceding example there is no such point.

Example 1.3.5. Solve the following system of linear equations if possible,

$$x - y + z = 0$$

$$3x - 3y = 0$$

$$2x - 2y - 3z = 0$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ 2 & -2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x - y = 0 \\ z = 0 \end{bmatrix}$$

The row of zeros indicates that we will not find a unique solution. We have a choice to make, either x or y can be stated as a function of the other. Typically in linear algebra we will solve for the variables that correspond to the pivot columns in terms of the non-pivot column variables. In this problem the pivot columns are the first column which corresponds to the variable x and the third column which corresponds the variable z. The variables x, z are called **basic variables** while y is called a **free** variable. The solution set is $[(y, y, 0) | y \in \mathbb{R}]$; in other words, x = y, y = y and z = 0 for all $y \in \mathbb{R}$.

You might object to the last example. You might ask why is y the free variable and not x. This is roughly equivalent to asking the question why is y the dependent variable and x the independent variable in the usual calculus. However, the roles are reversed. In the preceding example the variable x depends on y. Physically there may be a reason to distinguish the roles of one variable over another. There may be a clear cause-effect relationship which the mathematics fails to capture. For example, velocity of a ball in flight depends on time, but does time depend on the ball's velocty ? I'm guessing no. So time would seem to play the role of independent variable. However, when we write equations such as $v = v_o - gt$ we can just as well write $t = \frac{v-v_o}{-g}$; the algebra alone does not reveal which variable should be taken as "independent". Hence, a choice must be made. In the case of infinitely many solutions, we customarily **choose** the pivot variables as the "dependent" or "basic" variables and the non-pivot variables as the "free" variables. Sometimes the word *parameter* is used instead of variable, it is synonomous.

Example 1.3.6. Solve the following (silly) system of linear equations if possible,

$$x = 0$$

$$0x + 0y + 0z = 0$$

$$3x = 0$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

we find the solution set is $\{(0, y, z) \mid y, z \in \mathbb{R}\}\$. No restriction is placed the free variables y and z.

Example 1.3.7. Solve the following system of linear equations if possible,

$$x_1 + x_2 + x_3 + x_4 = x_1 - x_2 + x_3 = 1 -x_1 + x_3 + x_4 = 1$$

1

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.6 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{bmatrix}$$

We find solutions of the form $x_1 = 0$, $x_2 = -x_4/2$, $x_3 = 1 - 3x_4/4$ where $x_4 \in \mathbb{R}$ is free. The solution set is a subset of \mathbb{R}^4 , namely $[(0, -2s, 1 - 3s, 4s) | s \in \mathbb{R}]]$ (I used $s = 4x_4$ to get rid of the annoying fractions).

Remark 1.3.8.

The geometric interpretation of the last example is difficult to visualize. Equations of the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ represent volumes in \mathbb{R}^4 , they're called *hyperplanes*. The solution is parametrized by a single free variable, this means it is a line. We deduce that the three hyperplanes corresponding to the given system intersect along a line. Geometrically solving two equations and two unknowns isn't too hard with some graph paper and a little patience you can find the solution from the intersection of the two lines. When we have more equations and unknowns the geometric solutions are harder to grasp. Analytic geometry plays a secondary role in this course so if you have not had calculus III then don't worry too much. I should tell you what you need to know in these notes.

Example 1.3.9. Solve the following system of linear equations if possible,

$$x_1 + x_4 = 0$$

$$2x_1 + 2x_2 + x_5 = 0$$

$$4x_1 + 4x_2 + 4x_3 = 1$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.7 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & | & 0 \\ 2 & 2 & 0 & 0 & 1 & | & 0 \\ 4 & 4 & 4 & 0 & 0 & | & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & -1 & 1/2 & | & 0 \\ 0 & 0 & 1 & 0 & -1/2 & | & 1/4 \end{bmatrix}$$

Consequently, x_4, x_5 are free and solutions are of the form

$$x_1 = -x_4 x_2 = x_4 - \frac{1}{2}x_5 x_3 = \frac{1}{4} + \frac{1}{2}x_5$$

for all $x_4, x_5 \in \mathbb{R}$.

Example 1.3.10. Solve the following system of linear equations if possible,

```
x_1 + x_3 = 0

2x_2 = 0

3x_3 = 1

3x_1 + 2x_2 = 0
```

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.8 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, there are no solutions .

Example 1.3.11. Solve the following system of linear equations if possible,

$$x_1 + x_3 = 0$$

$$2x_2 = 0$$

$$3x_3 + x_4 = 0$$

$$3x_1 + 2x_2 = 0$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 2 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 1 & | & 0 \\ 3 & 2 & 0 & 0 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \end{bmatrix}$$

Therefore, the unique solution is $x_1 = x_2 = x_3 = x_4 = 0$. The solution set here is rather small, it's $\{(0,0,0,0)\}$.

1.4 applications to curve fitting and circuits

We explore a few fun simple examples in this section. I don't intend for you to master the outs and in's of circuit analysis, those examples are for site-seeing purposes.².

Example 1.4.1. Find a polynomial P(x) whose graph y = P(x) fits through the points (0, -2.7), (2, -4.5) and (1, 0.97). We expect a quadratic polynomial will do nicely here: let A, B, C be the coefficients so $P(x) = Ax^2 + Bx + C$. Plug in the data,

P(0) = C = -2.7		$\int A$				
	\Rightarrow	0	0	1	-2.7	
P(2) = 4A + 2B + C = -4.5	\Rightarrow	4	2	1	-4.5	
P(1) = A + B + C = 0.97		1	1	1	-2.7 -4.5 0.97	

I put in the A, B, C labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

rref	$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$	${0 \\ 2 \\ 1}$	1 1 1	-2.7 -4.5 0.97] =	$\left[\begin{array}{c}1\\0\\0\end{array}\right]$	$\begin{array}{c} 0 \\ 1 \\ 0 \end{array}$	$egin{array}{c} 0 \ 0 \ 1 \end{array}$:	\Rightarrow	A = -4.52 $B = 8.14$ $C = -2.7$
The requested poly	non	nial	l is	P(x) =	= -4.	52x	$^{2} +$	8.1	4x - 2.7	•		

Example 1.4.2. Find which cubic polynomial Q(x) have a graph y = Q(x) which fits through the points (0, -2.7), (2, -4.5) and (1, 0.97). Let A, B, C, D be the coefficients of $Q(x) = Ax^3 + Bx^2 + Cx + D$. Plug in the data,

Q(0) = D = -2.7					D	
)	0	0	1	-2.7
$Q(2) = 8A + 4B + 2C + D = -4.5 \qquad \Rightarrow \qquad Q(1) = A + B + C + D = 0.07$	8	3	4	2	1	-4.5
Q(1) = A + B + C + D = 0.97		-	1	1	1	$\begin{bmatrix} -2.7 \\ -4.5 \\ 0.97 \end{bmatrix}$

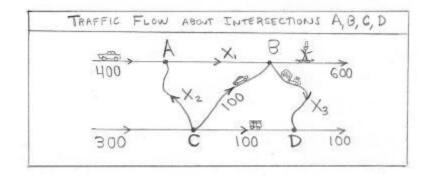
I put in the A, B, C, D labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

 $rref \begin{bmatrix} 0 & 0 & 0 & 1 & | & -2.7 \\ 8 & 4 & 2 & 1 & | & -4.5 \\ 1 & 1 & 1 & 1 & | & 0.97 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5 & 0 & | & -4.07 \\ 0 & 1 & 1.5 & 0 & | & 7.69 \\ 0 & 0 & 0 & 1 & | & -2.7 \end{bmatrix} \Rightarrow \begin{array}{c} A = -4.07 + 0.5C \\ B = 7.69 - 1.5C \\ C = C \\ D = -2.7 \end{array}$

It turns out there is a whole family of cubic polynomials which will do nicely. For each $C \in \mathbb{R}$ the polynomial is $Q_C(x) = (c - 4.07)x^3 + (7.69 - 1.5C)x^2 + Cx - 2.7$ fits the given points. We asked a question and found that it had infinitely many answers. Notice the choice C = 4.07 gets us back to the last example, in that case $Q_C(x)$ is not really a cubic polynomial.

²...well, modulo that homework I asked you to do, but it's not that hard, even a Sagittarian could do it.

Example 1.4.3. Consider the following traffic-flow pattern. The diagram indicates the flow of cars between the intersections A, B, C, D. Our goal is to analyze the flow and determine the missing pieces of the puzzle, what are the flow-rates x_1, x_2, x_3 . We assume all the given numbers are cars per hour, but we omit the units to reduce clutter in the equations.



We model this by one simple principle: conservation of vehicles

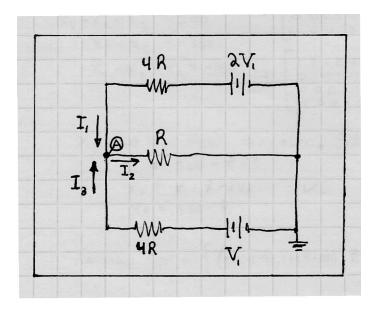
 $A: \quad x_1 - x_2 - 400 = 0$ $B: \quad -x_1 + 600 - 100 + x_3 = 0$ $C: \quad -300 + 100 + 100 + x_2 = 0$ $D: \quad -100 + 100 + x_3 = 0$

This gives us the augmented-coefficient matrix and Gaussian elimination that follows (we have to rearrange the equations to put the constants on the right and the variables on the left before we translate to matrix form)

$$rref \begin{bmatrix} 1 & -1 & 0 & | & 400 \\ -1 & 0 & 1 & | & -500 \\ 0 & 1 & 0 & | & 100 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 500 \\ 0 & 1 & 0 & | & 100 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this we conclude, $x_3 = 0, x_2 = 100, x_1 = 500$. By the way, this sort of system is called **overdetermined** because we have more equations than unknowns. If such a system is consistent they're often easy to solve. In truth, the rref business is completely uncessary here. I'm just trying to illustrate what can happen.

Example 1.4.4. Let $R = 1\Omega$ and $V_1 = 8V$. Determine the voltage V_A and currents I_1, I_2, I_3 flowing in the circuit as pictured below:



Conservation of charge implies the sum of currents into a node must equal the sum of the currents flowing out of the node. We use Ohm's Law V = IR to set-up the currents, here V should be the voltage dropped across the resistor R.

Substitute the first three equations into the fourth to obtain

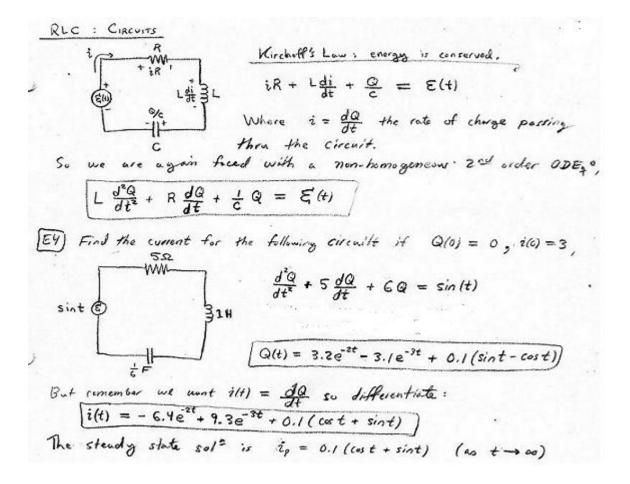
$$\frac{V_A}{R} = \frac{2V_1 - V_A}{4R} + \frac{V_1 - V_A}{4R}$$

Multiply by 4R and we find

$$4V_A = 2V_1 - V_A + V_1 - V_A \Rightarrow 6V_A = 3V_1 \Rightarrow V_A = V_1/2 = 4V.$$

Substituting back into the Ohm's Law equations we determine $I_1 = \frac{16V-4V}{4\Omega} = 3A$, $I_2 = \frac{4V}{1\Omega} = 4A$ and $I_3 = \frac{8V-4V}{4\Omega} = 1A$. This obvious checks with $I_2 = I_1 + I_3$. In practice it's not always best to use the full-power of the rref.

Example 1.4.5. The following is borrowed from my NCSU calculus II notes. The mathematics to solve 2nd order ODEs is actually really simple, but I just quote it here to illustrate something known as the phasor method in electrical engineering.



The basic idea is that a circuit with a sinusoidal source can be treated like a DC circuit if we replace the concept of resistance with the "impedance". The basic formulas are

$$Z_{resistor} = R$$
 $Z_{inductor} = j\omega L$ $Z_{capacitor} = \frac{1}{j\omega C}$

where $j^2 = -1$ and the complex voltage dropped across Z from a sinuisoidal source $\tilde{V} = V_o exp(j\omega t)$ follows a generalized Ohm's Law of $\tilde{V} = \tilde{I}Z$. The picture circuit has $\omega = 1$ and R = 5, L = 1 and C = 1/6 (omitting units) thus the total impedence of the circuit is

$$Z = R + j\omega L + \frac{1}{j\omega C} = 5 + j - 6j = 5 - 5j$$

Then we can calculate $\tilde{I} = \tilde{V}/Z$,

$$\tilde{I} = \frac{exp(jt)}{5-5j} = \frac{(5+5j)exp(jt)}{(5+j)(5-j)} = (0.1+0.1j)exp(jt)$$

Now, this complex formalism actually simultaneously treats a sine and cosine source; $\tilde{V} = exp(jt) = \cos(t) + j\sin(t)$ the term we are interested in are the imaginary components: notice that

$$\tilde{I} = (0.1 + 0.1j)e^{jt} = (0.1 + 0.1j)(\cos(t) + j\sin(t)) = 0.1[\cos(t) - \sin(t)] + 0.1j[\cos(t) + \sin(t)]$$

implies $Im((I)) = 0.1[\cos(t) + \sin(t)]$. We find the steady-state solution $I_p(t) = 0.1[\cos(t) + \sin(t)]$ (this is the solution for large times, there are two other terms in the solution are called transient). The phasor method has replaced differential equations argument with a bit of complex arithmetic. If we had a circuit with several loops and various inductors, capacitors and resistors we could use the complex Ohm's Law and conservation of charge to solve the system in the same way we solved the previous example. If you like this sort of thing you're probably an EE major.

1.5 conclusions

We concluded the last section with a rather believable (but tedious to prove) Theorem. We do the same here,

Theorem 1.5.1.

Given a system of m linear equations and n unknowns the solution set falls into one of the following cases:

- 1. the solution set is empty.
- 2. the solution set has only one element.
- 3. the solution set is infinite.

Proof: Consider the augmented coefficient matrix $[A|b] \in \mathbb{R}^{m \times (n+1)}$ for the system (Theorem 1.2.11 assures us it exists and is unique). Calculate rref[A|b]. If rref[A|b] contains a row of zeros with a 1 in the last column then the system is inconsistent and we find no solutions thus the solution set is empty.

Suppose rref[A|b] does not contain a row of zeros with a 1 in the far right position. Then there are less than n + 1 pivot columns. Suppose there are n pivot columns, let c_i for i = 1, 2, ..., m be the entries in the rightmost column. We find $x_1 = c_1, x_2 = c_2, ..., x_n = c_m$. Consequently the solution set is $\{(c_1, c_2, ..., c_m)\}$.

If rref[A|b] has k < n pivot columns then there are (n + 1 - k)-non-pivot positions. Since the last column corresponds to b it follows there are (n - k) free variables. But, k < n implies 0 < n - k hence there is at least one free variable. Therefore there are infinitely many solutions. \Box

Theorem 1.5.2.

Suppose that $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times p}$ then the first *n* columns of rref[A] and rref[A|B] are identical.

Proof: The forward pass of the elimination proceeds from the leftmost-column to the rightmostcolumn. The matrices A and [A|B] have the same *n*-leftmost columns thus the *n*-leftmost columns are identical after the forward pass is complete. The backwards pass acts on column at a time just clearing out above the pivots. Since the ref(A) and ref[A|B] have identical *n*-leftmost columns the backwards pass modifies those columns in the same way. Thus the *n*-leftmost columns of Aand [A|B] will be identical. \Box

The proofs in the Appendix of the text may appeal to you more if you are pickier on these points.

Theorem 1.5.3.

Given *n*-linear equations in *n*-unknowns Ax = b, a unique solution x exists iff rref[A|b] = [I|x]. Moreover, if $rref[A] \neq I$ then there is no unique solution to the system of equations.

Proof: If a unique solution $x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$ exists for a given system of equations Ax = b then we know

$$A_{i1}c_1 + A_{i2}c_2 + \dots + A_{in}c_n = b_i$$

for each i = 1, 2, ..., n and this is the only ordered set of constants which provides such a solution. Suppose that $rref[A|b] \neq [I|c]$. If rref[A|b] = [I|d] and $d \neq c$ then d is a new solution thus the solution is not unique, this contradicts the given assumption. Consider, on the other hand, the case rref[A|b] = [J|f] where $J \neq I$. If there is a row where f is nonzero and yet J is zero then the system is inconsistent. Otherwise, there are infinitely many solutions since J has at least one non-pivot column as $J \neq I$. Again, we find contradictions in every case except the claimed result. It follows if x = c is the unique solution then rref[A|b] = [I|c]. The converse follows essentially the same argument, if rref[A|b] = [I|c] then clearly Ax = b has solution x = c and if that solution were not unique then we be able to find a different rref for [A|b] but that contradicts the uniqueness of rref. \Box

There is much more to say about the meaning of particular patterns in the reduced row echelon form of the matrix. We will continue to mull over these matters in later portions of the course. Theorem 1.5.1 provides us the big picture. Again, I find it remarkable that two equations and two unknowns already revealed these patterns.

Remark 1.5.4.

Incidentally, you might notice that the Gauss-Jordan algorithm did not assume all the structure of the real numbers. For example, we never needed to use the ordering relations $\langle \text{ or } \rangle$. All we needed was addition, subtraction and the ability to multiply by the inverse of a nonzero number. Any **field** of numbers will likewise work. Theorems 1.5.1 and 1.2.11 also hold for matrices of rational (\mathbb{Q}) or complex (\mathbb{C}) numbers. We will encounter problems which require calculation in \mathbb{C} . If you are interested in encryption then calculations over a finite field \mathbb{Z}_p are necessary. In contrast, Gaussian elimination does not work for matrices of integers since we do not have fractions to work with in that context. Some such questions are dealt with in Abstract Algebra I and II.

Chapter 2

matrix arithmetic

In the preceding chapter I have used some matrix terminolgy in passing as if you already knew the meaning of such terms as "row", "column" and "matrix". I do hope you have had some previous exposure to basic matrix math, but this chapter should be self-contained. I'll start at the beginning and define all the terms.

2.1 basic terminology and notation

Definition 2.1.1.

An $m \times n$ matrix is an array of numbers with m rows and n columns. The elements in the array are called entries or components. If A is an $m \times n$ matrix then A_{ij} denotes the number in the *i*-th row and the *j*-th column. The label *i* is a row index and the index *j* is a column index in the preceding sentence. We usually denote $A = [A_{ij}]$. The set $m \times n$ of matrices with real number entries is denoted $\mathbb{R}^{m \times n}$. The set of $m \times n$ matrices with complex entries is $\mathbb{C}^{m \times n}$. If a matrix has the same number of rows and columns then it is called a square matrix.

Matrices can be constructed from set-theoretic arguments in much the same way as Cartesian Products. I will not pursue those matters in these notes. We will assume that everyone understands how to construct an array of numbers.

Example 2.1.2. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. We see that A has 2 rows and 3 columns thus $A \in \mathbb{R}^{2 \times 3}$. Moreover, $A_{11} = 1$, $A_{12} = 2$, $A_{13} = 3$, $A_{21} = 4$, $A_{22} = 5$, and $A_{23} = 6$. It's not usually possible to find a formula for a generic element in the matrix, but this matrix satisfies $A_{ij} = 3(i-1) + j$ for all i, j.

In the statement "for all i, j" it is to be understood that those indices range over their allowed values. In the preceding example $1 \le i \le 2$ and $1 \le j \le 3$.

Definition 2.1.3.

Two matrices A and B are equal if and only if they have the same size and $A_{ij} = B_{ij}$ for all i, j.

If you studied vectors before you should identify this is precisely the same rule we used in calculus III. Two vectors were equal iff all the components matched. Vectors are just specific cases of matrices so the similarity is not surprising.

Definition 2.1.4.

Let $A \in \mathbb{R}^{m \times n}$ then a submatrix of A is a matrix which is made of some rectangle of elements in A. Rows and columns are submatrices. In particular,

1. An $m \times 1$ submatrix of A is called a column vector of A. The *j*-th **column vector** is denoted $col_j(A)$ and $(col_j(A))_i = A_{ij}$ for $1 \le i \le m$. In other words,

$$col_{k}(A) = \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{bmatrix} \Rightarrow A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = [col_{1}(A)|col_{2}(A)|\cdots|col_{n}(A)]$$

2. An $1 \times n$ submatrix of A is called a row vector of A. The *i*-th **row vector** is denoted $row_i(A)$ and $(row_i(A))_j = A_{ij}$ for $1 \le j \le n$. In other words,

$row_k(A) = \begin{bmatrix} A_{k1} & A_{k2} & \cdots \end{bmatrix}$	$A_{kn}] \Rightarrow A =$	$\begin{array}{c} A_{11} \\ A_{21} \end{array}$	$\begin{array}{c} A_{21} \\ A_{22} \end{array}$	· · · ·	$\begin{array}{c} A_{1n} \\ A_{2n} \end{array}$		$\frac{row_1(A)}{row_2(A)}$
		\vdots A_{m1}	\vdots A_{m2}	•••	\vdots A_{mn}	=	$\boxed{row_m(A)}$

Suppose $A \in \mathbb{R}^{m \times n}$, note for $1 \leq j \leq n$ we have $col_j(A) \in \mathbb{R}^{m \times 1}$ whereas for $1 \leq i \leq m$ we find $row_i(A) \in \mathbb{R}^{1 \times n}$. In other words, an $m \times n$ matrix has n columns of length m and n rows of length m.

Example 2.1.5. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The columns of A are,

$$col_1(A) = \begin{bmatrix} 1\\4 \end{bmatrix}$$
, $col_2(A) = \begin{bmatrix} 2\\5 \end{bmatrix}$, $col_3(A) = \begin{bmatrix} 3\\6 \end{bmatrix}$.

The rows of A are

$$row_1(A) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$
, $row_2(A) = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$

Definition 2.1.6.

Let $A \in \mathbb{R}^{m \times n}$ then $A^T \in \mathbb{R}^{n \times m}$ is called the **transpose** of A and is defined by $(A^T)_{ji} = A_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Example 2.1.7. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. Notice that $row_1(A) = col_1(A^T), row_2(A) = col_2(A^T)$

and

$$col_1(A) = row_1(A^T), \ col_2(A) = row_2(A^T), \ col_3(A) = row_3(A^T)$$

Notice $(A^T)_{ij} = A_{ji} = 3(j-1) + i$ for all i, j; at the level of index calculations we just switch the indices to create the transpose.

The preceding example shows us that we can quickly create the transpose of a given matrix by switching rows to columns. The transpose of a row vector is a column vector and vice-versa.

Remark 2.1.8.

It is customary in analytic geometry to denote $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i\}$ as the set of points in *n*-dimensional space. There is a natural correspondence between points and vectors. Notice that $\mathbb{R}^{1 \times n} = \{[x_1 \ x_2 \ \cdots \ x_n] \mid x_i \in \mathbb{R} \text{ for all } i\}$ and $\mathbb{R}^{n \times 1} = \{[x_1 \ x_2 \ \cdots \ x_n]^T \mid x_i \in \mathbb{R} \text{ for all } i\}$ are naturally identified with \mathbb{R}^n . There is a bijection between points and row or column vectors. For example, $\Phi : \mathbb{R}^{n \times 1} \to \mathbb{R}^{1 \times n}$ defined by transposition

 $\Phi[x_1 \ x_2 \ \dots \ x_n] = [x_1 \ x_2 \ \cdots x_n]^T$

gives a one-one correspondence between row and column vectors. It is customary to use \mathbb{R}^n in the place of $\mathbb{R}^{1 \times n}$ or $\mathbb{R}^{n \times 1}$ when it is convenient. This means I can express solutions to linear systems as a column vector or as a point. For example,

$$x+y=2, \ x-y=0$$

has solution can be denoted by "x = 1, y = 1", or (1, 1), or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. In these notes I tend to be rather pedantic, other texts use \mathbb{R}^n where I have used $\mathbb{R}^{n \times 1}$ to be precise. However, this can get annoying. I you prefer you may declare $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ at the beginning of a problem and use that notation throughout.

2.2 addition and multiplication by scalars

Definition 2.2.1.

Let $A, B \in \mathbb{R}^{m \times n}$ then $A + B \in \mathbb{R}^{m \times n}$ is defined by $(A + B)_{ij} = A_{ij} + B_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$. If two matrices A, B are not of the same size then there sum is not defined.

Example 2.2.2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate

$$A+B = \left[\begin{array}{rrr} 1 & 2\\ 3 & 4 \end{array}\right] + \left[\begin{array}{rrr} 5 & 6\\ 7 & 8 \end{array}\right] = \left[\begin{array}{rrr} 6 & 8\\ 10 & 12 \end{array}\right].$$

Definition 2.2.3.

Let $A, B \in \mathbb{R}^{m \times n}, c \in \mathbb{R}$ then $cA \in \mathbb{R}^{m \times n}$ is defined by $(cA)_{ij} = cA_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$. We call the process of multiplying A by a number c **multiplication by a scalar**. We define $A-B \in \mathbb{R}^{m \times n}$ by A-B = A+(-1)B which is equivalent to $(A-B)_{ij} = A_{ij}-B_{ij}$ for all i, j.

Example 2.2.4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}.$$

Now multiply A by the scalar 5,

$$5A = 5\begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10\\ 15 & 20 \end{bmatrix}$$

Example 2.2.5. Let $A, B \in \mathbb{R}^{m \times n}$ be defined by $A_{ij} = 3i + 5j$ and $B_{ij} = i^2$ for all i, j. Then we can calculate $(A + B)_{ij} = 3i + 5j + i^2$ for all i, j.

Definition 2.2.6.

The **zero matrix** in $\mathbb{R}^{m \times n}$ is denoted 0 and defined by $0_{ij} = 0$ for all i, j. The additive inverse of $A \in \mathbb{R}^{m \times n}$ is the matrix -A such that A + (-A) = 0. The components of the additive inverse matrix are given by $(-A)_{ij} = -A_{ij}$ for all i, j.

The zero matrix joins a long list of other objects which are all denoted by 0. Usually the meaning of 0 is clear from the context, the size of the zero matrix is chosen as to be consistent with the equation in which it is found.

Example 2.2.7. Solve the following matrix equation,

$$0 = \left[\begin{array}{cc} x & y \\ z & w \end{array} \right] + \left[\begin{array}{cc} -1 & -2 \\ -3 & -4 \end{array} \right]$$

Equivalently,

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} x-1 & y-2 \\ z-3 & w-4 \end{array}\right]$$

The definition of matrix equality means this single matrix equation reduces to 4 scalar equations: 0 = x - 1, 0 = y - 2, 0 = z - 3, 0 = w - 4. The solution is x = 1, y = 2, z = 3, w = 4.

Theorem 2.2.8.

If $A \in \mathbb{R}^{m \times n}$ then 1. $0 \cdot A = 0$, (where 0 on the L.H.S. is the number zero) 2. 0A = 0, 3. A + 0 = 0 + A = A.

Proof: I'll prove (2.). Let $A \in \mathbb{R}^{m \times n}$ and consider

$$(0A)_{ij} = \sum_{k=1}^{m} 0_{ik} A_{kj} = \sum_{k=1}^{m} 0A_{kj} = \sum_{k=1}^{m} 0 = 0$$

for all i, j. Thus 0A = 0. I leave the other parts to the reader, the proofs are similar. \Box

2.3 matrix multiplication

The definition of matrix multiplication is natural for a variety of reasons. See your text for a less abrupt definition. I'm getting straight to the point here.

Definition 2.3.1.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the product of A and B is denoted by juxtaposition AB and $AB \in \mathbb{R}^{m \times p}$ is defined by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for each $1 \le i \le m$ and $1 \le j \le p$. In the case m = p = 1 the indices i, j are omitted in the equation since the matrix product is simply a number which needs no index.

This definition is very nice for general proofs, but pragmatically I usually think of matrix multiplication in terms of *dot-products*.

Definition 2.3.2.

Let $v, w \in \mathbb{R}^{n \times 1}$ then $v \cdot w = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \sum_{k=1}^n v_k w_k$

Proposition 2.3.3.

Let $v, w \in \mathbb{R}^{n \times 1}$ then $v \cdot w = v^T w$.

Proof: Since v^T is an $1 \times n$ matrix and w is an $n \times 1$ matrix the definition of matrix multiplication indicates $v^T w$ should be a 1×1 matrix which is a number. Note in this case the outside indices ij are absent in the boxed equation so the equation reduces to

$$v^{T}w = v^{T}_{1}w_{1} + v^{T}_{2}w_{2} + \dots + v^{T}_{n}w_{n} = v_{1}w_{1} + v_{2}w_{2} + \dots + v_{n}w_{n} = v \cdot w.\Box$$

Proposition 2.3.4.

The formula given below is equivalent to the Definition 2.3.1. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then

$$AB = \begin{bmatrix} row_1(A) \cdot col_1(B) & row_1(A) \cdot col_2(B) & \cdots & row_1(A) \cdot col_p(B) \\ row_2(A) \cdot col_1(B) & row_2(A) \cdot col_2(B) & \cdots & row_2(A) \cdot col_p(B) \\ \vdots & \vdots & \ddots & \vdots \\ row_m(A) \cdot col_1(B) & row_m(A) \cdot col_2(B) & \cdots & row_m(A) \cdot col_p(B) \end{bmatrix}$$

Proof: The formula above claims $(AB)_{ij} = row_i(A) \cdot col_j(B)$ for all i, j. Recall that $(row_i(A))_k = A_{ik}$ and $(col_j(B))_k = B_{kj}$ thus

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} (row_i(A))_k (col_j(B))_k$$

Hence, using definition of the dot-product, $(AB)_{ij} = row_i(A) \cdot col_j(B)$. This argument holds for all i, j therefore the Proposition is true. \Box

Example 2.3.5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} [1,2][5,7]^T & [1,2][6,8]^T \\ [3,4][5,7]^T & [3,4][6,8]^T \end{bmatrix}$$
$$= \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix}$$
$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Notice the product of square matrices is square. For numbers $a, b \in \mathbb{R}$ it we know the product of a

and b is commutative (ab = ba). Let's calculate the product of A and B in the opposite order,

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} [5,6][1,3]^T & [5,6][2,4]^T \\ [7,8][1,3]^T & [7,8][2,4]^T \end{bmatrix}$$
$$= \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix}$$
$$= \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Clearly $AB \neq BA$ thus matrix multiplication is noncommutative or nonabelian.

When we say that matrix multiplication is noncommutive that indicates that the product of two matrices does not *generally* commute. However, there are special matrices which commute with other matrices.

Example 2.3.6. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We calculate

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Likewise calculate,

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the matrix A was arbitrary we conclude that IA = AI for all $A \in \mathbb{R}^{2 \times 2}$.

Definition 2.3.7.

The identity matrix in $\mathbb{R}^{n \times n}$ is the $n \times n$ square matrix I which has components $I_{ij} = \delta_{ij}$. The notation I_n is sometimes used if the size of the identity matrix needs emphasis, otherwise the size of the matrix I is to be understood from the context.

	1	- 1	0	0 7		1	0	0	0	
$I_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$	т		1	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	т	0	1	0	0	
	$I_3 =$	0	1		$I_4 =$	0	0	1	0	
	l	- 0	0	ŢŢ	$I_4 =$	0	0	0	1	

Example 2.3.8. The product of a 3×2 and 2×3 is a 3×3

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} [1,0][4,7]^T & [1,0][5,8]^T & [1,0][6,9]^T \\ [0,1][4,7]^T & [0,1][5,8]^T & [0,1][6,9]^T \\ [0,0][4,7]^T & [0,0][5,8]^T & [0,0][6,9]^T \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.3.9. The product of a 3×1 and 1×3 is a 3×3

$$\begin{bmatrix} 1\\2\\3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 5 \cdot 1 & 6 \cdot 1\\4 \cdot 2 & 5 \cdot 2 & 6 \cdot 2\\4 \cdot 3 & 5 \cdot 3 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18 \end{bmatrix}$$

Example 2.3.10. The product of a 2×2 and 2×1 is a 2×1 . Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $v = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$,

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} [1,2][5,7]^T \\ [3,4][5,7]^T \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Likewise, define $w = \begin{bmatrix} 6\\ 8 \end{bmatrix}$ and calculate

$$Aw = \begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6\\ 8 \end{bmatrix} = \begin{bmatrix} [1,2][6,8]^T\\ [3,4][6,8]^T \end{bmatrix} = \begin{bmatrix} 22\\ 50 \end{bmatrix}$$

Something interesting to observe here, recall that in Example 2.3.5 we calculated $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$

 $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. But these are the same numbers we just found from the two matrix-vector products calculated above. We identify that B is just the **concatenation** of the vectors v and w; $B = [v|w] = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Observe that:

$$AB = A[v|w] = [Av|Aw].$$

The term **concatenate** is sometimes replaced with the word **adjoin**. I think of the process as gluing matrices together. This is an important operation since it allows us to lump together many solutions into a single matrix of solutions. (I will elaborate on that in detail in a future section)

Proposition 2.3.11.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then we can understand the matrix multiplication of A and B as the concatenation of several matrix-vector products,

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

Proof: see the Problem Set. You should be able to follow the same general strategy as the Proof of Proposition 2.3.4. Show that the i, j-th entry of the L.H.S. is equal to the matching entry on the R.H.S. Good hunting. \Box

Example 2.3.12. Consider A, v, w from Example 2.3.10.

$$v + w = \begin{bmatrix} 5\\7 \end{bmatrix} + \begin{bmatrix} 6\\8 \end{bmatrix} = \begin{bmatrix} 11\\15 \end{bmatrix}$$

Using the above we calculate,

$$A(v+w) = \begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix} \begin{bmatrix} 11\\ 15 \end{bmatrix} = \begin{bmatrix} 11+30\\ 33+60 \end{bmatrix} = \begin{bmatrix} 41\\ 93 \end{bmatrix}.$$

In constrast, we can add Av and Aw,

$$Av + Aw = \begin{bmatrix} 19\\43 \end{bmatrix} + \begin{bmatrix} 22\\50 \end{bmatrix} = \begin{bmatrix} 41\\93 \end{bmatrix}.$$

Behold, A(v+w) = Av + Aw for this example. It turns out this is true in general.

I collect all my favorite properties for matrix multiplication in the theorem below. To summarize, matrix math works as you would expect with the exception that matrix multiplication is not commutative. We must be careful about the order of letters in matrix expressions.

Theorem 2.3.13.

If
$$A, B, C \in \mathbb{R}^{m \times n}, X, Y \in \mathbb{R}^{n \times p}, Z \in \mathbb{R}^{p \times q}$$
 and $c_1, c_2 \in \mathbb{R}$ then
1. $(A + B) + C = A + (B + C),$
2. $(AX)Z = A(XZ),$
3. $A + B = B + A,$
4. $c_1(A + B) = c_1A + c_2B,$
5. $(c_1 + c_2)A = c_1A + c_2A,$
6. $(c_1c_2)A = c_1(c_2A),$
7. $(c_1A)X = c_1(AX) = A(c_1X) = (AX)c_1,$
8. $1A = A,$
9. $I_mA = A = AI_n,$
10. $A(X + Y) = AX + AY,$
11. $A(c_1X + c_2Y) = c_1AX + c_2AY,$
12. $(A + B)X = AX + BX,$

Proof: I will prove a couple of these and relegate most of the rest to the Problem Set. They actually make pretty fair proof-type test questions. Nearly all of these properties are proved by breaking the statement down to components then appealing to a property of real numbers. Just a reminder, we assume that it is known that \mathbb{R} is an ordered field. Multiplication of real numbers is commutative, associative and distributes across addition of real numbers. Likewise, addition of

real numbers is commutative, associative and obeys familar distributive laws when combined with addition.

Proof of (1.): assume A, B, C are given as in the statement of the Theorem. Observe that

$((A+B)+C)_{ij}$	$= (A+B)_{ij} + C_{ij}$	defn. of matrix add.
	$= (A_{ij} + B_{ij}) + C_{ij}$	defn. of matrix add.
	$= A_{ij} + (B_{ij} + C_{ij})$	assoc. of real numbers
	$= A_{ij} + (B + C)_{ij}$	defn. of matrix add.
	$= (A + (B + C))_{ij}$	defn. of matrix add.

for all i, j. Therefore (A + B) + C = A + (B + C). \Box Proof of (6.): assume c_1, c_2, A are given as in the statement of the Theorem. Observe that

 $\begin{array}{ll} ((c_1c_2)A)_{ij} &= (c_1c_2)A_{ij} & \text{defn. scalar multiplication.} \\ &= c_1(c_2A_{ij}) & \text{assoc. of real numbers} \\ &= (c_1(c_2A))_{ij} & \text{defn. scalar multiplication.} \end{array}$

for all i, j. Therefore $(c_1c_2)A = c_1(c_2A)$. \Box

Proof of (10.): assume A, X, Y are given as in the statement of the Theorem. Observe that

$$\begin{array}{ll} ((A(X+Y))_{ij} &= \sum_k A_{ik}(X+Y)_{kj} & \text{defn. matrix multiplication,} \\ &= \sum_k A_{ik}(X_{kj}+Y_{kj}) & \text{defn. matrix addition,} \\ &= \sum_k (A_{ik}X_{kj}+A_{ik}Y_{kj}) & \text{dist. of real numbers,} \\ &= \sum_k A_{ik}X_{kj} + \sum_k A_{ik}Y_{kj}) & \text{prop. of finite sum,} \\ &= (AX)_{ij} + (AY)_{ij} & \text{defn. matrix multiplication}(\times 2), \\ &= (AX + AY)_{ij} & \text{defn. matrix addition,} \end{array}$$

for all i, j. Therefore A(X + Y) = AX + AY. \Box

The proofs of the other items are similar, we consider the i, j-th component of the identity and then apply the definition of the appropriate matrix operation's definition. This reduces the problem to a statement about real numbers so we can use the properties of real numbers at the level of components. Then we reverse the steps. Since the calculation works for arbitrary i, j it follows the the matrix equation holds true. This Theorem provides a foundation for later work where we may find it convenient to prove a statement without resorting to a proof by components. Which method of proof is best depends on the question. However, I can't see another way of proving most of 2.3.13.

2.4 elementary matrices

Gauss Jordan elimination consists of three *elementary row operations*:

(1.)
$$r_i + ar_j \to r_i$$
, (2.) $br_i \to r_i$, (3.) $r_i \leftrightarrow r_j$

Left multiplication by **elementary matrices** will accomplish the same operation on a matrix.

Definition 2.4.1.

Let $[A: r_i + ar_j \to r_i]$ denote the matrix produced by replacing row *i* of matrix *A* with $row_i(A) + arow_j(A)$. Also define $[A: cr_i \to r_i]$ and $[A: r_i \leftrightarrow r_j]$ in the same way. Let $a, b \in \mathbb{R}$ and $b \neq 0$. The following matrices are called **elementary matrices**:

$$E_{r_i + ar_j \to r_i} = [I : r_i + ar_j \to r_i]$$
$$E_{br_i \to r_i} = [I : br_i \to r_i]$$
$$E_{r_i \leftrightarrow r_j} = [I : r_i \leftrightarrow r_j]$$

Example 2.4.2. Let $A = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}$

$$E_{r_2+3r_1\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3a+1 & 3b+2 & 3c+3 \\ u & m & e \end{bmatrix}$$
$$E_{7r_2\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 7 & 14 & 21 \\ u & m & e \end{bmatrix}$$
$$E_{r_2\to r_3}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & a & 1 \\ u & m & e \end{bmatrix}$$

Proposition 2.4.3.

Let
$$A \in \mathbb{R}^{m \times n}$$
 then there exist elementary matrices E_1, E_2, \ldots, E_k such that $rref(A) = E_1 E_2 \cdots E_k A$.

Proof: Gauss Jordan elimination consists of a sequence of k elementary row operations. Each row operation can be implemented by multiply the corresponding elementary matrix on the left. The Theorem follows. \Box

Example 2.4.4. Just for fun let's see what happens if we multiply the elementary matrices on the

right instead.

$$AE_{r_{2}+3r_{1}\rightarrow r_{2}} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & b & c \\ 1+6 & 2 & 3 \\ u+3m & m & e \end{bmatrix}$$
$$AE_{7r_{2}\rightarrow r_{2}} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 7b & c \\ 1 & 14 & 3 \\ u & 7m & e \end{bmatrix}$$
$$AE_{r_{2}\rightarrow r_{3}} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c & b \\ 1 & 3 & 2 \\ u & e & m \end{bmatrix}$$

Curious, they generate column operations, we might call these elementary column operations. In our notation the row operations are more important.

2.5 invertible matrices

Definition 2.5.1.

Let $A \in \mathbb{R}^{n \times n}$. If there exists $B \in \mathbb{R}^{n \times n}$ such that AB = I and BA = I then we say that A is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix.

Proposition 2.5.2.

Elementary matrices are invertible.

Proof: I list the inverse matrix for each below:

$$(E_{r_i+ar_j\to r_i})^{-1} = [I: r_i - ar_j \to r_i]$$
$$(E_{br_i\to r_i})^{-1} = [I: \frac{1}{b}r_i \to r_i]$$
$$(E_{r_i\leftrightarrow r_j})^{-1} = [I: r_j\leftrightarrow r_i]$$

I leave it to the reader to convince themselves that these are indeed inverse matrices. \Box

Example 2.5.3. Let me illustrate the mechanics of the proof above, $E_{r_1+3r_2\to r_1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{r_1-3r_2\to r_1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfy,

$$E_{r_1+3r_2 \to r_1} E_{r_1-3r_2 \to r_1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likewise,

$$E_{r_1-3r_2\to r_1}E_{r_1+3r_2\to r_1} = \begin{bmatrix} 1 & -3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $(E_{r_1+3r_2\to r_1})^{-1} = E_{r_1-3r_2\to r_1}$ just as we expected.

Theorem 2.5.4.

Let $A \in \mathbb{R}^{n \times n}$. The solution of Ax = 0 is unique iff A^{-1} exists.

Proof:(\Rightarrow) Suppose Ax = 0 has a unique solution. Observe A0 = 0 thus the only solution is the zero solution. Consequently, rref[A|0] = [I|0]. Moreover, by Proposition 2.4.3 there exist elementary matrices E_1, E_2, \dots, E_k such that $rref[A|0] = E_1E_2 \cdots E_k[A|0] = [I|0]$. Applying the concatenation Proposition 2.3.11 we find that $[E_1E_2 \cdots E_kA|E_1E_2 \cdots E_k0] = [I|0]$ thus $E_1E_2 \cdots E_kA = I$.

It remains to show that $AE_1E_2\cdots E_k = I$. Multiply $E_1E_2\cdots E_kA = I$ on the left by E_1^{-1} followed by E_2^{-1} and so forth to obtain

$$E_k^{-1} \cdots E_2^{-1} E_1^{-1} E_1 E_2 \cdots E_k A = E_k^{-1} \cdots E_2^{-1} E_1^{-1} I$$

this simplifies to

$$A = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.$$

Observe that

$$AE_1E_2\cdots E_k = E_k^{-1}\cdots E_2^{-1}E_1^{-1}E_1E_2\cdots E_k = I.$$

We identify that $A^{-1} = E_1 E_2 \cdots E_k$ thus A^{-1} exists.

(\Leftarrow) The converse proof is much easier. Suppose A^{-1} exists. If Ax = 0 then multiply by A^{-1} on the left, $A^{-1}Ax = A^{-1}0 \Rightarrow Ix = 0$ thus x = 0. \Box

Proposition 2.5.5.

Let $A \in \mathbb{R}^{n \times n}$. 1. If BA = I then AB = I. 2. If AB = I then BA = I.

Proof of (1.): Suppose BA = I. If Ax = 0 then BAx = B0 hence Ix = 0. We have shown that Ax = 0 only has the trivial solution. Therefore, Theorem 2.5.4 shows us that A^{-1} exists. Multiply BA = I on the left by A^{-1} to find $BAA^{-1} = IA^{-1}$ hence $B = A^{-1}$ and by definition it follows AB = I.

Proof of (2.): Suppose AB = I. If Bx = 0 then ABx = A0 hence Ix = 0. We have shown that Bx = 0 only has the trivial solution. Therefore, Theorem 2.5.4 shows us that B^{-1} exists. Multiply

AB = I on the right by B^{-1} to find $ABB^{-1} = IB^{-1}$ hence $A = B^{-1}$ and by definition it follows BA = I. \Box

Proposition 2.5.5 shows that we don't need to check both conditions AB = I and BA = I. If either holds the other condition automatically follows.

Proposition 2.5.6.

If $A \in \mathbb{R}^{n \times n}$ is invertible then its inverse matrix is unique.

Proof: Suppose B, C are inverse matrices of A. It follows that AB = BA = I and AC = CA = I thus AB = AC. Multiply B on the left of AB = AC to obtain BAB = BAC hence $IB = IC \Rightarrow B = C$. \Box

Example 2.5.7. In the case of a 2×2 matrix a nice formula to find the inverse is known:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It's not hard to show this formula works,

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+da \end{bmatrix}$$
$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

How did we know this formula? Can you derive it? To find the formula from first principles you could suppose there exists a matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that AB = I. The resulting algebra would lead you to conclude x = d/t, y = -b/t, z = -c/t, w = a/t where t = ad - bc. I leave this as an exercise for the reader.

There is a giant assumption made throughout the last example. What is it?

Example 2.5.8. A counterclockwise rotation by angle θ in the plane can be represented by a matrix $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. The inverse matrix corresponds to a rotation by angle $-\theta$ and (using the even/odd properties for cosine and sine) $R(-\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta)^{-1}$. Notice that $R(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ thus $R(\theta)R(-\theta) = R(0) = I$. We'll talk more about geometry in a later chapter. If you'd like to see how this matrix is related to the imaginary exponential $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ you can look at www.supermath.info/intro_to_complex.pdf where I show how the cosines and sines come from a rotation of the coordinate axes. If you draw the right picture you can understand why the formulas below describe changing the coordinates from (x, y) to $(\bar{x}, \bar{y}$ where the transformed coordinates are rotated by angle θ :

$$\bar{x} = \cos(\theta)x + \sin(\theta)y \\ \bar{y} = -\sin(\theta)x + \cos(\theta)y \qquad \Leftrightarrow \qquad \left[\begin{array}{c} \bar{x} \\ \bar{y} \end{array}\right] = \left[\begin{array}{c} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right]$$

Theorem 2.5.9.

If $A, B \in \mathbb{R}^{n \times n}$ are invertible, $X, Y \in \mathbb{R}^{m \times n}$, $Z, W \in \mathbb{R}^{n \times m}$ and nonzero $c \in \mathbb{R}$ then 1. $(AB)^{-1} = B^{-1}A^{-1}$, 2. $(cA)^{-1} = \frac{1}{c}A^{-1}$, 3. XA = YA implies X = Y, 4. AZ = AW implies Z = W,

Proof: To prove (1.) simply notice that

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

The proof of (2.) follows from the calculation below,

$$(\frac{1}{c}A^{-1})cA = \frac{1}{c}cA^{-1}A = A^{-1}A = I.$$

To prove (3.) assume that XA = YA and multiply both sides by A^{-1} on the right to obtain $XAA^{-1} = YAA^{-1}$ which reveals XI = YI or simply X = Y. To prove (4.) multiply by A^{-1} on the left. \Box

Remark 2.5.10.

The proofs just given were all matrix arguments. These contrast the *c*omponent level proofs needed for 2.3.13. We could give component level proofs for the Theorem above but that is not necessary and those arguments would only obscure the point. I hope you gain your own sense of which type of argument is most appropriate as the course progresses.

We have a simple formula to calculate the inverse of a 2×2 matrix, but sadly no such simple formula exists for bigger matrices. There is a nice method to calculate A^{-1} (if it exists), but we do not have all the theory in place to discuss it at this juncture.

Proposition 2.5.11.

If
$$A_1, A_2, \dots, A_k \in \mathbb{R}^{n \times n}$$
 are invertible then
 $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

Proof: Provided by you in the Problem Set. Your argument will involve induction on the index k. Notice you already have the cases k = 1, 2 from the arguments in this section. In particular, k = 1 is trivial and k = 2 is given by Theorem 2.5.11. \Box

2.6 systems of linear equations revisited

In the previous chapter we found that systems of equations could be efficiently solved by doing row operations on the augmented coefficient matrix. Let's return to that central topic now that we know more about matrix addition and multiplication. The proof of the proposition below is simply matrix multiplication.

Proposition 2.6.1.

Let x_1, x_2, \ldots, x_m be *m* variables and suppose $b_i, A_{ij} \in \mathbb{R}$ for $1 \le i \le m$ and $1 \le j \le n$ then recall that $A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1$

$$A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = b_2$$

: : : : :

$$A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = b_m$$

is called a system of linear equations. We define the coefficient matrix A, the inhomogeneous term b and the vector solution x as follows:

4 —	$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$	$a_{12} \\ a_{22}$	· · · ·	a_{1n} a_{2n}	$\begin{bmatrix} b_1\\b_2 \end{bmatrix}$	$\left[\begin{array}{c} x\\ x\end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
A =	\vdots a_{m1}	\vdots a_{m2}	· · · ·	\vdots a_{mn}	$b = \begin{bmatrix} \vdots \\ b_m \end{bmatrix}$		m

then the system of equations is equivalent to matrix form of the system Ax = b. (sometimes we may use v or \vec{x} if there is danger of confusion with the scalar variable x)

Definition 2.6.2.

Let Ax = b be a system of m equations and n-unknowns and x is in the solution set of the system. In particular, we denote the solution set by $Sol_{[A|b]}$ where

$$Sol_{[A|b]} = \{ x \in \mathbb{R}^{n \times 1} \mid Ax = b \}$$

We learned how to find the solutions to a system Ax = b in the last Chapter by performing Gaussian elimination on the augmented coefficient matrix [A|b]. We'll discuss the structure of matrix solutions further in the next Chapter. To give you a quick preview, it turns out that solutions to Ax = bhave the decomposition $x = x_h + x_p$ where the homogeneous term x_h satisfies $Ax_h = 0$ and the nonhomogeneous term x_p solves $Ax_p = b$.

2.6.1 concatenation for solving many systems at once

If we wish to solve $Ax = b_1$ and $Ax = b_2$ we use a concatenation trick to do both at once. In fact, we can do it for $k \in \mathbb{N}$ problems which share the same coefficient matrix but possibly differing inhomogeneous terms.

Proposition 2.6.3.

Let $A \in \mathbb{R}^{m \times n}$. Vectors v_1, v_2, \ldots, v_k are solutions of $Av = b_i$ for $i = 1, 2, \ldots k$ iff $V = [v_1|v_2|\cdots|v_k]$ solves AV = B where $B = [b_1|b_2|\cdots|b_k]$.

Proof: Let $A \in \mathbb{R}^{m \times n}$ and suppose $Av_i = b_i$ for i = 1, 2, ..., k. Let $V = [v_1|v_2|\cdots|v_k]$ and use the concatenation Proposition 2.3.11,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k] = [b_1|b_2|\cdots|b_k] = B.$$

Conversely, suppose AV = B where $V = [v_1|v_2|\cdots|v_k]$ and $B = [b_1|b_2|\cdots|b_k]$ then by Proposition 2.3.11 AV = B implies $Av_i = b_i$ for each i = 1, 2, ..., k. \Box

Example 2.6.4. Solve the systems given below,

x + y + z = 1		x + y + z = 1
x - y + z = 0	and	x - y + z = 1
-x+z=1		-x + z = 1

The systems above share the same coefficient matrix, however $b_1 = [1, 0, 1]^T$ whereas $b_2 = [1, 1, 1]^T$. We can solve both at once by making an extended augmented coefficient matrix $[A|b_1|b_2]$

	1	1	1	1	1	[1]	0	0	-1/4	0
$[A b_1 b_2] =$	1	-1	1	0	1	$rref[A b_1 b_2] = \begin{bmatrix} 1\\0\\0 \end{bmatrix}$	1	0	1/2	0
	1	0	1	1	1		0	1	3/4	1

We use Proposition 2.6.3 to conclude that

 $\begin{array}{l} x+y+z=1\\ x-y+z=0\\ -x+z=1 \end{array} \qquad has \ solution \ x=-1/4, y=1/2, z=3/4\\ x+y+z=1\\ x-y+z=1\\ -x+z=1 \end{array} \qquad has \ solution \ x=0, y=0, z=1. \end{array}$

2.7 how to calculate the inverse of a matrix

PROBLEM: how should we calculate A^{-1} for a 3×3 matrix ?

Consider that the Proposition 2.6.3 gives us another way to look at the problem,

$$AA^{-1} = I \iff A[v_1|v_2|v_3] = I_3 = [e_1|e_2|e_3]$$

Where $v_i = col_i(A^{-1})$ and $e_1 = [0 \ 0 \ 0]^T$, $e_2 = [0 \ 1 \ 0]^T$, $e_3 = [0 \ 0 \ 1]^T$. We observe that the problem of finding A^{-1} for a 3 × 3 matrix amounts to solving three separate systems:

$$Av_1 = e_1, Av_2 = e_2, Av_3 = e_3$$

when we find the solutions then we can construct $A^{-1} = [v_1|v_2|v_3]$. Think about this, if A^{-1} exists then it is unique thus the solutions v_1, v_2, v_3 are likewise unique. Consequently, by Theorem 1.5.3,

$$rref[A|e_1] = [I|v_1], \ rref[A|e_2] = [I|v_2], \ rref[A|e_3] = [I|v_3].$$

Each of the systems above required the same sequence of elementary row operations to cause $A \mapsto I$. We can just as well do them at the same time in one big matrix calculation:

$$rref[A|e_1|e_2|e_3] = [I|v_1|v_2|v_3]$$

While this discuss was done for n = 3 we can just as well do the same for n > 3. This provides the proof for the first sentence of the theorem below. Theorem 1.5.3 together with the discussion above proves the second sentence.

Theorem 2.7.1.

If $A \in \mathbb{R}^{n \times n}$ is invertible then $rref[A|I] = [I|A^{-1}]$. Otherwise, A^{-1} not invertible iff $rref(A) \neq I$ iff $rref[A|I] \neq [I|B]$.

This is perhaps the most pragmatic theorem so far stated in these notes. This theorem tells us how and when we can find an inverse for a square matrix.

Example 2.7.2. Recall that in Example 1.2.7 we worked out the details of

$$rref \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 2 & 0 & | & 0 & 1 & 0 \\ 4 & 4 & 4 & | & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1/2 & 0 \\ 0 & 0 & 1 & | & 0 & -1/2 & 1/4 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/4 \end{bmatrix}.$$

Example 2.7.3. I omit the details of the Gaussian elimination,

$$rref \begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & -1 & | & 0 & 1 & 0 \\ 6 & 2 & 3 & | & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & -2 & -3 & -1 \\ 0 & 1 & 0 & | & -3 & -3 & -1 \\ 0 & 0 & 1 & | & -2 & -4 & -1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

2.8 all your base are belong to us (e_i and E_{ij} that is)

It is convenient to define special notation for certain basic vectors. First I define a useful shorthand,

Definition 2.8.1.

The symbol
$$\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$$
 is called the **Kronecker delta**.

For example, $\delta_{22} = 1$ while $\delta_{12} = 0$.

Definition 2.8.2.

Let $e_i \in \mathbb{R}^{n \times 1}$ be defined by $(e_i)_j = \delta_{ij}$. The size of the vector e_i is determined by context. We call e_i the *i*-th standard basis vector.

Definition 2.8.3.

A linear combination of objects
$$A_1, A_2, \ldots, A_k$$
 is a sum
 $c_1A_1 + c_2A_2 + \cdots + c_kA_k = \sum_{i=1}^k c_iA_i$
where $c_i \in \mathbb{R}$ for each i .

We will look at linear combinations of vectors, matrices and even functions in this course. If $c_i \in \mathbb{C}$ then we call it a *complex linear combination*.

Proposition 2.8.4.

Every vector in $\mathbb{R}^{n \times 1}$ is a linear combination of e_1, e_2, \ldots, e_n .

Proof: Let $v = [v_1, v_2, \dots, v_n]^T \in \mathbb{R}^{n \times 1}$. By the definition of matrix addition we justify the following steps:

$$v = [v_1, v_2, \dots, v_n]^T$$

= $[v_1, 0, \dots, 0]^T + [0, v_2, \dots, v_n]^T$
= $[v_1, 0, \dots, 0]^T + [0, v_2, \dots, 0]^T + \dots + [0, 0, \dots, v_n]^T$
= $[v_1, 0 \cdot v_1, \dots, 0 \cdot v_1]^T + [0 \cdot v_2, v_2, \dots, 0 \cdot v_2]^T + \dots + [0 \cdot v_n, 0 \cdot v_n, \dots, v_n]^T$

In the last step I rewrote each zero to emphasize that the each entry of the k-th summand has a v_k factor. Continue by applying the definition of scalar multiplication to each vector in the sum above we find,

$$v = v_1[1, 0, \dots, 0]^T + v_2[0, 1, \dots, 0]^T + \dots + v_n[0, 0, \dots, 1]^T$$

= $v_1e_1 + v_2e_2 + \dots + v_ne_n$

Therefore, every vector in $\mathbb{R}^{n \times 1}$ is a linear combination of e_1, e_2, \ldots, e_n . For each $v \in \mathbb{R}^{n \times 1}$ we have $v = \sum_{i=1}^n v_n e_n$. \Box

Definition 2.8.5.

The *ij*-th standard basis matrix for $\mathbb{R}^{m \times n}$ is denoted E_{ij} for $1 \le i \le m$ and $1 \le j \le n$. The matrix E_{ij} is zero in all entries except for the (i, j)-th slot where it has a 1. In other words, we define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

Proposition 2.8.6.

Every matrix in $\mathbb{R}^{m \times n}$ is a linear combination of the E_{ij} where $1 \le i \le m$ and $1 \le j \le n$.

Proof: Let $A \in \mathbb{R}^{m \times n}$ then

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$
$$= A_{11} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + A_{12} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + A_{mn} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$
$$= A_{11}E_{11} + A_{12}E_{12} + \cdots + A_{mn}E_{mn}.$$

The calculation above follows from repeated mn-applications of the definition of matrix addition and another mn-applications of the definition of scalar multiplication of a matrix. We can restate the final result in a more precise langauge,

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} E_{ij}.$$

As we claimed, any matrix can be written as a linear combination of the E_{ij} . \Box

The term "basis" has a technical meaning which we will discuss at length in due time. For now, just think of it as part of the names of e_i and E_{ij} . These are the basic building blocks for matrix theory.

Example 2.8.7. Suppose $A \in \mathbb{R}^{m \times n}$ and $e_i \in \mathbb{R}^{n \times 1}$ is a standard basis vector,

$$(Ae_i)_j = \sum_{k=1}^n A_{jk}(e_i)k = \sum_{k=1}^n A_{jk}\delta_{ik} = A_{ji}$$

Thus, $[Ae_i] = col_i(A)$. We find that multiplication of a matrix A by the standard basis e_i yields the i - th column of A.

Example 2.8.8. Suppose $A \in \mathbb{R}^{m \times n}$ and $e_i \in \mathbb{R}^{m \times 1}$ is a standard basis vector,

$$(e_i^T A)_j = \sum_{k=1}^n (e_i)_k A_{kj} = \sum_{k=1}^n \delta_{ik} A_{kj} = A_{ij}$$

Thus, $\boxed{[e_i^T A] = row_i(A)}$. We find multiplication of a matrix A by the transpose of standard basis e_i yields the i - th row of A.

Example 2.8.9. Again, suppose $e_i, e_j \in \mathbb{R}^{n \times 1}$ are standard basis vectors. The product $e_i^T e_j$ of the $1 \times n$ and $n \times 1$ matrices is just a 1×1 matrix which is just a number. In particular consider,

$$e_i^T e_j = \sum_{k=1}^n (e_i^T)_k (e_j)_k = \sum_{k=1}^n \delta_{ik} \delta_{jk} = \delta_{ij}$$

The product is zero unless the vectors are identical.

Example 2.8.10. Suppose $e_i \in \mathbb{R}^{m \times 1}$ and $e_j \in \mathbb{R}^{n \times 1}$. The product of the $m \times 1$ matrix e_i and the $1 \times n$ matrix e_j^T is an $m \times n$ matrix. In particular,

$$(e_i e_j^T)_{kl} = (e_i^T)_k (e_j)_k = \delta_{ik} \delta_{jk} = (E_{ij})_{kl}$$

Thus we can construct the standard basis matrices by multiplying the standard basis vectors; $E_{ij} = e_i e_j^T$.

Example 2.8.11. What about the matrix E_{ij} ? What can we say about multiplication by E_{ij} on the right of an arbitrary matrix? Let $A \in \mathbb{R}^{m \times n}$ and consider,

$$(AE_{ij})_{kl} = \sum_{p=1}^{n} A_{kp}(E_{ij})_{pl} = \sum_{p=1}^{n} A_{kp}\delta_{ip}\delta_{jl} = A_{ki}\delta_{jl}$$

Notice the matrix above has zero entries unless j = l which means that the matrix is mostly zero except for the j-th column. We can select the j-th column by multiplying the above by e_j , using Examples 2.8.9 and 2.8.7,

$$(AE_{ij}e_j)_k = (Ae_ie_j^T e_j)_k = (Ae_i\delta_{jj})_k = (Ae_i)_k = (col_i(A))_k$$

This means,

$$AE_{ij} = \begin{bmatrix} & & column \ j & & \\ 0 & 0 & \cdots & A_{1i} & \cdots & 0 \\ 0 & 0 & \cdots & A_{2i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_{mi} & \cdots & 0 \end{bmatrix}$$

Right multiplication of matrix A by E_{ij} moves the *i*-th column of A to the *j*-th column of AE_{ij} and all other entries are zero. It turns out that left multiplication by E_{ij} moves the *j*-th row of A to the *i*-th row and sets all other entries to zero.

Example 2.8.12. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ consider multiplication by E_{12} ,

$$AE_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 \mid col_1(A) \end{bmatrix}$$

Which agrees with our general abstract calculation in the previous example. Next consider,

$$E_{12}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} row_2(A) \\ 0 \end{bmatrix}.$$

Example 2.8.13. Calculate the product of E_{ij} and E_{kl} .

$$(E_{ij}E_{kl})_{mn} = \sum_{p} (E_{ij})_{mp} (E_{kl})_{pn} = \sum_{p} \delta_{im} \delta_{jp} \delta_{kp} \delta_{ln} = \delta_{im} \delta_{jk} \delta_{ln}$$

For example,

$$(E_{12}E_{34})_{mn} = \delta_{1m}\delta_{23}\delta_{4n} = 0.$$

In order for the product to be nontrivial we must have j = k,

$$(E_{12}E_{24})_{mn} = \delta_{1m}\delta_{22}\delta_{4n} = \delta_{1m}\delta_{4n} = (E_{14})_{mn}.$$

We can make the same identification in the general calculation,

$$(E_{ij}E_{kl})_{mn} = \delta_{jk}(E_{il})_{mn}.$$

Since the above holds for all m, n,

$$E_{ij}E_{kl} = \delta_{jk}E_{il}$$

this is at times a very nice formula to know about.

Remark 2.8.14.

You may find the general examples in this portion of the notes a bit too much to follow. If that is the case then don't despair. Focus on mastering the numerical examples to begin with then work your way up to the general problems. These examples are actually not that hard, you just have to get used to index calculations. The proofs in these examples are much longer if written without the benefit of index notation. I was disappointed your text fails to use the index notation in it's full power. The text deliberately uses $+ \cdots$ rather than \sum . I will use both langauges.

Example 2.8.15. Let $A \in \mathbb{R}^{m \times n}$ and suppose $e_i \in \mathbb{R}^{m \times 1}$ and $e_i \in \mathbb{R}^{n \times 1}$. Consider,

$$(e_i)^T A e_j = \sum_{k=1}^m ((e_i)^T)_k (A e_j)_k = \sum_{k=1}^m \delta_{ik} (A e_j)_k = (A e_j)_i = A_{ij}$$

This is a useful observation. If we wish to select the (i, j)-entry of the matrix A then we can use the following simple formula,

$$A_{ij} = (e_i)^T A e_j$$

This is analogus to the idea of using dot-products to select particular components of vectors in analytic geometry; (reverting to calculus III notation for a moment) recall that to find v_1 of \vec{v} we learned that the dot product by $\hat{i} = <1, 0, 0>$ selects the first components $v_1 = \vec{v} \cdot \hat{i}$. The following theorem is simply a summary of our results for this section.

Theorem 2.8.16.

Assume $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^{n \times 1}$ and define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $(e_i)_j = \delta_{ij}$ as we previously discussed, $v = \sum_{i=1}^n v_n e_n \qquad A = \sum_{i=1}^m \sum_{j=1}^n A_{ij} E_{ij}.$ $[e_i^T A] = row_i(A) \qquad [Ae_i] = col_i(A) \qquad A_{ij} = (e_i)^T Ae_j$ $E_{ij} E_{kl} = \delta_{jk} E_{il} \qquad E_{ij} = e_i e_j^T \qquad e_i^e_j = \delta_{ij}$

2.9 matrices with names

In this section we learn about a few special types of matrices.

2.9.1 symmetric and antisymmetric matrices

Definition 2.9.1.

Let
$$A \in \mathbb{R}^{n \times n}$$
. We say A is symmetric iff $A^T = A$. We say A is antisymmetric iff $A^T = -A$.

At the level of components, $A^T = A$ gives $A_{ij} = A_{ji}$ for all i, j. Whereas, $A^T = -A$ gives $A_{ij} = -A_{ji}$ for all i, j. I should mention **skew-symmetric** is another word for antisymmetric. In physics, second rank (anti)symmetric tensors correspond to (anti)symmetric matrices. In electromagnetism, the electromagnetic field tensor has components which can be written as an antisymmetric 4×4 matrix. In classical mechanics, a solid propensity to spin in various directions is described by the intertia tensor which is symmetric. The energy-momentum tensor from electrodynamics is also symmetric. Matrices are everywhere if look for them.

Example 2.9.2. Some matrices are symmetric:

$$I, O, E_{ii}, \left[\begin{array}{rrr} 1 & 2\\ 2 & 0 \end{array}\right]$$

Some matrices are antisymmetric:

$$O, \left[\begin{array}{cc} 0 & 2 \\ -2 & 0 \end{array} \right]$$

Only 0 is both symmetric and antisymmetric (can you prove it?). Many other matrices are neither symmetric nor antisymmetric:

$$e_i, E_{i,i+1}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

I assumed n > 1 so that e_i is a column vector which is not square.

Proposition 2.9.3.

Let $A, B \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}$ then 1. $(A^T)^T = A$ 2. $(AB)^T = B^T A^T$ socks-shoes property for transpose of product 3. $(cA)^T = cA^T$ 4. $(A+B)^T = A^T + B^T$ 5. $(A^T)^{-1} = (A^{-1})^T$.

Proof: See the Problem Set. \Box

Proposition 2.9.4.

All square matrices are formed by the sum of a symmetric and antisymmetric matrix.

Proof: Let $A \in \mathbb{R}^{n \times n}$. Utilizing Proposition 2.9.3 we find

$$\left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A^T+(A^T)^T) = \frac{1}{2}(A^T+A) = \frac{1}{2}(A+A^T)$$

thus $\frac{1}{2}(A + A^T)$ is a symmetric matrix. Likewise,

$$\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T)$$

thus $\frac{1}{2}(A - A^T)$ is an antisymmetric matrix. Finally, note the identity below:

$$A = \frac{1}{2}(A + A^{T}) + \frac{1}{2}(A - A^{T})$$

The theorem follows. \Box

The proof that any function on \mathbb{R} is the sum of an even and odd function uses the same trick.

Example 2.9.5. The proof of the Proposition above shows us how to break up the matrix into its symmetric and antisymmetric pieces:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right) + \frac{1}{2} \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Example 2.9.6. What are the symmetric and antisymmetric parts of the standard basis E_{ij} in $\mathbb{R}^{n \times n}$? Here the answer depends on the choice of i, j. Note that $(E_{ij})^T = E_{ji}$ for all i, j. Suppose i = j then $E_{ij} = E_{ii}$ is clearly symmetric, thus there is no antisymmetric part. If $i \neq j$ we use the standard trick,

$$E_{ij} = \frac{1}{2}(E_{ij} + E_{ji}) + \frac{1}{2}(E_{ij} - E_{ji})$$

where $\frac{1}{2}(E_{ij}+E_{ji})$ is the symmetric part of E_{ij} and $\frac{1}{2}(E_{ij}-E_{ji})$ is the antisymmetric part of E_{ij} .

Proposition 2.9.7.

Let $A \in \mathbb{R}^{m \times n}$ then $A^T A$ is symmetric.

Proof: Proposition 2.9.3 yields $(A^T A)^T = A^T (A^T)^T = A^T A$. Thus $A^T A$ is symmetric. \Box

2.9.2 exponent laws for matrices

The power of a matrix is defined in the natural way. Notice we need for A to be square in order for the product AA to be defined.

Definition 2.9.8.

Let $A \in \mathbb{R}^{n \times n}$. We define $A^0 = I$, $A^1 = A$ and $A^m = AA^{m-1}$ for all $m \ge 1$. If A is invertible then $A^{-p} = (A^{-1})^p$.

As you would expect, $A^3 = AA^2 = AAA$.

Proposition 2.9.9.

Let $A, B \in \mathbb{R}^{n \times n}$ and $p, q \in \mathbb{N} \cup \{0\}$ 1. $(A^p)^q = A^{pq}$. 2. $A^p A^q = A^{p+q}$. 3. If A is invertible, $(A^{-1})^{-1} = A$.

Proof: left to reader. \Box

You should notice that $(AB)^p \neq A^p B^p$ for matrices. Instead,

$$(AB)^2 = ABAB, \qquad (AB)^3 = ABABAB, etc...$$

This means the binomial theorem will not hold for matrices. For example,

 $(A+B)^{2} = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB$

hence $(A+B)^2 \neq A^2 + 2AB + B^2$ as the matrix product is not generally commutative. If we have A and B commute then AB = BA and we can prove that $(AB)^p = A^p B^p$ and the binomial theorem holds true.

Proposition 2.9.10.

If A is symmetric then A^k is symmetric for all $k \in \mathbb{N}$.

Proof: Suppose $A^T = A$. Proceed inductively. Clearly k = 1 holds true since $A^1 = A$. Assume inductively that A^k is symmetric.

 $\begin{aligned} (A^{k+1})^T &= (AA^k)^T & \text{defn. of matrix exponents,} \\ &= (A^k)^T A^T & \text{socks-shoes prop. of transpose,} \\ &= A^k A & \text{using inducition hypothesis.} \\ &= A^{k+1} & \text{defn. of matrix exponents,} \end{aligned}$

thus by proof by mathematical induction A^k is symmetric for all $k \in \mathbb{N}$. \Box There are many other fun identities about symmetric and invertible matrices. I'll probably put a few in the Problem Set since they make nice easy proof problems.

2.9.3 diagonal and triangular matrices

Definition 2.9.11.

Let $A \in \mathbb{R}^{m \times n}$. If $A_{ij} = 0$ for all i, j such that $i \neq j$ then A is called a **diagonal** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \leq j$ then we call A a **upper triangular** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \geq j$ then we call A a **lower triangular** matrix.

Example 2.9.12. Let me illustrate a generic example of each case for 3×3 matrices:

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

As you can see the diagonal matrix only has nontrivial entries on the diagonal, and the names lower triangular and upper triangular are likewise natural.

If an upper triangular matrix has zeros on the diagonal then it is said to be **strictly upper triangular**. Likewise, if a lower triangular matrix has zeros on the diagonal then it is said to be **strictly lower triangular**. Obviously and matrix can be written as a sum of a diagonal and strictly upper and strictly lower matrix,

$$A = \sum_{i,j} A_{ij} E_{ij}$$
$$= \sum_{i} A_{ii} E_{ii} + \sum_{i < j} A_{ij} E_{ij} + \sum_{i > j} A_{ij} E_{ij}$$

There is an algorithm called LU-factorization which for many matrices A finds a lower triangular matrix L and an upper triangular matrix U such that A = LU. We may discuss it at the end of the course. It is one of several factorization schemes which is calculationally advantageous for large systems. There are many many ways to solve a system, but some are faster methods. Algorithmics is the study of which method is optimal.

Proposition 2.9.13.

Let $A, B \in \mathbb{R}^{n \times n}$.

- 1. If A, B are upper diagonal then AB is diagonal.
- 2. If A, B are upper triangular then AB is upper triangular.
- 3. If A, B are lower triangular then AB is lower triangular.

Proof of (1.): Suppose A and B are diagonal. It follows there exist a_i, b_j such that $A = \sum_i a_i E_{ii}$ and $B = \sum_j b_j E_{jj}$. Calculate,

$$AB = \sum_{i} a_{i}E_{ii} \sum_{j} b_{j}E_{jj}$$
$$= \sum_{i} \sum_{j} a_{i}b_{j}E_{ii}E_{jj}$$
$$= \sum_{i} \sum_{j} a_{i}b_{j}\delta_{ij}E_{ij}$$
$$= \sum_{i} a_{i}b_{i}E_{ii}$$

thus the product matrix AB is also diagonal and we find that the diagonal of the product AB is just the product of the corresponding diagonals of A and B.

Proof of (2.): Suppose A and B are upper diagonal. It follows there exist A_{ij}, B_{ij} such that $A = \sum_{i \leq j} A_{ij} E_{ij}$ and $B = \sum_{k \leq l} B_{kl} E_{kl}$. Calculate,

$$AB = \sum_{i \le j} A_{ij} E_{ij} \sum_{k \le l} B_{kl} E_{kl}$$
$$= \sum_{i \le j} \sum_{k \le l} A_{ij} B_{kl} E_{ij} E_{kl}$$
$$= \sum_{i \le j} \sum_{k \le l} A_{ij} B_{kl} \delta_{jk} E_{il}$$
$$= \sum_{i \le j} \sum_{j \le l} A_{ij} B_{jl} E_{il}$$

Notice that every term in the sum above has $i \leq j$ and $j \leq l$ hence $i \leq l$. It follows the product is upper triangular since it is a sum of upper triangular matrices. The proof of (3.) is similar. \Box .

2.10 applications

Definition 2.10.1.

Let $P \in \mathbb{R}^{n \times n}$ with $P_{ij} \ge 0$ for all i, j. If the sum of the entries in any column of P is one then we say P is a stochastic matrix.

Example 2.10.2. Stochastic Matrix: A medical researcher¹ is studying the spread of a virus in 1000 lab. mice. During any given week it's estimated that there is an 80% probability that a mouse will overcome the virus, and during the same week there is an 10% likelyhood a healthy mouse will become infected. Suppose 100 mice are infected to start, (a.) how many sick next week? (b.) how many sick in 2 weeks? (c.) after many many weeks what is the steady state solution?

$$I_{k} = infected mice at beginning of week k$$
$$P = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = [I_k, N_k]$ by the probability transition matrix P given above. Notice we are given that $X_1 = [100, 900]^T$. Calculate then,

$$X_2 = \left[\begin{array}{cc} 0.2 & 0.1\\ 0.8 & 0.9 \end{array}\right] \left[\begin{array}{c} 100\\ 900 \end{array}\right] = \left[\begin{array}{c} 110\\ 890 \end{array}\right]$$

After one week there are 110 infected mice Continuing to the next week,

$$X_3 = \left[\begin{array}{cc} 0.2 & 0.1 \\ 0.8 & 0.9 \end{array} \right] \left[\begin{array}{c} 110 \\ 890 \end{array} \right] = \left[\begin{array}{c} 111 \\ 889 \end{array} \right]$$

After two weeks we have 111 mice infected. What happens as $k \to \infty$? Generally we have $X_k = PX_{k-1}$. Note that as k gets large there is little difference between k and k-1, in the limit they both tend to infinity. We define the steady-state solution to be $X^* = \lim_{k\to\infty} X_k$. Taking the limit of $X_k = PX_{k-1}$ as $k \to \infty$ we obtain the requirement $X^* = PX^*$. In other words, the steady state solution is found from solving $(P-I)X^* = 0$. For the example considered here we find,

$$(P-I)X^* = \begin{bmatrix} -0.8 & 0.1\\ 0.8 & -0.1 \end{bmatrix} \begin{bmatrix} u\\ v \end{bmatrix} = 0 \qquad v = 8u \qquad X^* = \begin{bmatrix} u\\ 8u \end{bmatrix}$$

However, by conservation of mice, u + v = 1000 hence 9u = 1000 and $u = 111.\overline{11}$ thus the steady state can be shown to be $X^* = [111.\overline{11}, 888.\overline{88}]$

Example 2.10.3. Diagonal matrices are nice: Suppose that demand for doorknobs halves every week while the demand for yo-yos it cut to 1/3 of the previous week's demand every week due to

¹this example and most of the other applied examples in these notes are borrowed from my undergraduate linear algebra course taught from Larson's text by Dr. Terry Anderson of Appalachian State University

an amazingly bad advertising campaign². At the beginning there is demand for 2 doorknobs and 5 yo-yos.

$$\begin{array}{l} D_k = demand \ for \ doorknobs \ at \ beginning \ of \ week \ k \\ Y_k = demand \ for \ yo-yos \ at \ beginning \ of \ week \ k \end{array} \qquad P = \left[\begin{array}{c} 1/2 & 0 \\ 0 & 1/3 \end{array} \right]$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = [D_k, Y_k]$ by the transition matrix P given above. Notice we are given that $X_1 = [2, 5]^T$. Calculate then,

$$X_2 = \begin{bmatrix} 1/2 & 0\\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2\\ 5 \end{bmatrix} = \begin{bmatrix} 1\\ 5/3 \end{bmatrix}$$

Notice that we can actually calculate the k-th state vector as follows:

$$X_{k} = P^{k}X_{1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}^{k} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix}^{k} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k+1} \\ 5(3^{-k}) \end{bmatrix}$$

Therefore, assuming this silly model holds for 100 weeks, we can calculate the 100-the step in the process easily,

$$X_{100} = P^{100} X_1 = \begin{bmatrix} 2^{-101} \\ 5(3^{-100}) \end{bmatrix}$$

Notice that for this example the analogue of X^* is the zero vector since as $k \to \infty$ we find X_k has components which both go to zero.

Example 2.10.4. Naive encryption: in Example 2.7.3 we found observed that the matrix A has inverse matrix A^{-1} where:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

We use the alphabet code

$$A = 1, B = 2, C = 3, \ldots, Y = 25, Z = 26$$

and a space is encoded by 0. The words are parsed into row vectors of length 3 then we multiply them by A on the right; [decoded]A = [coded]. Suppose we are given the string, already encoded by A

$$[9, -1, -9], [38, -19, -19], [28, -9, -19], [-80, 25, 41], [-64, 21, 31], [-7, 4, 7], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4], [-7, 4],$$

Find the hidden message by undoing the multiplication by A. Simply multiply by A^{-1} on the right,

$$[9, -1, -9]A^{-1}, [38, -19, -19]A^{-1}, [28, -9, -19]A^{-1},$$

 $^{^{2}}$ insert your own more interesting set of quantities that doubles/halves or triples during some regular interval of time

$$[-80, 25, 41]A^{-1}, [-64, 21, 31]A^{-1}, [-7, 4, 7]A^{-1}$$

This yields,

[19, 19, 0], [9, 19, 0], [3, 1, 14], [3, 5, 12], [12, 5, 4]

which reads CLASS IS CANCELLED³.

If you enjoy this feel free to peruse my Math 121 notes, I have additional examples of this naive encryption. I say it's naive since real encryption has much greater sophistication by this time. There are many other applications, far too many to mention here. If students were willing to take an "applied linear algebra course" as a follow-up to this course it would be possible to cover many more "real-world" applications. I have several other undergraduate linear algebra texts which have tons of applications we don't cover. A parial list: matrix multiplication of the input by the transfer matrix gives the output signal for linear systems in electrical engineering. There are coincidence matrices, permutation matrices, Leslie matrices, tri-banded matrices, shear transformation matrices, matrices to model an affine transformation of computer graphics... My goal in this section is simply to show you a few simple applications and also to invite you to study more as your interests guide you.

³Larson's pg. 100-102 # 22

2.11 conclusions

The theorem that follows here collects the various ideas we have discussed concerning an $n \times n$ matrix and invertibility and solutions of Ax = b.

Theorem 2.11.1.

Let A be a real $n \times n$ matrix then the following are equivalent: (a.) A is invertible, (b.) rref[A|0] = [I|0] where $0 \in \mathbb{R}^{n \times 1}$, (c.) Ax = 0 iff x = 0, (d.) A is the product of elementary matrices, (e.) there exists $B \in \mathbb{R}^{n \times n}$ such that AB = I, (f.) there exists $B \in \mathbb{R}^{n \times n}$ such that BA = I, (g.) rref[A] = I, (h.) rref[A|b] = [I|x] for an $x \in \mathbb{R}^{n \times 1}$, (i.) Ax = b is consistent for every $b \in \mathbb{R}^{n \times 1}$, (j.) Ax = b has exactly one solution for every $b \in \mathbb{R}^{n \times 1}$, (k.) A^{T} is invertible.

These are in no particular order. If you examine the arguments in this chapter you'll find we've proved most of this theorem. What did I miss? 4

⁴teaching moment or me trying to get you to do my job, you be the judge.

Chapter 3

determinants

I should warn you there are some difficult calculations in this Chapter. However, the good news is these are primarily to justify the various properties of the determinant. The determinant of a square matrix is simply a number which contains lots of useful information. We will conclude this Chapter with a discussion of what the determinant says about systems of equations. There are a lot of different ways to introduce the determinant, my approach is rooted in my love of index calculations from physics. A pure mathematician would likely take another approach (mine is better). Geometrically, determinants are used to capture the idea of an oriented volume. I illustrate this with several examples before we get too deep into the more esoteric calculations.

3.1 determinants and geometry

The determinant of a square matrix can be defined by the following formulas. I'll give the general formula in the next section, but more often than not the formulas given here are more than enough. Well, this one is just silly:

$$\det a = a.$$

Then the 2×2 case is perhaps more familar,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

we've seen this before somewhere. Then the 3×3 formula is:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

and finally the 4×4 determinant is given by

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \cdot \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \cdot \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix}$$
(3.1)

$$+ c \cdot \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \cdot \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$
(3.2)

What do these formulas have to do with geometry?

Example 3.1.1. Consider the vectors $[l \ 0]$ and $[0 \ w]$. They make two sides of a rectangle with length l and width w. Notice

$$det \left[\begin{array}{cc} l & 0\\ 0 & w \end{array} \right] = lw.$$

In contrast,

$$det \left[\begin{array}{cc} 0 & w \\ l & 0 \end{array} \right] = -lw.$$

Interestingly this works for parallellograms with sides [a/b] and [c/d] the area is given by $\pm det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The sign indicates the **orientation** of the paralellogram. If the paralellogram lives in the xy-plane then it has an up-ward pointing normal if the determinant is positive whereas it has a downward pointing normal if the determinant is negative. (in calculus III we learned a parallelogram can be parametrized by $\vec{r}(u, v) = u\vec{A} + v\vec{B}$ and our formula above has $\vec{A} = [a \ b]$ and $\vec{B} = [c \ d]$.)

Example 3.1.2. If we look at a three dimensional box with vectors $\vec{A}, \vec{B}, \vec{C}$ pointing along three edges with from a common corner then it can be shown that the volume V is given by the determinant

$$V = \pm det \begin{bmatrix} \vec{A} \\ \vec{B} \\ \hline \vec{C} \end{bmatrix}$$

Of course it's easy to see that V = lwh if the sides have length l, width w and height h. However, this formula is more general than that, it also holds if the vectors lie along a paralell piped. Again the sign of the determinant has to do with the **orientation** of the box. If the determinant is positive then that means that the set of vectors $\{\vec{A}, \vec{B}, \vec{C}\}$ forms a righted-handed set of vectors. In terms of calculus III, $\vec{A} \times \vec{B}$ is in the direction of (\vec{C}) and so forth; the ordering of the vectors is consistent with the right-hand rule. If the determinant of the three vectors is negative then they will be consistent with the (inferior and evil) left-hand rule.

I derive these facts in calculus III or the calculus III notes. Ask me in office hours if you'd like to know more. We cover the geometry of vector addition and cross-products etc... in calculus III. I will not repeat all that here. That said, I will show how determinants give the formulas we used in calculus III. If you have not take calculus III, my apologies.

Example 3.1.3. To calculate the cross-product of \vec{A} and \vec{B} we can use the heuristic rule

$$\vec{A} \times \vec{B} = det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$$

technically this is not a real "determinant" because there are vectors in the top row but numbers in the last two rows.

Example 3.1.4. The infinitesimal area element for polar coordinate is calculated from the Jacobian:

$$dS = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} drd\theta = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta = rdrd\theta$$

Example 3.1.5. The infinitesimal volume element for cylindrical coordinate is calculated from the Jacobian:

$$dV = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) & 0\\ \cos(\theta) & \sin(\theta) & 0\\ 0 & 0 & 1 \end{bmatrix} drd\theta dz = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta dz = rdrd\theta dz$$

Jacobians are needed to change variables in multiple integrals. The Jacobian is a determinant which measures how a tiny volume is rescaled under a change of coordinates. Each row in the matrix making up the Jacobian is a tangent vector which points along the direction in which a coordinate increases when the other two coordinates are fixed. (The discussion on page 206-208 is nice, I avoid discussing linear transformations until a little later in the course.)

3.2 cofactor expansion for the determinant

The precise definition of the determinant is intrinsically combinatorial. A permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ is a bijection. Every permutation can be written as a product of an even or odd composition of transpositions. The $sgn(\sigma) = 1$ if σ is formed from an even product of transpositions. The $sgn(\sigma) = -1$ if σ is formed from an odd product of transpositions. The sum below is over all possible permutations,

$$det(A) = \sum_{\sigma} sgn(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

this provides an explicit definition of the determinant. For example, in the n = 2 case we have $\sigma_o(x) = x$ or $\sigma_1(1) = 2, \sigma_1(2) = 1$. The sum over all permutations has just two terms in the n = 2 case,

$$det(A) = sgn(\sigma_o)A_{1\sigma_o(1)}A_{2\sigma_o(2)} + sgn(\sigma_1)A_{1\sigma_1(1)}A_{2\sigma_1(2)} = A_{11}A_{22} - A_{12}A_{21}$$

In the notation $A_{11} = a, A_{12} = b, A_{21} = c, A_{22} = d$ the formula above says det(A) = ad - bc.

Pure mathematicians tend to prefer the definition above to the one I am preparing below. I would argue mine has the advantage of not summing over functions. My sums are simply over integers.

The calculations I make in the proofs in this Chapter may appear difficult to you, but if you gain a little more experience with index calculations I think you would find them accessible. I will not go over them all in lecture. I would recommend you at least read over them.

Definition 3.2.1.

Let $\epsilon_{i_1i_2...i_n}$ be defined to be the completely antisymmetric symbol in *n*-indices. We define $\epsilon_{12...n} = 1$ then all other values are generated by demanding the interchange of any two indices is antisymmetric. This is also known as the **Levi-Civita** symbol.

We have nice formulas for the determinant with the help of the Levi-Civita symbol, the following is yet another way of stating the definition for det(A),

$$det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

Example 3.2.2. I prefer this definition. I can actually calculate it faster, for example the n = 1 case is pretty quick:

$$det(A) = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{12}A_{23}A_{31} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32}$$

In principle there are 27 terms above but only these 6 are nontrivial because if any index is repeated the ϵ_{ijk} is zero. The only nontrivial terms are $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. Thus,

$$det(A) = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}$$
$$-A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{23}A_{32}$$

This formula is much closer to the trick-formula for calculating the determinant without using minors. (I'll put it on the board in class, it is above my skill-level for these notes)

The formalism above will be used in all my proofs. I take the Levi-Civita definition as the primary definition for the determinant. All other facts flow from that source. The cofactor expansions of the determinant could also be used as a definition.

Definition 3.2.3.

Let $A = [A_{ij}] \in \mathbb{R}^{n \times n}$. The minor of A_{ij} is denoted M_{ij} which is defined to be the determinant of the $\mathbb{R}^{(n-1)\times(n-1)}$ matrix formed by deleting the *i*-th column and the *j*-th row of A. The (i, j)-th co-factor of A is $(-1)^{i+j}M_{ij}$.

Theorem 3.2.4.

The determinant of $A \in \mathbb{R}^{n \times n}$ can be calculated from a sum of cofactors either along any row or column;

1.
$$det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \dots + A_{in}C_{in}$$
 (*i*-th row expansion)
2. $det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \dots + A_{nj}C_{nj}$ (*j*-th column expansion)

Proof: I'll attempt to sketch a proof of (2.) directly from the general definition. Let's try to identify A_{1i_1} with A_{1j} then A_{2i_2} with A_{2j} and so forth, keep in mind that j is a fixed but arbitrary index, it is not summed over.

$$det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

$$= \sum_{i_2, \dots, i_n} \epsilon_{j, i_2, \dots, i_n} A_{1j} A_{2i_2} \cdots A_{ni_n} + \sum_{i_1 \neq j, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

$$= \sum_{i_2, \dots, i_n} \epsilon_{j, i_2, \dots, i_n} A_{1j} A_{2i_2} \cdots A_{ni_n} + \sum_{i_1 \neq j, i_3, \dots, i_n} \epsilon_{i_1, j, \dots, i_n} A_{1i_1} A_{2j} \cdots A_{ni_n}$$

$$+ \cdots + \sum_{i_1 \neq j, i_2 \neq j, \dots, i_{n-1} \neq j} \epsilon_{i_1, i_2, \dots, i_{n-1}, j} A_{1i_1} \cdots A_{n-1, i_{n-1}} A_{nj}$$

$$+ \sum_{i_1 \neq j, \dots, i_n \neq j} \epsilon_{i_1, \dots, i_n} A_{1i_1} A_{1i_2} \cdots A_{ni_n}$$

Consider the summand. If all the indices $i_1, i_2, \ldots, i_n \neq j$ then there must be at least one repeated index in each list of such indices. Consequently the last sum vanishes since $\epsilon_{i_1,\ldots,i_n}$ is zero if any two indices are repeated. We can pull out A_{1j} from the first sum, then A_{2j} from the second sum, and so forth until we eventually pull out A_{nj} out of the last sum.

$$det(A) = A_{1j} \left(\sum_{i_2, \dots, i_n} \epsilon_{j, i_2, \dots, i_n} A_{2i_2} \cdots A_{ni_n} \right) + A_{2j} \left(\sum_{i_1 \neq j, \dots, i_n} \epsilon_{i_1, j, \dots, i_n} A_{1i_1} \cdots A_{ni_n} \right) + \cdots + A_{nj} \left(\sum_{i_1 \neq j, i_2 \neq j, \dots, j \neq i_{n-1}} \epsilon_{i_1, i_2, \dots, j} A_{1i_1} A_{2i_2} \cdots A_{n-1, i_{n-1}} \right)$$

The terms appear different, but in fact there is a hidden symmetry. If any index in the summations above takes the value j then the Levi-Civita symbol with have two j's and hence those terms are zero. Consequently we can just as well take all the sums over all values **except** j. In other words, each sum is a completely antisymmetric sum of products of n - 1 terms taken from all columns except j. For example, the first term has an antisymmetrized sum of a product of n - 1 terms not including column j or row 1.Reordering the indices in the Levi-Civita symbol generates a sign of

 $(-1)^{1+j}$ thus the first term is simply $A_{1j}C_{1j}$. Likewise the next summand is $A_{2j}C_{2j}$ and so forth until we reach the last term which is $A_{nj}C_{nj}$. In other words,

$$det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \dots + A_{nj}C_{nj}$$

The proof of (1.) is probably similar. We will soon learn that $det(A^T) = det(A)$ thus (2.) \implies (1.). since the *j*-th row of A^T is the *j*-th columns of A.

All that remains is to show why $det(A) = det(A^T)$. Recall $(A^T)_{ij} = A_{ji}$ for all i, j, thus

$$det(A^{T}) = \sum_{i_{1}, i_{2}, \dots, i_{n}} \epsilon_{i_{1}, i_{2}, \dots, i_{n}} (A^{T})_{1i_{1}} (A^{T})_{2i_{2}} \cdots (A^{T})_{ni_{n}}$$
$$= \sum_{i_{1}, i_{2}, \dots, i_{n}} \epsilon_{i_{1}, i_{2}, \dots, i_{n}} A_{i_{1}1} A_{i_{2}2} \cdots A_{i_{n}n}$$
$$= \sum_{i_{1}, i_{2}, \dots, i_{n}} \epsilon_{i_{1}, i_{2}, \dots, i_{n}} A_{1i_{1}} A_{2i_{2}} \cdots A_{ni_{n}} = det(A)$$

to make the last step one need only see that both sums contain all the same terms just written in a different order. Let me illustrate explicitly how this works in the n = 3 case,

$$det(A^{T}) = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{21}A_{32}A_{13} + \epsilon_{312}A_{31}A_{12}A_{23} + \epsilon_{321}A_{31}A_{22}A_{13} + \epsilon_{213}A_{21}A_{12}A_{33} + \epsilon_{132}A_{11}A_{32}A_{23}$$

The I write the entries so the column indices go 1, 2, 3

$$det(A^{T}) = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{13}A_{21}A_{32} + \epsilon_{312}A_{12}A_{23}A_{31} + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32}$$

But, the indices of the Levi-Civita symbol are not in the right order yet. Fortunately, we have identities such as $\epsilon_{231} = \epsilon_{312}$ which allow us to reorder the indices without introducing any new signs,

$$det(A^{T}) = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{231}A_{12}A_{23}A_{31} + \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32}$$

But, these are precisely the terms in det(A) just written in a different order (see Example 3.2.2). Thus $det(A^T) = det(A)$. I leave the details of how to reorder the order n sum to the reader. \Box

Remark 3.2.5.

Your text does not prove (2.). Instead it refers you to the grown-up version of the text by the same authors. If you look in *L*inear Algebra by Insel, Spence and Friedberg you'll find the proof of the co-factor expansion is somewhat involved. However, the heart of the proof involves multilinearity. Multilinearity is practically manifest with our Levi-Civita definition. Anywho, a better definition for the determinant is as follows: **the determinant is the alternating**, *n*-**multilinear**, **real valued map such that** det(I) = 1. It can be shown this uniquely defines the determinant. All these other things like permutations and the Levi-Civita symbol are just notation.

Remark 3.2.6.

The best way to prove things about determinants is likely the wedge product formalism. In that notation the Levi-Civita symbol is implicit within the so-called wedge product of vectors. For a $n \times n$ matrix the det(A) is defined implicitly by the formula $col_1(A) \wedge col_2(A) \wedge \cdots \wedge col_n(A) = det(A)e_1 \wedge e_2 \wedge \cdots \wedge e_n$. One nice place to read more about these things from a purely linear-algebraic perspective is the text *Abstract Linear Algebra* by Morton L. Curtis.

Example 3.2.7. I suppose it's about time for an example. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

I usually calculate by expanding across the top row out of habit,

$$det(A) = 1det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$
$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$
$$= -3 + 12 - 9$$
$$= 0.$$

Now, we could also calculate by expanding along the middle row,

$$det(A) = -4det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 5det \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} - 6det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$
$$= -4(18 - 24) + 5(9 - 21) - 6(8 - 14)$$
$$= 24 - 60 + 36$$
$$= 0.$$

Many other choices are possible, for example expan along the right column,

$$det(A) = 3det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} - 6det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} + 9det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$
$$= 3(32 - 35) - 6(8 - 14) + 9(5 - 8)$$
$$= -9 + 36 - 27$$
$$= 0.$$

which is best? Certain matrices might have a row or column of zeros, then it's easiest to expand along that row or column.

Example 3.2.8. Let's look at an example where we can exploit the co-factor expansion to greatly reduce the difficulty of the calculation. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 9 & 3 & 4 & 0 \\ -1 & -2 & -3 & 0 & 1 \end{bmatrix}$$

Begin by expanding down the 4-th column,

$$det(A) = (-1)^{4+4} M_{44} = 4det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 \\ 6 & 7 & 8 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

Next expand along the 2-row of the remaining determinant,

$$det(A) = (4)(5(-1)^{2+3}M_{23}) = -20det \begin{bmatrix} 1 & 2 & 4 \\ 6 & 7 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Finish with the trick for 3×3 determinants, it helps me to write out

then calculate the products of the three down diagonals and the three upward diagonals. Subtract the up-diagonals from the down-diagonals.

$$det(A) = -20(7 + 0 - 48 - (-28) - (0) - (12)) = -20(-25) = 500$$

3.3 adjoint matrix

Definition 3.3.1.

Let $A \in \mathbb{R}^{n \times n}$ the the matrix of cofactors is called the **adjoint** of A. It is denoted adj(A) and is defined by and $adj(A)_{ij} = C_{ij}^T$ where C_{ij} is the (i, j)-th cofactor.

I'll keep it simple here, lets look at the 2×2 case:

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$$

has cofactors $C_{11} = (-1)^{1+1} det(d) = d$, $C_{12} = (-1)^{1+2} det(c) = -c$, $C_{21} = (-1)^{2+1} det(b) = -b$ and $C_{22} = (-1)^{2+2} det(a) = a$. Collecting these results,

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is interesting. Recall we found a formula for the inverse of A (if it exists). The formula was

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that det(A) = ad - bc thus in the 2×2 case the relation between the inverse and the adjoint is rather simple:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

It turns out this is true for larger matrices as well:

Proposition 3.3.2.

If A is invertible then $A^{-1} = \frac{1}{det(A)} adj(A)$.

Proof: Calculate the product of A and adj(A),

$$Aadj(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{T}$$

The (i, j)-th component of the product above is

$$(Aadj(A))_{ij} = A_{i1}C_{j1} + A_{i2}C_{j2} + \dots + A_{in}C_{jn}$$

Suppose that i = j then the sum above is precisely the co-factor expansion for det(A). If $i \neq j$ then the sum vanishes. I leave the details to the reader. \Box

Remark 3.3.3.

If you want to find the adjoint in your text, see exercise 82 from section 3.2. I think that's it. There are better ways to calculate A^{-1} so the text does not feel obligated to discuss this material.

Example 3.3.4. Let's calculate the general formula for the inverse of a 3×3 matrix. Assume it exists for the time being. (the criteria for the inverse existing is staring us in the face everywhere here). Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$C_{11} = det \begin{bmatrix} e & f \\ h & i \end{bmatrix} = ei - fh,$$
$$C_{12} = -det \begin{bmatrix} d & f \\ g & i \end{bmatrix} = fg - di,$$
$$C_{13} = det \begin{bmatrix} d & e \\ g & h \end{bmatrix} = dh - eg,$$
$$C_{21} = -det \begin{bmatrix} b & c \\ h & i \end{bmatrix} = ch - bi,$$
$$C_{22} = det \begin{bmatrix} a & c \\ g & h \end{bmatrix} = ai - cg,$$
$$C_{23} = -det \begin{bmatrix} a & b \\ g & h \end{bmatrix} = bg - ah,$$
$$C_{31} = det \begin{bmatrix} b & c \\ e & f \end{bmatrix} = bf - ce,$$
$$C_{32} = -det \begin{bmatrix} a & c \\ d & f \end{bmatrix} = cd - af,$$
$$C_{33} = det \begin{bmatrix} a & b \\ d & e \end{bmatrix} = ae - bd.$$

Calculate the cofactors,

Hence the adjoint is $% \left(f_{i} \right) = \left(f_{i} \right) \left($

$$adj(A) = \begin{bmatrix} ei - fh & fg - di & dh - eg\\ ch - bi & ai - cg & bg - ah\\ bf - ce & cd - af & ae - bd \end{bmatrix}^{T}$$

Thus, using the $A^{-1} = det(A)adj(A)$

· ·		<i>c</i> -] -1	1			dh - eg	
d	e	f	=	$\overline{aei + bfg + cdh - gec - hfa - idb}$	ch - bi	ai-cg	bg - ah	
$\lfloor g$	h	<i>i</i> _		act + ofg + can gee hija tao	bf - ce	cd-af	ae-bd]

You should notice that are previous method for finding A^{-1} is far superior to this method. It required much less calculation. Let's check my formula¹ in the case A = 3I, this means a = e = i = 3 and the others are zero.

$$I^{-1} = \frac{1}{27} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \frac{1}{3}I$$

This checks, $(3I)(\frac{1}{3}I) = \frac{3}{3}II = I$. I do not recommend that you memorize this formula to calculate inverses for 3×3 matrices.

3.4 Kramer's Rule

The numerical methods crowd seem to think this is a loathsome brute. It is an incredibly clumsy way to calculate the solution of a system of equations Ax = b. Moreover, Kramer's rule fails in the case det(A) = 0 so it's not nearly as general as our other methods. However, it does help calculate the variation of parameters formulas in differential equations so it is still of theoretical interest at a minimum. Students sometimes like it because it gives you a *formula* to find the solution. Students sometimes incorrectly jump to the conclusion that a formula is easier than say a *method*. It is certainly wrong here, the method of Gaussian elimination beats Kramer's rule by just about every objective criteria in so far as concrete numerical examples are concerned.

Proposition 3.4.1.

If Ax = b is a linear system of equations with $x = [x_1 \ x_2 \ \cdots \ x_n]^T$ and $A \in \mathbb{R}^{n \times n}$ such that $det(A) \neq 0$ then we find solutions

$$x_1 = \frac{\det(A_1)}{\det(A)}, \ x_2 = \frac{\det(A_2)}{\det(A)}, \ \dots, \ x_n = \frac{\det(A_n)}{\det(A)}$$

where we define A_k to be the $n \times n$ matrix obtained by replacing the k-th column of A by the inhomogeneous term b.

Proof: Assume that Ax = b has a unique solution. Then we may calculate:

$$Ax = b \Leftrightarrow x = A^{-1}b = \frac{1}{det(A)}adj(A)b$$

¹in the first version of these notes I missed a transpose, I'm not certain if the adjoint technically includes that transpose or if in fact we should write $A^{-1} = \frac{1}{\det(A)} [adj(A)]^T$, however I am confident in the validity of the explicit examples in this section, my concern is the universality of the notation, I could be in error on this point

Thus,

$$\begin{aligned} x_k &= \frac{1}{\det(A)} \sum_j adj(A)_{kj} b_j \\ &= \frac{1}{\det(A)} \sum_j C_{kj} b_j \\ &= \frac{1}{\det(A)} det[col_1(A)| \cdots |col_{k-1}(A)| b|col_{k+1}(A)| \cdots |col_n(A)] \\ &= \frac{det(A_k)}{det(A)} \end{aligned}$$

Clearly Kramer's rule and the adj(A) formula for the inverse matrix are closely connected. \Box

Example 3.4.2. Solve Ax = b given that

$$A = \begin{bmatrix} 1 & 3\\ 2 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} 1\\ 5 \end{bmatrix}$$

where $x = [x_1 \ x_2]^T$. Apply Kramer's rule, note det(A) = 2,

$$x_1 = \frac{1}{2}det \begin{bmatrix} 1 & 3\\ 5 & 8 \end{bmatrix} = \frac{1}{2}(8-15) = \frac{-7}{2}$$

and,

$$x_2 = \frac{1}{2}det \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = \frac{1}{2}(5-2) = \frac{3}{2}.$$

The original system of equations would be $x_1 + 3x_2 = 1$ and $2x_1 + 8x_2 = 5$. As a quick check we can substitute in our answers $x_1 = -7/2$ and $x_2 = 3/2$ and see if they work.

Example 3.4.3. An nonhomogeneous system of linear, constant coefficient ordinary differential equations can be written as a matrix differential equation:

$$\frac{dx}{dt} = Ax + f$$

It turns out we'll be able to solve the homogeneous system dx/dt = Ax via something called the matrix exponential. Long story short, we'll find n-solutions which we can concatenate into one big matrix solution X. To solve the given nonhomogeneous problem one makes the ansatz that x = Xv is a solution for some yet unknown vector of functions. Then calculus leads to the problem of solving

$$X\frac{dv}{dt} = f$$

where X is matrix of functions, dv/dt and f are vectors of functions. X is invertible so we expect to find a unique solution dv/dt. Kramer's rule says,

$$\left(\frac{dv}{dt}\right)_{i} = \frac{1}{det(X)}det[\vec{x}_{1}|\cdots|g|\cdots|\vec{x}_{n}] = \frac{W_{i}[f]}{det(X)} \quad defining \ W_{i} \ in \ the \ obvious \ way$$

For each *i* we integrate the equation above,

$$v_i(t) = \int \frac{W_i[f]dt}{det(X)}.$$

The general solution is thus,

$$x = Xv = X\left[\int \frac{W_i[f]dt}{det(X)}\right].$$

The first component of this formula justifies n-th order variation of parameters. For example in the n = 2 case you may have learned that $y_p = y_1v_1 + y_2v_2$ solves ay'' + by' + cy = g if

$$v_1 = \int \frac{-gy'_2 dt}{y_1 y'_2 - y_2 y'_1} \qquad v_2 = \int \frac{gy'_1 dt}{y_1 y'_2 - y_2 y'_1}$$

These come from the general result above. I'll explain more in a later chapter. However, it turns out our focus will be on solving the dx/dt = Ax problem. We do not treat systems of differential equations in math 334 because we thought it is better to treat them here where we can apply the full force of linear algebra without any extra work.

3.5 properties of determinants

We're finally getting towards the good part.

addition and I assume k < j.

Proposition 3.5.1.

Let
$$A \in \mathbb{R}^{n \times n}$$
,
1. $det(A^T) = det(A)$,
2. If there exists j such that $row_j(A) = 0$ then $det(A) = 0$,
3. If there exists j such that $col_j(A) = 0$ then $det(A) = 0$,
4. $det[A_1|A_2|\cdots|aA_k+bB_k|\cdots A_n] = adet[A_1|\cdots|A_k|\cdots|A_n] + bdet[A_1|\cdots|B_k|\cdots|A_n]$,
5. $det(kA) = k^n det(A)$
6. if $B = \{A : r_k \leftrightarrow r_j\}$ then $det(B) = -det(A)$,
7. if $B = \{A : r_k + ar_j \to r_k\}$ then $det(B) = det(A)$,
8. if $row_i(A) = krow_j(A)$ for $i \neq j$ then $det(A) = 0$
where I mean to denote $r_k \leftrightarrow r_j$ as the row interchange and $r_k + ar_j \to r_k$ as a column

Proof: we already proved (1.) in the proof of the cofactor expansion Theorem 3.2.4. The proof of (2.) and (3.) follows immediately from the cofactor expansion if we expand along the zero row or column. The proof of (4.) is not hard given our Levi-Civita definition, let

$$C = [A_1|A_2|\cdots|aA_k + bB_k|\cdots|A_n]$$

Calculate from the definition,

$$det(C) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} C_{1i_1} \cdots C_{ki_k} \cdots C_{ni_n}$$

= $\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} \cdots (aA_{ki_k} + bB_{ki_k}) \cdots A_{ni_n}$
= $a \left(\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n} \right)$
+ $b \left(\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} \cdots B_{ki_k} \cdots A_{ni_n} \right)$
= $a det[A_1|A_2| \cdots |A_k| \cdots |A_n] + b det[A_1|A_2| \cdots |B_k| \cdots |A_n].$

by the way, the property above is called multilinearity. The proof of (5.) is similar,

$$det(kA) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} kA_{1i_1} kA_{2i_2} \cdots kA_{ni_n}$$
$$= k^n \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$
$$= k^n det(A)$$

Let B be as in (6.), this means that $col_k(B) = col_j(A)$ and vice-versa,

$$det(B) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, \dots, i_k, \dots, i_j, \dots, i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n}$$
$$= \sum_{i_1, i_2, \dots, i_n} -\epsilon_{i_1, \dots, i_j, \dots, i_k, \dots, i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n}$$
$$= -det(A)$$

where the minus sign came from interchanging the indices i_j and i_k .

To prove (7.) let us define B as in the Proposition: let $row_k(B) = row_k(A) + arow_j(A)$ and $row_i(B) = row_i(A)$ for $i \neq k$. This means that $B_{kl} = A_{kl} + aA_{jl}$ and $B_{il} = A_{il}$ for each l.

3.5. PROPERTIES OF DETERMINANTS

Consequently,

$$det(B) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, \dots, i_k, \dots, i_n} A_{1i_1} \cdots (A_{ki_k} + aA_{ji_k}) \cdots A_{ni_n}$$

= $\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, \dots, i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n}$
+ $a \bigg(\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, \dots, i_j, \dots, i_k, \dots, i_n} A_{1i_1} \cdots A_{j, i_j} \cdots A_{ji_k} \cdots A_{ni_n} \bigg)$
= $\sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, \dots, i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n}$
= $det(A).$

The term in parenthesis vanishes because it has the sum of an antisymmetric tensor in i_j , i_k against a symmetric tensor in i_j , i_k . Here is the pattern, suppose $S_{ij} = S_{ji}$ and $T_{ij} = -T_{ji}$ for all i, j then consider

$$\sum_{i} \sum_{j} S_{ij} T_{ij} = \sum_{j} \sum_{i} S_{ji} T_{ji}$$
 switched indices
$$= \sum_{j} \sum_{i} -S_{ij} T_{ij}$$
 used sym. and antisym.
$$= -\sum_{i} \sum_{j} S_{ij} T_{ij}$$
 interchanged sums.

thus we have $\sum S_{ij}T_{ij} = -\sum S_{ij}T_{ij}$ which indicates the sum is zero. We can use the same argument on the pair of indices i_j, i_k in the expression since $A_{ji_j}A_{ji_k}$ is symmetric in i_j, i_k whereas the Levi-Civita symbol is antisymmetric in i_j, i_k .

We get (8.) as an easy consequence of (2.) and (7.), just subtract one row from the other so that we get a row of zeros. \Box

Proposition 3.5.2.

The determinant of a diagonal matrix is the product of the diagonal entries.

Proof: Use multilinearity on each row,

$$det \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \cdots = d_1 d_2 \cdots d_n det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus $det(D) = d_1 d_2 \cdots d_n$ as claimed. \Box

Proposition 3.5.3.

Let L be a lower triangular square matric and U be an upper triangular square matrix. 1. $det(L) = L_{11}L_{22}\cdots L_{nn}$ 2. $det(U) = U_{11}U_{22}\cdots U_{nn}$

Proof: I'll illustrate the proof of (2.) for the 3×3 case. We use the co-factor expansion across the first column of the matrix to begin,

$$det \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = A_{11}det \begin{bmatrix} U_{22} & U_{23} \\ 0 & U_{33} \end{bmatrix} = U_{11}U_{22}U_{33}$$

The proof of the $n \times n$ case is essentially the same. For (1.) use the co-factor expansion across the top row of L, to get $det(L) = L_{11}C_{11}$. Not the submatrix for calculating C_{11} is again has a row of zeros across the top. We calculate $C_{11} = L_{22}C_{22}$. This continues all the way down the diagonal. We find $det(L) = L_{11}L_{22}\cdots L_{nn}$. \Box

Proposition 3.5.4.

Let $A \in \mathbb{R}^{n \times n}$ and $k \neq 0 \in \mathbb{R}$, 1. $det(E_{r_i \leftrightarrow r_j}) = -1$, 2. $det(E_{kr_i \rightarrow r_i}) = k$, 3. $det(E_{r_i+br_j \rightarrow r_i}) = 1$, 4. for any square matrix B and elementary matrix E, det(EB) = det(E)det(B)5. if E_1, E_2, \dots, E_k are elementary then $det(E_1E_2 \cdots E_k) = det(E_1)det(E_2) \cdots det(E_k)$

Proof: Proposition 3.3.2 shows us that det(I) = 1 since $I^{-1} = I$ (there are many easier ways to show that). Note that $E_{r_i \leftrightarrow r_j}$ is a row-swap of the identity matrix thus by Proposition 3.5.1 we find $det(E_{r_i \leftrightarrow r_j}) = -1$. To prove (2.) we use multilinearity from Proposition 3.5.1. For (3.) we use multilinearity again to show that:

$$det(E_{r_i+br_j\to r_i}) = det(I) + bdet(E_{ij})$$

Again det(I) = 1 and since the unit matrix E_{ij} has a row of zeros we know by Proposition 3.5.1 $det(E_{ij}) = 0$.

To prove (5.) we use Proposition 3.5.1 multiple times in the arguments below. Let $B \in \mathbb{R}^{n \times n}$ and suppose E is an elementary matrix. If E is multiplication of a row by k then det(E) = kfrom (2.). Also EB is the matrix B with some row multiplied by k. Use multiplicative to see that det(EB) = kdet(B). Thus det(EB) = det(E)det(B). If E is a row interchange then EB is B with a row swap thus det(EB) = -det(B) and det(E) = -1 thus we again find det(EB) = det(E)det(B). Finally, if E is a row addition then EB is B with a row addition and det(EB) = det(B) and det(E) = 1 hence det(EB) = det(E)det(B). Notice that (6.) follows by repeated application of (5.). \Box

Proposition 3.5.5.

A square matrix A is invertible iff $det(A) \neq 0$.

Proof: recall there exist elementary matrices E_1, E_2, \ldots, E_k such that $rref(A) = E_1 E_2 \cdots E_k A$. Thus $det(rref(A)) = det(E_1)det(E_2) \cdots det(E_k)det(A)$. Either det(rref(A)) = 0 and det(A) = 0 or they are both nonzero.

Suppose A is invertible. Then Ax = 0 has a unique solution and thus rref(A) = I hence $det(rref(A)) = 1 \neq 0$ implying $det(A) \neq 0$.

Conversely, suppose $det(A) \neq 0$, then $det(rref(A)) \neq 0$. But this means that rref(A) does not have a row of zeros. It follows rref(A) = I. Therefore $A^{-1} = E_1 E_2 \cdots E_k$. \Box

Proposition 3.5.6.

If $A, B \in \mathbb{R}^{n \times n}$ then det(AB) = det(A)det(B).

Proof: If either A or B is not invertible then the reduced row echelon form of the nonivertible matrix will have a row of zeros hence det(A)det(B) = 0. Without loss of generality, assume A is not invertible. Note $rref(A) = E_1E_2\cdots E_kA$ hence $E_3^{-1}E_2^{-1}E_1^{-1}rref(A)B = AB$. Notice that rref(A)B will have at least one row of zeros since rref(A) has a row of zeros. Thus $det(E_3^{-1}E_2^{-1}E_1^{-1}rref(A)B) = det(E_3^{-1}E_2^{-1}E_1^{-1})det(rref(A)B) = 0$.

Suppose that both A and B are invertible. Then there exist elementary matrices such that $A = E_1 \cdots E_p$ and $B = E_{p+1} \cdots E_{p+q}$ thus

$$det(AB) = det(E_1 \cdots E_p E_{p+1} \cdots E_{p+q})$$

= $det(E_1 \cdots E_p)det(E_{p+1} \cdots E_{p+q})$
= $det(A)det(B)$.

We made repeated use of (6.) in Proposition 3.5.4. \Box

Proposition 3.5.7.

If $A \in \mathbb{R}^{n \times n}$ is invertible then $det(A^{-1}) = \frac{1}{det(A)}$.

Proof: If A is invertible then there exists $A^{-1} \in \mathbb{R}^{n \times n}$ such that $AA^{-1} = I$. Apply Proposition 3.5.6 to see that

$$det(AA^{-1}) = det(A)det(A^{-1}) = det(I) \implies det(A)det(A^{-1}) = 1.$$

Thus, $det(A^{-1}) = 1/det(A)$

Many of the properties we used to prove det(AB) = det(A)det(B) are easy to derive if you were simply given the assumption det(AB) = det(A)det(B). When you look at what went into the proof of Proposition 3.5.6 it's not surprising that det(AB) = det(A)det(B) is a powerful formula to know.

3.6 examples of determinants

In the preceding section I worked pretty hard to prove a number of useful properties for determinants. I show how to use them in this section.

Example 3.6.1. Notice that row 2 is twice row 1,

$$det \left[\begin{array}{rrr} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{array} \right] = 0.$$

Example 3.6.2. To calculate this one we make a single column swap to get a diagonal matrix. The determinant of a diagonal matrix is the product of the diagonals, thus:

$$det \begin{bmatrix} 0 & 6 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = -det \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 48.$$

Example 3.6.3. I choose the column/row for the co-factor expansion to make life easy each time:

$$det \begin{bmatrix} 0 & 1 & 0 & 2 \\ 13 & 71 & 5 & \pi \\ 0 & 3 & 0 & 4 \\ -2 & e & 0 & G \end{bmatrix} = -5det \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ -2 & e & G \end{bmatrix}$$
$$= -5(-2)det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= 10(4-6)$$
$$= -20.$$

Example 3.6.4. Find the values of λ such that the matrix $A - \lambda I$ is singular given that

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The matrix $A - \lambda I$ is singular iff $det(A - \lambda I) = 0$,

$$det(A - \lambda I) = det \begin{bmatrix} 1 - \lambda & 0 & 2 & 3\\ 1 & -\lambda & 0 & 0\\ 0 & 0 & 2 - \lambda & 0\\ 0 & 0 & 0 & 3 - \lambda \end{bmatrix}$$
$$= (3 - \lambda)det \begin{bmatrix} 1 - \lambda & 0 & 2\\ 1 & \lambda & 0\\ 0 & 0 & 2 - \lambda \end{bmatrix}$$
$$= (3 - \lambda)(2 - \lambda)det \begin{bmatrix} 1 - \lambda & 0\\ 1 & \lambda \end{bmatrix}$$
$$= (3 - \lambda)(2 - \lambda)(1 - \lambda)(-\lambda)$$
$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Thus we need $\lambda = 0, 1, 2$ or 3 in order that $A - \lambda I$ be a noninvertible matrix. These values are called the **eigenvalues** of A. We will have much more to say about that later.

Example 3.6.5. Suppose we are given the LU-factorization of a particular matrix (stolen from the text see Example 2 on pg. 154-155 of Spence, Insel and Friedberg.)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

The LU-factorization is pretty easy to find, we may discuss that at the end of the course. It is an important topic if you delve into serious numerical work where you need to write your own code and so forth. Anyhow, notice that L, U are triangular so we can calculate the determinant very easily,

$$det(A) = det(L)det(U) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 4$$

From a numerical perspective, the LU-factorization is a superior method for calculating det(A) as compared to the co-factor expansion. It has much better "convergence" properties. Incidentally, your text has much more to say about algorithmics so please keep that in mind if my comments here leave you wanting more.

3.7 applications

The determinant is a convenient mneumonic to create expressions which are antisymmetric. The key property is that if we switch a row or column it creates a minus sign. This means that if any two rows are repeated then the determinant is zero. Notice this is why the cross product of two vectors is naturally phrased in terms of a determinant. The antisymmetry of the determinant insures the formula for the cross-product will have the desired antisymmetry. In this section we examine a few more applications for the determinant.

Example 3.7.1. The Pauli's exclusion principle in quantum mechanics states that the wave function of a system of fermions is antisymmetric. Given N-electron wavefunctions $\chi_1, \chi_2, \ldots, \chi_N$ the following is known as the **Slater Determinant**

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = det \begin{bmatrix} \chi_1(\vec{r}_1) & \chi_2(\vec{r}_1) & \cdots & \chi_N(\vec{r}_1) \\ \chi_1(\vec{r}_2) & \chi_2(\vec{r}_2) & \cdots & \chi_N(\vec{r}_2) \\ \vdots & \vdots & \cdots & \vdots \\ \chi_1(\vec{r}_N) & \chi_2(\vec{r}_N) & \cdots & \chi_N(\vec{r}_N) \end{bmatrix}$$

Notice that $\Psi(\vec{r}_1, \vec{r}_1, \dots, \vec{r}_N) = 0$ and generally if any two of the position vectors $\vec{r}_i = \vec{r}_j$ then the total wavefunction $\Psi = 0$. In quantum mechanics the wavefunction's modulus squared gives the probability density of finding the system in a particular circumstance. In this example, the fact that any repeated entry gives zero means that no two electrons can share the same position. This is characteristic of particles with half-integer spin, such particles are called fermions. In contrast, bosons are particles with integer spin and they can occupy the same space. For example, light is made of photons which have spin 1 and in a laser one finds many waves of light traveling in the same space.

Example 3.7.2. The Vandermonde determinant is discussed further in the Problem Set. I will simply point out a curious formula:

$$det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} = 0$$

Let's reduce this by row-operations²

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2}_{r_3 - r_1 \to r_3} \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix}$$

Notice that the row operations above could be implemented by multiply on the left by $E_{r_2-r_1 \to r_2}$ and $E_{r_3-r_1 \to r_3}$. These are invertible matrices and thus $det(E_{r_2-r_1 \to r_2}) = k_1$ and $det(E_{r_3-r_1 \to r_3}) = k_2$

 $^{^{2}}$ of course we could calculate it straight from the co-factor expansion, I merely wish to illustrate how we can use row operations to simplify a determinant

for some pair of nonzero constants k_1, k_2 . If X is the given matrix and Y is the reduced matrix above then $Y = E_{r_3-r_1 \to r_3} E_{r_2-r_1 \to r_2} X$ thus,

$$0 = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} = k_1 k_2 det \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix}$$
$$= k_1 k_2 [(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)]$$

Divide by k_1k_2 and rearrange to find:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1) \qquad \Rightarrow \qquad y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_2 - x_1)$$

The boxed equation is the famous two-point formula for a line.

3.8 conclusions

We continue Theorem 2.11.1 from the previous chapter.

Theorem 3.8.1.

Let A be a real $n \times n$ matrix then the following are equivalent: (a.) A is invertible, (b.) rref[A|0] = [I|0] where $0 \in \mathbb{R}^{n \times 1}$, (c.) Ax = 0 iff x = 0, (d.) A is the product of elementary matrices, (e.) there exists $B \in \mathbb{R}^{n \times n}$ such that AB = I, (f.) there exists $B \in \mathbb{R}^{n \times n}$ such that BA = I, (g.) rref[A] = I, (h.) rref[A|b] = [I|x] for an $x \in \mathbb{R}^{n \times 1}$, (i.) Ax = b is consistent for every $b \in \mathbb{R}^{n \times 1}$, (j.) Ax = b has exactly one solution for every $b \in \mathbb{R}^{n \times 1}$, (k.) A^{T} is invertible, (l.) $det(A) \neq 0$, (m.) Kramer's rule yields solution of Ax = b for every $b \in \mathbb{R}^{n \times 1}$.

It's a small addition, however the determinant is a nice tool for small systems since it's pretty easy to calculate. Also, Kramer's rule is nice for small systems since it just gives us the solution. This is all a very special case, in general we could have an inconsistent system or infinitely many solutions.

Theorem 3.8.2.

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Let A be a real n \times n matrix then the following are equivalent:
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- (a.) A is not invertible,
- (b.) Ax = 0 has at least one nontrivial solution.,
- (c.) there exists $b \in \mathbb{R}^{n \times 1}$ such that Ax = b is inconsistent,
- (d.) det(A) = 0,

It turns out this theorem is also useful. We shall see it is fundamental to the theory of eigenvectors.

Chapter 4

linear algebra

I have strayed from the ordering of topics in the text by Spence, Insel and Friedberg. We introduce vector spaces to begin, then subspaces. Subspaces give us a natural reason to think about spans and linear independence. Then the concept of a basis unifies spanning and linear independence into a common cause. The theorems and overall layout of this chapter mirror Chapter 5 of Anton and Rorres' *Elementary Linear Algebra* (9-th ed.).

Up to this point the topics we have discussed loosely fit into the category of matrix theory. The concept of a matrix is milienia old. If I trust my source, and I think I do, the Chinese even had an analogue of Gaussian elimination about 2000 years ago. The modern notation likely stems from the work of Cauchy in the 19-th century. Cauchy's prolific work colors much of the notation we still use. The theory of determinants occupied much of the professional mathematicians' for a large part of the 19-th century. Determinants produce all sorts of useful formulae, but in the modern view they play a secondary role. The concept of coordinate geometry as introduced by Descartes and Fermat around 1644 is what ultimately led to the concept of a vector space.¹. Grassmann, Hamilton, and many many others worked out volumous work detailing possible transformations on what we now call $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$,. Argand(complex numbers) and Hamilton(quaternions) had more than what we would call a vector space. They had a linear structure plus some rule for multiplication of vectors. A vector space with a multiplication is called an *algebra* in the modern terminology.

Honestly, I think once the concept of the Cartesian plane was discovered the concept of a vector space almost certainly must follow. That said, it took a while for the definition I state in the next section to appear. Giuseppe Peano gave the modern definition for a vector space in 1888². In addition he put forth some of the ideas concerning linear transformations which we discuss in the next chapter. Peano is also responsible for the modern notations for intersection and unions of sets³. He made great contributions to proof by induction and the construction of the natural

¹ Bourbaki 1969, ch. "Algebre lineaire et algebre multilineaire", pp. 7891.

 $^{^{2}}$ Peano, Giuseppe (1888), Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle Operazioni della Logica Deduttiva, Turin

 $^{^3 \}mathrm{see}$ Pg 87 of A Transition to Advanced Mathematics: A Survey Course By William Johnston

numbers from basic set theory.

Finally, I should mention the work of Hilbert, Lebesque, Fourier, Banach and others were greatly influential in the formation of infinite dimensional vector spaces. Our focus is on the finite dimensional case.⁴

Finally, let me summarize what a vector space is before we define it. In short, a vector space over \mathbb{F} is simply a set which allows you to add its elements and multiply by the numbers in \mathbb{F} .

4.1 definition and examples

Axioms are not derived from a more basic logic. They are the starting point. Their validity is ultimately judged by their use.

Definition 4.1.1.

A vector space V over \mathbb{R} is a set V together with a function $+: V \times V \to V$ called **vector** addition and another function $\cdot: \mathbb{R} \times V \to V$ called **scalar multiplication**. We require that the operations of vector addition and scalar multiplication satisfy the following 10 axioms: for all $x, y, z \in V$ and $a, b \in \mathbb{R}$,

- 1. (A1) x + y = y + x for all $x, y \in V$,
- 2. (A2) (x+y) + z = x + (y+z) for all $x, y, z \in V$,
- 3. (A3) there exists $0 \in V$ such that x + 0 = x for all $x \in V$,
- 4. (A4) for each $x \in V$ there exists $-x \in V$ such that x + (-x) = 0,
- 5. (A5) $1 \cdot x = x$ for all $x \in V$,
- 6. (A6) $(ab) \cdot x = a \cdot (b \cdot x)$ for all $x \in V$ and $a, b \in \mathbb{R}$,
- 7. (A7) $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $x, y \in V$ and $a \in \mathbb{R}$,
- 8. (A8) $(a+b) \cdot x = a \cdot x + b \cdot x$ for all $x \in V$ and $a, b \in \mathbb{R}$,
- 9. (A9) If $x, y \in V$ then x + y is a single element in V, (we say V is closed with respect to addition)
- 10. (A10) If $x \in V$ and $c \in \mathbb{R}$ then $c \cdot x$ is a single element in V. (we say V is closed with respect to scalar multiplication)

We call 0 in axiom 3 the **zero vector** and the vector -x is called the **additive inverse** of x. We will sometimes omit the \cdot and instead denote scalar multiplication by juxtaposition; $a \cdot x = ax$.

⁴this history is flawed, one-sided and far too short. You should read a few more books if you're interested.

Axioms (9.) and (10.) are admittably redundant given that those automatically follow from the statements that $+: V \times V \to V$ and $\cdot: \mathbb{R} \times V \to V$ are <u>functions</u>. I've listed them so that you are less likely to forget they must be checked.

The terminology "vector" does not necessarily indicate an explicit geometric interpretation in this general context. Sometimes I'll insert the word "abstract" to emphasize this distinction. We'll see that matrices, polynomials and functions in general can be thought of as abstract vectors.

Example 4.1.2. \mathbb{R} is a vector space if we identify addition of real numbers as the vector addition and multiplication of real numbers as the scalar multiplication.

The preceding example is very special because we can actually multiply the vectors. Usually we cannot multiply vectors.

Example 4.1.3. \mathbb{R}^n forms a vector space if we define addition by:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \equiv (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, and scalar multiplication by

$$c \cdot (x_1, x_2, \dots, x_n) = (cx_1, cx_2, \dots, cx_n)$$

for all $c \in \mathbb{R}$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Verifying the axioms is a straightforward, yet tedious, task. I'll show how we prove A1 let $(x_i), (y_i) \in \mathbb{R}^n$

$$\begin{aligned} (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) & defn. \ of + \\ &= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) & \mathbb{R} \ has \ commutative + \\ &= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n) & defn. \ of +. \end{aligned}$$

The crucial step hinges on the corresponding property of real numbers. I leave the other axioms to the reader. $\mathbb{R}^{n \times 1}$ also forms a vector space, its operations are similar. Some books freely interchange \mathbb{R}^n and $\mathbb{R}^{n \times 1}$ since they are essentially the same space. I'm probably being needlessly pedantic on this point.

Example 4.1.4. The set of all $m \times n$ matrices is denoted $\mathbb{R}^{m \times n}$. It forms a vector space with respect to matrix addition and scalar multiplication as we defined previously. Notice that we cannot mix matrices of differing sizes since we have no natural way of adding them.

Example 4.1.5. Let $\mathcal{F}(\mathbb{R})$ denote the set of all functions with domain \mathbb{R} . Let $f, g \in \mathcal{F}(\mathbb{R})$ and suppose $c \in \mathbb{R}$, define addition of functions by

$$(f+g)(x) \equiv f(x) + g(x)$$

for all $x \in \mathbb{R}$. Likewise for $f \in \mathcal{F}(\mathbb{R})$ and $c \in \mathbb{R}$ define scalar multiplication of a function by a constant in the obvious way:

$$(cf)(x) = cf(x)$$

for all $x \in \mathbb{R}$. In short, we define addition and scalar multiplication by the natural "point-wise" rules. Notice we must take functions which share the same domain since otherwise we face difficulty in choosing the domain for the new function f+g, we can also consider functions sharing a common domain $I \subset \mathbb{R}$ and denote that by $\mathcal{F}(I)$. These are called function spaces.

Example 4.1.6. Let $P_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$, the set of all polynomials up to quadratic order. Define addition and scalar multiplication by the usual operations on polynomials. Notice that if $ax^2 + bx + c$, $dx^2 + ex + f \in P_2$ then

$$(ax^{2} + bx + c) + (dx^{2} + ex + f) = (a + d)x^{2} + (b + e)x + (c + f) \in P_{2}$$

thus $+: P_2 \times P_2 \rightarrow P_2$ (it is a binary operation on P_2). Similarly,

$$d(ax^2 + bx + c) = dax^2 + dbx + dc \in P_2$$

thus scalar multiplication maps $\mathbb{R} \times P_2 \to P_2$ as it ought. Verification of the other 8 axioms is straightfoward. We denote the set of polynomials of order n or less via $P_n = \{a_n x^n + \cdots + a_1 x + a_0 | a_i \in \mathbb{R}\}$. Naturally, P_n also forms a vector space. Finally, if we take the set of all polynomials P it forms a vector space. Notice,

$$P_2 \subset P_3 \subset P_4 \subset \cdots \subset P$$

The theorem that follows is full of seemingly obvious facts. Each of these facts can be **derived** from the vector space axioms.

Theorem 4.1.7.

Let V be a vector space with zero vector 0 and let $c \in \mathbb{R}$, 1. $0 \cdot x = 0$ for all $x \in V$, 2. $c \cdot 0 = 0$ for all $c \in \mathbb{R}$, 3. $(-1) \cdot x = -x$ for all $x \in V$, 4. if cx = 0 then c = 0 or x = 0.

Lemma 4.1.8. Law of Cancellation:

Let a, x, y be vectors in a vector space V. If x + a = y + a then x = y.

Proof of Lemma: Suppose x + a = y + a. By A4 there exists -a such that a + (-a) = 0. Thus x + a = y + a implies (x + a) + (-a) = (y + a) + (-a). By A2 we find x + (a + (-a)) = y + (a + (-a)) which gives x + 0 = y + 0. Continuing we use A3 to obtain x + 0 = 0 and y + 0 = y and consequently x = y. \Box .

Proof of Theorem: Begin with (1.). Let $x \in V$, notice that by A6,

$$2 \cdot (0 \cdot x) = (2 \cdot 0) \cdot x = 0 \cdot x.$$

By A8 and A6,

$$2 \cdot (0 \cdot x) = (1+1) \cdot (0 \cdot x) = 1 \cdot (0 \cdot x) + 1 \cdot (0 \cdot x) = (1 \cdot 0) \cdot x + (1 \cdot 0) \cdot x = 0 \cdot x + 0 \cdot x.$$

Thus, $0 \cdot x = 0 \cdot x + 0 \cdot x$ and by A3 we find $0 + 0 \cdot x = 0 \cdot x + 0 \cdot x$. Using the Lemma we cancel off the $0 \cdot x$ on both sides leaving $0 = 0 \cdot x$. Since x was arbitrary it follows that $0 \cdot x = 0$ for all $x \in V$.

I'm leaving (2.), (3.) and (4.) as exercises for the reader. What makes these challenging is you have to fight the urge to use things we have yet to prove true. Essentially, all we have to work with is the cancellation Lemma and the vector space axioms. \Box

4.2 subspaces

Definition 4.2.1.

Let V be a vector space. If $W \subseteq V$ such that W is a vector space with respect to the operations of V restricted to W then we say that W is a **subspace** of V and we write $W \leq V$.

Example 4.2.2. Let V be a vector space. Notice that $V \subseteq V$ and obviously V is a vector space with respect to its operations. Therefore $V \leq V$. Likewise, the set containing the zero vector $\{0\} \leq V$. Notice that 0 + 0 = 0 and $c \cdot 0 = 0$ so Axioms 9 and 10 are satisfied. I leave the other axioms to the reader. The subspaces $\{0\}$ is called the **trivial subspace**.

Example 4.2.3. Let $L = \{(x, y) \in \mathbb{R}^2 | ax + by = 0\}$. Define addition and scalar multiplication by the natural rules in \mathbb{R}^2 . Note if $(x, y), (z, w) \in L$ then (x, y) + (z, w) = (x + z, y + w) and a(x + z) + b(y + w) = ax + by + az + bw = 0 + 0 = 0 hence $(x, y) + (z, w) \in L$. Likewise, if $c \in \mathbb{R}$ and $(x, y) \in L$ then ax + by = 0 implies acx + bcy = 0 thus $(cx, cy) = c(x, y) \in L$. We find that L is closed under vector addition and scalar multiplication. The other 8 axioms are naturally inherited from \mathbb{R}^2 .

Example 4.2.4. If $V = \mathbb{R}^3$ then

- 1. $\{(0,0,0)\}$ is a subspace,
- 2. any line through the origin is a subspace,
- 3. any plane through the origin is a subspace.

Example 4.2.5. Let $W = \{(x, y, z) \mid x + y + z = 1\}$. Is this a subspace of \mathbb{R}^3 The answer is no. There are many reasons,

- 1. $(0,0,0) \notin W$ thus W has no zero vector, axiom 3 fails. Notice we cannot change the idea of "zero" for the subspace, if (0,0,0) is zero for \mathbb{R}^3 then it is the only zero for potential subspaces. Why? Because subspaces inherit their structure from the vector space which contains them.
- 2. let $(u, v, w), (a, b, c) \in W$ then u + v + w = 1 and a + b + c = 1, however $(u + a, v + b, w + c) \notin W$ since (u + a) + (v + b) + (w + c) = (u + v + w) + (a + b + c) = 1 + 1 = 2.
- 3. let $(u, v, w) \in W$ then notice that 2(u, v, w) = (2u, 2v, 2w). Observe that 2u + 2v + 2w = 2(u + v + w) = 2 hence $(2u, 2v, 2w) \notin W$. Thus axiom 10 fails, the subset W is not closed under scalar multiplication.

Of course, one reason is all it takes.

My focus on the last two axioms is not without reason. Let me explain this obsession.

Theorem 4.2.6.

Let V be a vector space and suppose $W \subset V$ with $W \neq \emptyset$ then $W \leq V$ if and only if the following two conditions hold true

1. if $x, y \in W$ then $x + y \in W$ (W is closed under addition),

2. if $x \in W$ and $c \in \mathbb{R}$ then $c \cdot x \in W$ (W is closed under scalar multiplication).

Proof: (\Rightarrow) If $W \leq V$ then W is a vector space with respect to the operations of addition and scalar multiplication thus (1.) and (2.) hold true.

(\Leftarrow) Suppose W is a nonempty set which is closed under vector addition and scalar multiplication of V. We seek to prove W is a vector space with respect to the operations inherited from V. Let $x, y, z \in W$ then $x, y, z \in V$. Use A1 and A2 for V (which were given to begin with) to find

$$x + y = y + x$$
 and $(x + y) + z = x + (y + z)$.

Thus A1 and A2 hold for W. By (3.) of Theorem 4.1.7 we know that $(-1) \cdot x = -x$ and $-x \in W$ since we know W is closed under scalar multiplication. Consequently, $x + (-x) = 0 \in W$ since W is closed under addition. It follows A3 is true for W. Then by the arguments just given A4 is true for W. Let $a, b \in \mathbb{R}$ and notice that by A5,A6,A7,A8 for V we find

$$1 \cdot x = x, \quad (ab) \cdot x = a \cdot (b \cdot x), \quad a \cdot (x+y) = a \cdot x + a \cdot y, \quad (a+b) \cdot x = a \cdot x + b \cdot x.$$

Thus A5,A6,A7,A8 likewise hold for W. Finally, we assumed closure of addition and scalar multiplication on W so A9 and A10 are likewise satisfied and we conclude that W is a vector space. Thus $W \leq V$. (if you're wondering where we needed W nonempty it was to argue that there exists at least one vector x and consequently the zero vector is in W.) \Box

Remark 4.2.7.

The application of Theorem 4.2.6 is a four-step process

- 1. check that $W \subset V$
- 2. check that $0 \in W$

3. take arbitrary $x, y \in W$ and show $x + y \in W$

4. take arbitrary $x \in W$ and $c \in \mathbb{R}$ and show $cx \in W$

Step (2.) is just for convenience, you could just as well find another vector in W. We need to find at least one to show that W is nonempty.

Example 4.2.8. The function space $\mathcal{F}(\mathbb{R})$ has many subspaces.

- 1. continuous functions: $C(\mathbb{R})$
- 2. differentiable functions: $C^1(\mathbb{R})$
- 3. smooth functions: $C^{\infty}(\mathbb{R})$
- 4. polynomial functions
- 5. analytic functions
- 6. solution set of a linear homogeneous ODE with no singular points

The proof that each of these follows from Theorem 4.2.6. For example, f(x) = x is continuous therefore $C(\mathbb{R}) \neq \emptyset$. Moreover, the sum of continuous functions is continuous and a scalar multiple of a continuous function is continuous. Thus $C(\mathbb{R}) \leq \mathcal{F}(\mathbb{R})$. The arguments for (2.), (3.), (4.), (5.)and (6.) are identical. The solution set example is one of the most important examples for engineering and physics, linear ordinary differential equations naturally invoke a great deal of linear algebra.

Example 4.2.9. The null space of a matrix $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^{m \times 1}$ defined as follows:

$$Null(A) \equiv \{x \in \mathbb{R}^{m \times 1} \mid Ax = 0\}$$

Let's prove $Null(A) \leq \mathbb{R}^{m \times 1}$. Observe that A0 = 0 hence $0 \in Null(A)$ so the nullspace is nonempty. Suppose $x, y \in Null(A)$ and $c \in \mathbb{R}$,

$$A(x + cy) = Ax + cAy = 0 + c(0) = 0$$

thus $x + cy \in Null(A)$. Closure of addition for Null(A) follows from c = 1 and closure of scalar multiplication follows from x = 0 in the just completed calculation. \Box

Sometimes it's easier to check both scalar multiplication and addition at once. It saves some writing. If you don't understand it then don't use the trick I just used, we should understand what we are doing. **Example 4.2.10.** Let $W = \{A \in \mathbb{R}^{n \times n} \mid A^T = A\}$. This is the set of symmetric matrices, it is nonempty since $I^T = I$ (of course there are many other examples, we only need one to show it's nonempty). Let $A, B \in W$ and suppose $c \in \mathbb{R}$ then

$$(A+B)^T = A^T + B^T \quad prop. of transpose = A+B \quad since A, B \in W$$

thus $A + B \in W$ and we find W is closed under addition. Likewise let $A \in W$ and $c \in \mathbb{R}$,

$$(cA)^T = cA^T$$
 prop. of transpose
= cA since $A, B \in W$

thus $cA \in W$ and we find W is closed under scalar multiplication. Therefore, by the subspace test Theorem 4.2.6, $W \leq \mathbb{R}^{n \times n}$.

Example 4.2.11. Let $W = \{f \in \mathcal{F}(\mathbb{R}) \mid \int_{-1}^{1} f(x) dx = 0\}$. Notice the zero function 0(x) = 0 is in W since $\int_{-1}^{1} 0 dx = 0$. Let $f, g \in W$, use linearity property of the definite integral to calculate

$$\int_{-1}^{1} (f(x) + g(x)) \, dx = \int_{-1}^{1} f(x) \, dx + \int_{-1}^{1} g(x) \, dx = 0 + 0 = 0$$

thus $f + g \in W$. Likewise, if $c \in \mathbb{R}$ and $f \in W$ then

$$\int_{-1}^{1} cf(x) \, dx = c \int_{-1}^{1} f(x) \, dx = c(0) = 0$$

thus $cf \in W$ and by subspace test Theorem 4.2.6 $W \leq \mathcal{F}(\mathbb{R})$.

4.3 spanning sets and subspaces

The expression x+cy is a "linear combination" of x and y. Subspaces must keep linear combinations of subspace vectors from escaping the subspace. We defined linear combinations in a previous chapter (see 2.8.3). Can we use linear combinations to form a subspace?

Theorem 4.3.1.

Let V be a vector space which contains vectors v_1, v_2, \ldots, v_k then

- 1. the set of all linear combinations of v_1, v_2, \ldots, v_k forms a subspace of V, call it W_o
- 2. W_o is the smallest subspace of V which contains v_1, v_2, \ldots, v_k . Any other subspace which contains v_1, v_2, \ldots, v_k also contains W_o .

Proof: Define $W_o = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k \mid c_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, k\}$. Notice $0 \cdot v_1 = 0$ hence $0 \in W_o$. Suppose that $x, y \in W_o$ then there exist constants c_i and b_i such that

$$x = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$
 $y = b_1 v_1 + b_2 v_2 + \dots + b_k v_k$

Consider the sum of x and y,

$$\begin{aligned} x+y &= c_1v_1 + c_2v_2 + \cdots + c_kv_k + b_1v_1 + b_2v_2 + \cdots + b_kv_k \\ &= (c_1+b_1)v_1 + (c_2+b_2)v_2 + \cdots + (c_k+b_k)v_k \end{aligned}$$

thus $x + y \in W_o$ for all $x, y \in W_o$. Let $a \in \mathbb{R}$ and observe

$$ax = a(c_1v_1 + c_2v_2 + \dots + c_kv_k) = ac_1v_1 + ac_2v_2 + \dots + ac_kv_k$$

thus $cx \in W_o$ for all $x \in W_o$ and $c \in \mathbb{R}$. Thus by the subspace test theorem we find $W_o \leq V$.

To prove (2.) we suppose R is any subspace of V which contains v_1, v_2, \ldots, v_k . By definition R is closed under scalar multiplication and vector addition thus all linear combinations of v_1, v_2, \ldots, v_k must be in R hence $W_o \subseteq R$. Finally, it is clear that $v_1, v_2, \ldots, v_k \in W_o$ since $v_1 = 1v_1+0v_2+\cdots+0v_k$ and $v_2 = 0v_1 + 1v_2 + \cdots + 0v_k$ and so forth. \Box

Definition 4.3.2.

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a finite set of vectors in a vector space V then span(S) is defined to be the set of all linear combinations of S:

$$span\{v_1, v_2, \dots, v_k\} = \{\sum_{i=1}^k c_i v_i \mid c_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, k\}$$

If W = span(S) then we say that S is a generating set for W.

In view of Theorem 4.3.1 the definition above is equivalent to defining span(S) to be the smallest subspace which contains S.

Example 4.3.3. The following claims should be familar from calculus III:

- 1. a line through the origin is spanned by its direction vector.
- 2. a plane through the origin is spanned by any two non-paralell vectors that lie in that plane.
- 3. three dimensional space is spanned by three non-coplanar vectors. For example, $\hat{i}, \hat{j}, \hat{k}$ span \mathbb{R}^3 .

Example 4.3.4. You might have noticed we already discussed the concept of spanning in a couple particular cases in an earlier chapter. I just didn't bother to call it a "span". We should recall Propositions 2.8.4 and 2.8.6 at this point. Proposition 2.8.4 stated that each vector $v \in \mathbb{R}^{n \times 1}$ was a linear combination of the standard basis vectors e_1, e_2, \ldots, e_n :

$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n \implies v \in span\{e_1, e_2, \dots, e_n\}$$

Since v was arbitrary we have shown $\mathbb{R}^{n \times 1} \subseteq span\{e_i\}_{i=1}^n$. The converse inclusion $span\{e_i\}_{i=1}^n \subseteq \mathbb{R}^{n \times 1}$ follows from basic matrix properties. Therefore, $\mathbb{R}^{n \times 1} = span\{e_i\}_{i=1}^n$. Similar arguments show that $\mathbb{R}^{m \times n} = span\{E_{ij}\}_{i,j=1}^n$

Example 4.3.5. Let $S = \{1, x, x^2, ..., x^n\}$ then $span(S) = P_n$. For example,

$$span\{1, x, x^2\} = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\} = P_2$$

The set of all polynomials is spanned by $\{1, x, x^2, x^3, ...\}$. We are primarily interested in the span of finite sets however this case is worth mentioning.

The following definition explains what is meant by the span of an infinite set. In words, the span is the set of all finite linear combinations in the possibly infinite set. (I used this definition implicitly in the preceding example)

Definition 4.3.6.

Let S be a set of vectors. We say that S is **spanned** by a set of vectors B iff each $v \in S$ is a finite linear combination of the vectors in B. Moreover, given vectors v_1, v_2, \ldots, v_k of the same type,

 $span\{v_1, v_2, \dots, v_k\} = \{w \mid \exists c_i \text{ such that } w = c_1v_1 + c_2v_2 + \dots + c_kv_k\}$

Example 4.3.7. Let $W = \{[s + t, 2s + t, 3s + t]^T \mid s, t \in \mathbb{R}\}$. We can show W is a subspace of $\mathbb{R}^{3 \times 1}$. What is a generating set of W? Let $w \in W$ then by definition there exist $s, t \in \mathbb{R}$ such that

$$w = \begin{bmatrix} s+t\\2s+t\\3s+t \end{bmatrix} = s \begin{bmatrix} 1\\2\\3 \end{bmatrix} + t \begin{bmatrix} 1\\1\\1 \end{bmatrix}$$

Thus $w \in span\{[1,2,3]^T, [1,1,1]^T\}$ and it follows $W \subseteq span\{[1,2,3]^T, [1,1,1]^T\}$. Conversely, if $y \in span\{[1,2,3]^T, [1,1,1]^T\}$ then there exist $c_1, c_2 \in \mathbb{R}$ such that $y = c_1[1,2,3]^T + c_2[1,1,1]^T$. But then $y = [c_1 + c_2, 2c_1 + c_2, 3c_1 + c_2]^T$ so it is clear $y \in W$, therefore $span\{[1,2,3]^T, [1,1,1]^T\} \subseteq W$. It follows that $W = span\{[1,2,3]^T, [1,1,1]^T\}$. Finally, Theorem 4.3.1 gaurantees that $W \leq \mathbb{R}^{3 \times 1}$.

The lesson of the last example is that we can show a particular space is a subspace by finding its generating set. Theorem 4.3.1 tells us that any set generated by a span is a subspace. This test is only convenient for subspaces which are defined as some sort of span. In that case we can immediately conclude the subset is in fact a subspace.

Example 4.3.8. Let $A \in \mathbb{R}^{m \times n}$. Define the column space of A as the span of the columns of A:

$$Col(A) = span\{col_j(A) \mid j = 1, 2, \dots, n\}$$

this is clearly a subspace of $\mathbb{R}^{n \times 1}$ since it is constructed as a span of vectors in E_{11} and E_{22} . We also can define row space as the span of the rows:

$$Row(A) = span\{row_i(A) \mid i = 1, 2, \dots, m\}$$

this is clearly a subspace of $\mathbb{R}^{1 \times m}$ since it is formed as a span of vectors. Since the columns of A^T are the rows of A and the rows of A^T are the columns of A we can conclude that $Col(A^T) = Row(A)$ and $Row(A^T) = Col(A)$.

Time for examples with more bite.

Example 4.3.9. Let $b_1 = e_1 + e_2, b_2 = e_2 + e_3, b_3 = e_2 - e_3$ be 3×1 vectors. Does $\{b_1, b_2, b_3\}$ span $\mathbb{R}^{3 \times 1}$? Let $v = [v_1 \ v_2 \ v_3]^T \in \mathbb{R}^{3 \times 1}$, we seek to find if there exist $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = x_1(e_1 + e_2) + x_2(e_2 + e_3) + x_3(e_2 - e_3)$$

This yields

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_3 \\ -x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Or in matrix notation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

We find b_1, b_2, b_3 span $\mathbb{R}^{3\times 1}$ if the matrix equation $[b_1|b_2|b_3]x = v$ is consistent for all vectors v. We learned in the previous work that this is only the case if $rref[b_1|b_2|b_3] = I$ since then $[b_1|b_2|b_3]$ is invertible and we'll find $x = [b_1|b_2|b_3]^{-1}v$ is the solution of $[b_1|b_2|b_3]x = v$ for arbitrary v. Observe,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\frac{-r_3/2 \to r_3}{r_2 - r_3 \to r_2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{r_1 - r_3 \to r_1}{r_1 \to r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $rref[b_1|b_2|b_3] = I$ and we find b_1, b_2, b_3 span $\mathbb{R}^{3 \times 1}$. Question: how would you calculate $x, y, z \in \mathbb{R}$ such that $v = xb_1 + yb_2 + zb_3$?

Example 4.3.10. Let b_1, b_2, b_3 be defined in the same way as the preceding example. Is $[0 \ 0 \ 1] = e_3 \in span\{b_1, b_2, b_3\}$? Find the explicit linear combination of b_1, b_2, b_3 that produces e_3 . Following the same arguments as the last example this boils down to the question of which v provides a solution of $[b_1|b_2|b_3]v = e_3$? We can calculate the solution by the same row reduction as the preceding example if we simply concatenate e_3 onto the right of the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 1 \end{bmatrix} \xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -1 & | & 1 \end{bmatrix} \xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & | & 1 \end{bmatrix} \xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & -1/2 \end{bmatrix}$$

We find that $e_3 = \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3$ thus $e_3 \in span\{b_1, b_2, b_3\}$. Question: how would you calculate $x, y, z \in \mathbb{R}$ such that $v = xb_1 + yb_2 + zb_3$? If $v = [v_1 \ v_2 \ v_3]^T$ then the answer will give x, y, z as functions of v_1, v_2, v_3 .

The power of the matrix technique shines bright in the next example. Sure you could guess the last two, but as things get messy we'll want a refined efficient algorithm to dispatch spanning questions with ease.

Example 4.3.11. Let $b_1 = [1 \ 2 \ 3 \ 4]^T$, $b_2 = [0 \ 1 \ 0 \ 1]^T$ and $b_3 = [0 \ 0 \ 1 \ 1]^T$. Is $w = [1 \ 1 \ 4 \ 4]^T \in span\{b_1, b_2, b_3\}$? We seek to find $x = [x_1 \ x_2 \ x_3]^T$ such that $x_1b_1 + x_2b_2 + x_3b_3 = w$. In other words, can we solve $[b_1|b_2|b_3]x = w$? Use the aug. coeff. matrix as is our custom:

$$\begin{bmatrix} b_1 | b_2 | b_3 | w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 2 & 1 & 0 & | & 1 \\ 3 & 0 & 1 & | & 4 \\ 4 & 1 & 1 & | & 4 \end{bmatrix} \xrightarrow{\begin{array}{c} r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3 \\ r_4 - 4r_1 \to r_4 \end{array}} \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \xrightarrow{\begin{array}{c} r_4 - r_2 \to r_4 \\ r_4 - r_2 \to r_4 \end{array}} \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \xrightarrow{\begin{array}{c} r_4 - r_2 \to r_4 \\ r_4 - r_3 \to r_4 \end{array}} \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \xrightarrow{\begin{array}{c} r_4 - r_3 \to r_4 \\ r_4 - r_3 \to r_4 \end{array}} \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = rref[b_1 | b_2 | b_3 | w]$$

We find $x_1 = 1, x_2 = -1, x_3 = 1$ thus $w = b_1 - b_2 + b_3$. Therefore, $w \in span\{b_1, b_2, b_3\}$.

Remark 4.3.12.

If we are given $B = \{b_1, b_2, \ldots, b_k\} \subset \mathbb{R}^{n \times 1}$ and $T = \{w_1, w_2, \ldots, w_r\} \subset \mathbb{R}^{n \times 1}$ and we wish to determine if $T \subset span(B)$ then we can answer the question by examining if $[b_1|b_2|\cdots|b_k]x = w_j$ has a solution for each $j = 1, 2, \ldots r$. Or we could make use of Proposition 2.6.3 and solve it in one sweeping matrix calculation;

$$rref[b_1|b_2|\cdots|b_k|w_1|w_2|\cdots|w_r]$$

If there is a row with zeros in the first k-columns and a nonzero entry in the last r-columns then this means that at least one vector w_k is not in the span of B (moreover, the vector not in the span corresponds to the nonzero entrie(s)). Otherwise, each vector is in the span of B and we can read the precise linear combination from the matrix. I will illustrate this in the example that follows. **Example 4.3.13.** Let $W = span\{e_1 + e_2, e_2 + e_3, e_1 - e_3\}$ and suppose $T = \{e_1, e_2, e_3 - e_1\}$. Is $T \leq W$? If not, which vectors in T are not in W? Consider,

$$\begin{bmatrix} e_1 + e_1 | e_2 + e_3 | e_1 - e_3 | | e_1 | e_2 | e_3 - e_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & -1 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & 0 & -1 \\ 0 & 1 & -1 & | & -1 & 1 & 1 \\ 0 & 1 & -1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & -1 \\ 0 & 1 & -1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 + r_3 \to r_2} \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & -1 \\ 0 & 1 & -1 & | & 0 & 0 & | \\ 1 & 0 & 1 & | & 1 & -1 & 0 \end{bmatrix} \xrightarrow{r_2 + r_3 \to r_2} \xrightarrow{r_1 \to r_2 \to r_3} \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & -1 \\ 0 & 1 & -1 & | & 0 & 0 & | \\ 1 & 0 & 1 & | & 1 & -1 & 0 \end{bmatrix} \xrightarrow{r_2 + r_3 \to r_2} \xrightarrow{r_1 \to r_2} \xrightarrow{r_1 \to r_2} \xrightarrow{r_2 \to r_3} \xrightarrow{r_1 \to r_2} \xrightarrow{r_2 \to r_3} \xrightarrow{r_1 \to r_2} \xrightarrow{r_2 \to r_3} \xrightarrow{r_1 \to r_2} \xrightarrow{r_2 \to r_3} \xrightarrow{r_2 \to r_3} \xrightarrow{r_3 \to r_4} \xrightarrow{r_4 \to r_4} \xrightarrow{r_4$$

Let me summarize the calculation:

$$rref[e_1 + e_2|e_2 + e_3||e_1 - e_3|e_1|e_2|e_3 - e_1] = \begin{bmatrix} 1 & 0 & 1 & | & 0 & 1 & -1 \\ 0 & 1 & -1 & | & 0 & 0 & 1 \\ 0 & 0 & 0 & | & 1 & -1 & 0 \end{bmatrix}$$

We deduce that e_1 and e_2 are not in W. However, $e_1 - e_3 \in W$ and we can read from the matrix $-(e_1 + e_2) + (e_2 + e_3) = e_3 - e_1$. I added the double vertical bar for book-keeping purposes, as usual the vertical bars are just to aid the reader in parsing the matrix.

Proposition 4.3.14.

If
$$S = \{v_1, v_2, \dots, v_m\} \subset \mathbb{R}^{n \times 1}$$
 and $k < n$ then $span(S) \neq \mathbb{R}^{n \times 1}$.

Proof: I'll sketch the proof. If $A \in \mathbb{R}^{n \times m}$ is a matrix with m < n then we cannot have *m*-pivot columns. It follows rref[A|b] can represent an inconsistent system Ax = b since *b* could put a pivot column in the rightmost column. Thus, there are $b \in \mathbb{R}^{m \times 1}$ such that $Ax \neq b$ for any possible *x*. So, given the vectors in *S* if we concantenate them we'll find a matrix [S] in $\mathbb{R}^{n \times m}$ with m < n and as such there will be some choice of the vector *b* for which $[S]x \neq b$. It follows $b \notin span(S)$ hence $span(S) \neq \mathbb{R}^{n \times 1}$. \Box

The tricks we've developed in these last few examples really only work for vectors in $\mathbb{R}^{n\times 1}$. If we have abstract vectors, or even just row vectors, then we'll need to deal with spanning questions by other methods. However, once we have the idea of *coordinates* ironed out then we can use the tricks on the coordinate vectors then push back the result to the world of abstract vectors.

Example 4.3.15. Is $E_{11} \in span\{E_{12} + 2E_{11}, E_{12} - E_{11}\}$? Assume $E_{ij} \in \mathbb{R}^{2 \times 2}$ for all i, j. We seek to find solutions of

$$E_{11} = a(E_{12} + 2E_{11}) + b(E_{12} - E_{11})$$

in explicit matrix form the equation above reads:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right) + b \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 2a & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -b & b \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2a - b & a + b \\ 0 & 0 \end{bmatrix}$$

thus 1 = 2a - b and 0 = a + b. Substitute a = -b to find 1 = 3a hence $a = \frac{1}{3}$ and $b = \frac{-1}{3}$. Indeed,

$$\frac{1}{3}(E_{12} + 2E_{11}) - \frac{1}{3}(E_{12} - E_{11}) = \frac{2}{3}E_{11} + \frac{1}{3}E_{11} = E_{11}.$$

Therefore, $E_{11} \in span\{E_{12} + 2E_{11}, E_{12} - E_{11}\}.$

Example 4.3.16. Find a generating set for the set of symmetric 2×2 matrices. That is find a set S of matrices such that $span(S) = \{A \in \mathbb{R}^{2 \times 2} \mid A^T = A\} = W$. There are many approaches, but I find it most natural to begin by studying the condition which defines W. Let $A \in W$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

this means we need b = c but we find no particular condition on a or d. Notice $A \in W$ implies

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aE_{11} + b(E_{12} + E_{21}) + dE_{22}$$

Thus $A \in W$ implies $A \in span\{E_{11}, E_{12} + E_{21}, E_{22}\}$, hence $W \subseteq span\{E_{11}, E_{12} + E_{21}, E_{22}\}$. Conversely, if $B \in span\{E_{11}, E_{12} + E_{21}, E_{22}\}$ then there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$B = c_1 E_{11} + c_2 (E_{12} + E_{21}) + c_3 E_{22}$$

but this means

$$B = \left[\begin{array}{cc} c_1 & c_2 \\ c_2 & c_3 \end{array} \right]$$

so B is symmetric and it follows $span\{E_{11}, E_{12}+E_{21}, E_{22}\} \subseteq W$. Consequently $W = span\{E_{11}, E_{12}+E_{21}, E_{22}\}$ and the set $\{E_{11}, E_{12}+E_{21}, E_{22}\}$ generates W. This is not unique, there are many other sets which also generate W. For example, if we took $\overline{S} = \{E_{11}, E_{12}+E_{21}, E_{22}, E_{11}+E_{22}\}$ then the span of \overline{S} would still work out to W.

Theorem 4.3.17.

If $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_r\}$ are subsets of a vector space V then span(S) = span(T) if and only if every vector in S is a linear combination of vectors in T and every vector in T is a linear combination of vectors in S.

Proof: (\Rightarrow) Assume span(S) = span(T). If $v \in S$ then $v \in span(S)$ hence $v \in span(T)$ and it follows that v is a linear combination of vectors in T. If $w \in T$ then $w \in span(T)$ hence $w \in span(S)$ and by definition of the span(S) we find w is a linear combination of vectors in S.

(\Leftarrow) Assume every vector in S is a linear combination of vectors in T and every vector in T is a linear combination of vectors in S. Suppose $v \in Span(S)$ then v is a linear combination of vectors in S, say

$$v = c_1 s_1 + c_2 s_2 + \dots + c_k s_k.$$

Furthermore, each vector in S is a linear combination of vectors in T by assumption so there exist constants d_{ij} such that

$$s_i = d_{i1}t_1 + d_{i2}t_2 + \dots + d_{ir}t_r$$

for each $i = 1, 2, \ldots, k$. Thus,

$$v = c_1 s_1 + c_2 s_2 + \dots + c_k s_k.$$

= $c_1 (d_{11}t_1 + d_{12}t_2 + \dots + d_{1r}t_r) + c_2 (d_{21}t_1 + d_{22}t_2 + \dots + d_{2r}t_r) + \dots + c_k (d_{k1}t_1 + d_{k2}t_2 + \dots + d_{kr}t_r)$
= $(c_1 d_{11} + c_2 d_{21} + \dots + c_k d_{k1})t_1 + (c_1 d_{12} + c_2 d_{22} + \dots + c_k d_{k2})t_2 + \dots + (c_1 d_{1r} + c_2 d_{2r} + \dots + c_k d_{kr})t_r$

thus v is a linear combination of vectors in T, in other words $v \in span(T)$ and we find $span(S) \subseteq span(T)$. Notice, we just proved that a linear combination of linear combinations is again a linear combination. Almost the same argument shows $span(T) \subseteq span(S)$ hence span(S) = span(T). \Box .

4.4 linear independence

We have seen a variety of generating sets in the preceding section. In the last example I noted that if we added an additional vector $E_{11} + E_{22}$ then the same span would be created. The vector $E_{11} + E_{22}$ is **redundant** since we already had E_{11} and E_{22} . In particular, $E_{11} + E_{22}$ is a linear combination of E_{11} and E_{22} so adding it will not change the span. How can we decide if a vector is absolutely necessary for a span? In other words, if we want to span a subspace W then how do we find a **minimal spanning set**? We want a set of vectors which does not have any linear dependencies. For example, $\hat{i}, \hat{j}, \hat{k}$ spans \mathbb{R}^3 however if we took any one of these away we would only generate a plane. We say such vectors are linearly independent. Let me be precise:

Definition 4.4.1.

If a vector v_k can be written as a linear combination of vectors $\{v_1, v_2, \ldots, v_{k-1}\}$ then we say that the vectors $\{v_1, v_2, \ldots, v_{k-1}, v_k\}$ are **linearly dependent**. If the vectors $\{v_1, v_2, \ldots, v_{k-1}, v_k\}$ are not linear dependent then they are said to be **linearly independent**.

Example 4.4.2. Let $v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ and $w = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}^T$. Clearly v, w are linearly dependent since w = 2v.

I often quote the following proposition as the definition of linear independence, it is an equivalent statement and as such can be used as the definition. If this was our definition then our definition would become a proposition. Math always has a certain amount of this sort of ambiguity.

Proposition 4.4.3.

 c_1, c_2, \ldots, c_k such that

Let $v_1, v_2, \ldots, v_k \in V$ a vector space. The set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent iff $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \implies c_1 = c_2 = \cdots = c_k = 0.$

Proof: (\Rightarrow) Suppose $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Assume that there exist constants

$$c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$$

and at least one constant, say c_i , is nonzero. Then we can divide by c_i to obtain

$$\frac{c_1}{c_j}v_1 + \frac{c_2}{c_j}v_2 + \dots + v_j + \dots + \frac{c_k}{c_j}v_k = 0$$

solve for v_i , (we mean for $\hat{v_i}$ to denote the deletion of v_i from the list)

$$v_j = -\frac{c_1}{c_j}v_1 - \frac{c_2}{c_j}v_2 - \dots - \hat{v_j} - \dots - \frac{c_k}{c_j}v_k$$

but this means that v_j linearly depends on the other vectors hence $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent. This is a contradiction, therefore $c_j = 0$. Note j was arbitrary so we may conclude $c_j = 0$

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for all *j*. Therefore, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \implies c_1 = c_2 = \cdots = c_k = 0$.

Proof: (\Leftarrow) Assume that

 $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0 \implies c_1 = c_2 = \dots = c_k = 0.$

If $v_j = b_1v_1 + b_2v_2 + \cdots + \widehat{b_jv_j} + \cdots + b_kv_k$ then $b_1v_1 + b_2v_2 + \cdots + b_jv_j + \cdots + b_kv_k = 0$ where $b_j = -1$, this is a contradiction. Therefore, for each j, v_j is not a linear combination of the other vectors. Consequently, $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proposition 4.4.4.

S is a linearly independent set of vectors iff for all $v_1, v_2, \ldots, v_k \in S$,

 $a_1v_1 + a_2v_2 + \dots + a_kv_k = b_1v_1 + b_2v_2 + \dots + b_kv_k$

implies $a_i = b_i$ for each i = 1, 2, ..., k. In other words, we can <u>equate coefficients</u> of linearly indpendent vectors. And, conversely if a set of vectors allows for equating coefficients then it is linearly independent.

Proof: see the Problem Set. This is important, you need to prove it for yourself. This is yet another equivalent definition of linear independence, as such I will at times say we can equation coefficients since the vectors are linearly independent. \Box

Proposition 4.4.5.

If S is a finite set of vectors which contains the zero vector then S is linearly dependent.

Proof: Let $\{\vec{0}, v_2, \dots, v_k\} = S$ and observe that

$$1\vec{0} + 0v_2 + \dots + 0v_k = 0$$

Thus $c_1\vec{0} + c_2v_2 + \cdots + c_kv_k = 0$ does not imply $c_1 = 0$ hence the set of vectors is not linearly independent. Thus S is linearly dependent. \Box

Proposition 4.4.6.

Let v and w be nonzero vectors.

v, w are linearly dependent $\Leftrightarrow \exists k \neq 0 \in \mathbb{R}$ such that v = kw.

Proof: Suppose v, w are linearly dependent then there exist constants c_1, c_2 , not all zero, such that $c_1v + c_2w = 0$. Suppose that $c_1 = 0$ then $c_2w = 0$ hence $c_2 = 0$ or w = 0 by (4.) of Theorem 4.1.7. But this is a contradiction since v, w are nonzero and at least one of c_1, c_2 must be nonzero. Therefore, $c_1 \neq 0$. Likewise, if $c_2 = 0$ we find a similar contradiction. Hence c_1, c_2 are both nonzero and we calculate $v = (-c_2/c_1)w$, identify that $k = -c_2/c_1$. \Box

Remark 4.4.7.

For two vectors the term "linearly dependent" can be taken quite literally: two vectors are linearly dependent if they point along the same line. For three vectors they are linearly dependent if they point along the same line or possibly lay in the same plane. When we get to four vectors we can say they are linearly dependent if they reside in the same volume, plane or line. As I mentioned previously I don't find the geometric method terribly successful for dimensions higher than two. However, it is neat to think about the geometric meaning of certain calculations in dimensions higher than 3. We can't even draw it but we can eulicidate all sorts of information with mathematics.

Example 4.4.8. Let $v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ and $w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Let's prove these are linearly independent. Assume that $c_1v + c_2w = 0$, this yields

	$\begin{bmatrix} 1 \end{bmatrix}$		1		[0]
c_1	2	$+ c_2$	0	=	0
	3		0		

thus $c_1 + c_2 = 0$ and $2c_1 = 0$ and $3c_1 = 0$. We find $c_1 = c_2 = 0$ thus v, w are linearly independent. Alternatively, you could explain why there does not exist any $k \in \mathbb{R}$ such that v = kw

Think about this, if the set of vectors $\{v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^{n \times 1}$ is linearly independent then the equation $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ has the unique solution $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. Notice we can reformulate the problem as a matrix equation:

$$c_1v_1 + c_2v_2 + \dots + c_kv_k = 0 \iff [v_1|v_2|\cdots|v_k][c_1 \ c_2 \ \cdots \ c_k]^T = 0$$

The matrix $[v_1|v_2|\cdots|v_k]$ is an $n \times k$. This is great. We can use the matrix techniques we already developed to probe for linear independence of a set of vectors.

Proposition 4.4.9.

Let $\{v_1, v_2, \ldots, v_k\}$ be a set of vectors in $\mathbb{R}^{n \times 1}$.

- 1. If $rref[v_1|v_2|\cdots|v_k]$ has less than k pivot columns then the set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.
- 2. If $rref[v_1|v_2|\cdots|v_k]$ has k pivot columns then the set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof: Denote $V = [v_1|v_2|\cdots|v_k]$ and $c = [c_1, c_2, \ldots, c_k]^T$. If V contains a linearly independent set of vectors then we must find that Vc = 0 implies c = 0. Consider Vc = 0, this is equivalent to using Gaussian elimination on the augmented coefficient matrix [V|0]. We know this system is consistent since c = 0 is a solution. Thus Theorem 1.5.1 tells us that there is either a unique solution or infinitely many solutions.

Clearly if the solution is unique then c = 0 is the only solution and hence the implication Av = 0 implies c = 0 holds true and we find the vectors are linearly independent. In this case we would find

$$rref[v_1|v_2|\cdots|v_k] = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

where there are *n*-rows in the matrix above. If n = k then there would be no zeros below the *k*-th row.

If there are infinitely many solutions then there will be free variables in the solution of Vc = 0. If we set the free variables to 1 we then find that Vc = 0 does not imply c = 0 since at least the free variables are nonzero. Thus the vectors are linearly dependent in this case, proving (2.). Rather than attempting to sketch a general rref[V|0] I will illustrate with several examples to follow. \Box

Before I get to the examples let me glean one more fairly obvious statement from the proof above:

Corollary 4.4.10.

If $\{v_1, v_2, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^{n \times 1}$ and k > n then the vectors are linearly dependent.

Proof: Proposition 4.4.9 tells us that the set is linearly independent if there are k pivot columns in $[v_1|\cdots|v_k]$. However, that is impossible since k > n this means that there will be at least one column of zeros in $rref[v_1|\cdots|v_k]$. Therefore the vectors are linearly dependent. \Box

This Proposition is obvious but useful. We may have at most 2 linearly independent vectors in \mathbb{R}^2 , 3 in \mathbb{R}^3 , 4 in \mathbb{R}^4 , and so forth...

Example 4.4.11. Determine if v_1, v_2, v_3 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.

$$v_1 = \begin{bmatrix} 1\\1\\1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2\\1\\0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3\\2\\1 \end{bmatrix}$$

We seek to use the Proposition 4.4.9. Consider then,

$$\begin{bmatrix} v_1 | v_2 | v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3\\ 1 & 1 & 2\\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 3\\ 0 & -1 & -1\\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{r_1 + 2r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1\\ 0 & -1 & -1\\ 0 & 0 & 0 \end{bmatrix}$$

Thus we find that,

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the variable c_3 is free in the solution of Vc = 0. We find solutions of the form $c_1 = -c_3$ and $c_2 = -c_3$. This means that

 $-c_3v_1 - c_3v_2 + c_3v_3 = 0$

for any value of c_3 . I suggest $c_3 = 1$ is easy to plug in,

$$-v_1 - v_2 + v_3 = 0$$
 or we could write $v_3 = v_1 + v_2$

We see clearly that v_3 is a linear combination of v_1, v_2 .

Example 4.4.12. Determine if v_1, v_2, v_3, v_4 (given below) are linearly independent or dependent.

$$v_1 = \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1\\1\\0\\0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1\\1\\1\\0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix}$$

We seek to use the Proposition 4.4.9. Notice that $[v_1|v_2|v_3|v_4]$ is upper triangular. Recall Proposition 3.5.3 states the determinant of an upper triangular matrix is the product of the diagonal entries. Thus $det[v_1|v_2|v_3|v_4] = 1$ and by Theorem 3.8.1 the system has a unique solution. Hence the vectors are linearly independent. (I don't need to write the following matrix equation but I will just to illustrate the Proposition in this case)

$$rref[v_1|v_2|v_3|v_4] = rref\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case no variables are free, the only solution is $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$.

Example 4.4.13. Determine if v_1, v_2, v_3 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.

$$v_1 = \begin{bmatrix} 1\\0\\0\\3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3\\1\\2\\0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 2\\1\\2\\-3 \end{bmatrix}$$

We seek to use the Proposition 4.4.9. Consider $[v_1|v_2|v_3] =$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & -3 \end{bmatrix} \xrightarrow{r_4 - 3r_1 \to r_4} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -9 & -9 \end{bmatrix} \xrightarrow{r_1 - 3r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ r_4 + 9r_2 \to r_4 \end{array} = rref[V].$$

Hence the variable c_3 is free in the solution of Vc = 0. We find solutions of the form $c_1 = c_3$ and $c_2 = -c_3$. This means that

$$c_3v_1 - c_3v_2 + c_3v_3 = 0$$

for any value of c_3 . I suggest $c_3 = 1$ is easy to plug in,

$$v_1 - v_2 + v_3 = 0$$
 or we could write $v_3 = v_2 - v_1$

We see clearly that v_3 is a linear combination of v_1, v_2 .

Example 4.4.14. Determine if v_1, v_2, v_3, v_4 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.

$$v_1 = \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0\\1\\1\\0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0\\1\\2\\0 \end{bmatrix}$$

We seek to use the Proposition 4.4.9. Consider $[v_1|v_2|v_3|v_4] =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \rightarrow r_1} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref[v_1|v_2|v_3|v_4]$$

Hence the variables c_3 and c_4 are free in the solution of Vc = 0. We find solutions of the form $c_1 = -c_3 + c_4$ and $c_2 = -c_3 - c_4$. This means that

$$(c_4 - c_3)v_1 - (c_3 + c_4)v_2 + c_3v_3 + c_4v_4 = 0$$

for any value of c_3 or c_4 . I suggest $c_3 = 1, c_4 = 0$ is easy to plug in,

$$-v_1 - v_2 + v_3 = 0$$
 or we could write $v_3 = v_2 + v_1$

Likewise select $c_3 = 0, c_4 = 1$ to find

$$v_1 - v_2 + v_4 = 0$$
 or we could write $v_4 = v_2 - v_1$

We find that v_3 and v_4 are linear combinations of v_1 and v_2 .

If you pay particular attention to the last few examples you may have picked up on a pattern. The columns of the $rref[v_1|v_2|\cdots|v_k]$ depend on each other in the same way that the vectors $v_1, v_2, \ldots v_k$ depend on each other. These provide examples of the so-called "column correspondence property". In a nutshell, the property says you can read the linear dependencies right off the $rref[v_1|v_2|\cdots|v_k]$.

Proposition 4.4.15.

Let $A = [col_1(A)| \cdots |col_n(A)] \in \mathbb{R}^{m \times n}$ and $R = rref[A] = [col_1(R)| \cdots |col_n(R)]$. There exist constants $c_1, c_2, \ldots c_k$ such that $c_1col_1(A) + c_2col_2(A) + \cdots + c_kcol_k(A) = 0$ if and only if $c_1col_1(R) + c_2col_2(R) + \cdots + c_kcol_k(R) = 0$. If $col_j(rref[A])$ is a linear combination of other columns of rref[A] then $col_j(A)$ is likewise the same linear combination of columns of A.

We prepare for the proof of the Proposition by establishing a useful Lemma.

Lemma 4.4.16.

Let $A \in \mathbb{R}^{m \times n}$ then there exists an invertible matrix E such that $col_j(rref(A)) = Ecol_j(A)$ for all j = 1, 2, ..., n.

Proof of Lemma: Recall that there exist elementary matrices E_1, E_2, \ldots, E_r such that $A = E_1E_2\cdots E_r rref(A) = E^{-1}rref(A)$ where I have defined $E^{-1} = E_1E_2\cdots E_k$ for convenience. Recall the concatenation proposition: $X[b_1|b_2|\cdots|b_k] = [Xb_1|Xb_2|\cdots|Xb_k]$. We can unravel the Gaussian elimination in the same way,

$$EA = E[col_1(A)|col_2(A)|\cdots|col_n(A)]$$
$$= [Ecol_1(A)|Ecol_2(A)|\cdots|Ecol_n(A)]$$

Observe that EA = rref(A) hence we find the above equation says $col_j(rref(A)) = Ecol_j(A)$ for all j. \Box

Proof of Proposition: Suppose that there exist constants c_1, c_2, \ldots, c_k such that $c_1 col_1(A) + c_2 col_2(A) + \cdots + c_k col_k(A) = 0$. By the Lemma we know there exists E such that $col_j(rref(A)) = Ecol_j(A)$. Multiply linear combination by E to find:

$$c_1 E col_1(A) + c_2 E col_2(A) + \dots + c_k E col_k(A) = 0$$

which yields

$$c_1 col_1(rref(A)) + c_2 col_2(rref(A)) + \dots + c_k col_k(rref(A)) = 0.$$

Likewise, if we are given a linear combination of columns of rref(A) we can multiply by E^{-1} to recover the same linear combination of columns of A. \Box

Example 4.4.17. I will likely use the abbreviation "CCP" for column correspondence property. We could have deduced all the linear dependencies via the CCP in Examples 4.4.11,4.4.13 and 4.4.14. We found in 4.4.11 that

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously $col_3(R) = col_1(R) + col_2(R)$ hence by CCP $v_3 = v_1 + v_2$. We found in 4.4.13 that

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $col_3(R) = col_2(R) - col_1(R)$ hence by CCP $v_3 = v_2 - v_1$. We found in 4.4.14 that

$$rref[v_1|v_2|v_3|v_4] = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By inspection, $col_3(R) = col_1(R) + col_2(R)$ hence by CCP $v_3 = v_1 + v_2$. Likewise by inspection, $col_4(R) = col_2(R) - col_1(R)$ hence by CCP $v_4 = v_2 - v_1$.

You should notice that the CCP saves us the trouble of expressing how the constants c_i are related. If we are only interested in how the vectors are related the CCP gets straight to the point quicker. We should pause and notice another pattern here while were thinking about these things.

Proposition 4.4.18.

The non-pivot columns of a matrix can be written as linear combinations of the pivot columns and the pivot columns of the matrix are linearly independent.

Proof: Let A be a matrix. Notice the Proposition is clearly true for rref(A). Hence, using Lemma 4.4.16 we find the same is true for the matrix A. \Box

Proposition 4.4.19.

The rows of a matrix A can be written as linear combinations of the transposes of pivot columns of A^T , and the rows which are transposes of the pivot columns of A^T are linearly independent.

Proof: Let A be a matrix and A^T its transpose. Apply Proposition 4.4.15 to A^T to find pivot columns which we denote by $col_{i_j}(A^T)$ for j = 1, 2, ..., k. These columns are linearly independent and they span $Col(A^T)$. Suppose,

$$c_1 row_{i_1}(A) + c_2 row_{i_2}(A) + \dots + c_k row_{i_k}(A) = 0.$$

Take the transpose of the equation above, use Proposition 2.9.3 to simplify:

$$c_1(row_{i_1}(A))^T + c_2(row_{i_2}(A))^T + \dots + c_k(row_{i_k}(A))^T = 0$$

Recall $(row_j(A))^T = col_j(A^T)$ thus,

$$c_1 col_{i_1}(A^T) + c_2 col_{i_2}(A^T) + \dots + c_k col_{i_k}(A^T) = 0.$$

hence $c_1 = c_2 = \cdots = c_k = 0$ as the pivot columns of A^T are linearly independendent. This shows the corresponding rows of A are likewise linearly independent. The proof that these same rows span Row(A) is similar. \Box

4.5 bases and dimension

We have seen that linear combinations can generate vector spaces. We have also seen that sometimes we can remove a vector from the generating set and still generate the whole vector space. For example,

$$span\{e_1, e_2, e_1 + e_2\} = \mathbb{R}^{2 \times 1}$$

and we can remove any one of these vector and still span $\mathbb{R}^{2\times 1}$,

$$span\{e_1, e_2\} = span\{e_1, e_1 + e_2\} = span\{e_2, e_1 + e_2\} = \mathbb{R}^{2 \times 1}$$

However, if we remove another vector then we will not span $\mathbb{R}^{2\times 1}$. A generating set which is just big enough is called a basis. We can remove vectors which are linearly dependent on the remaining vectors without changing the span. Therefore, we should expect that a minimal spanning set is linearly independent.

Definition 4.5.1.

A **basis** for a vector space V is a set of vectors S such that

1. V = span(S),

2. S is linearly independent.

Example 4.5.2. It is not hard to show that $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1, e_1 + e_2\}$ and $B_3 = \{e_2, e_1 + e_2\}$ are linearly independent sets. Furthermore, each spans $\mathbb{R}^{2 \times 1}$. Therefore, B_1, B_2, B_3 are bases for $\mathbb{R}^{2 \times 1}$. In particular, $B_1 = \{e_1, e_2\}$ is called the <u>standard basis</u>.

Example 4.5.3. I called $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathbb{R}^{n \times 1}$. Since $v \in \mathbb{R}^{n \times 1}$ can be written as

$$v = \sum_{i} v_i e_i$$

it follows $\mathbb{R}^{n \times 1} = span\{e_i \mid 1 \le i \le n\}$. Moreover, linear independence of $\{e_i \mid 1 \le i \le n\}$ follows from a simple calculation:

$$0 = \sum_{i} c_{i} e_{i} \Rightarrow 0 = \left[\sum_{i} c_{i} e_{i}\right]_{k} = \sum_{i} c_{i} \delta_{ik} = c_{k}$$

hence $c_k = 0$ for all k. Thus $\{e_i \mid 1 \leq i \leq n\}$ is a basis for $\mathbb{R}^{n \times 1}$, we continue to call it the standard basis of $\mathbb{R}^{n \times 1}$. The vectors e_i are also called "unit-vectors".

Example 4.5.4. Since $A \in \mathbb{R}^{m \times n}$ can be written as

$$A = \sum_{i,j} A_{ij} E_{ij}$$

it follows $\mathbb{R}^{m \times n} = span\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Moreover, linear independence of $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ follows from a simple calculation:

$$0 = \sum_{i,j} c_{ij} E_{ij} \Rightarrow 0 = \left[\sum_{i,j} c_{ij} E_{ij}\right]_{kl} = \sum_{i,j} c_{ij} \delta_{ik} \delta_{jl} = c_{kl}$$

hence $c_{kl} = 0$ for all k, l. Thus $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $\mathbb{R}^{m \times n}$, we continue to call it the standard basis of $\mathbb{R}^{m \times n}$. The matrices E_{ij} are also called "unit-matrices".

Definition 4.5.5.

Suppose
$$B = \{f_1, f_2, \dots, f_n\}$$
 is a basis for V . If $v \in V$ has
 $v = v_1 f_1 + v_2 f_2 + \dots + v_n f_n$
then $[v]_B = [v_1 \ v_2 \ \dots \ v_n]^T \in \mathbb{R}^{n \times 1}$ is called the **coordinate vector** of v with respect to
 B

Technically, the each basis considered in the course is an "ordered basis". This means the set of vectors that forms the basis has an ordering to it. This is more structure than just a plain set since basic set theory does not distinguish $\{1,2\}$ from $\{2,1\}$. I should always say "we have an ordered basis" but I will not (and most people do not) say that in this course. Let it be understood that when we list the vectors in a basis they are listed in order and we cannot change that order without changing the basis. For example $v = [1, 2, 3]^T$ has coordinate vector $[v]_{B_1} = [1, 2, 3]^T$ with respect to $B_1 = \{e_1, e_2, e_3\}$. On the other hand, if $B_2 = \{e_2, e_1, e_3\}$ then the coordinate vector of v with respect to B_2 is $[v]_{B_2} = [2, 1, 3]^T$.

Proposition 4.5.6.

Suppose $B = \{f_1, f_2, \ldots, f_n\}$ is a basis for V. Let $v \in V$, if $[x_i]$ and $[y_i]$ are coordinate vectors of v then $[x_i] = [y_i]$. In other words, the coordinates of a vector with respect to a basis are unique.

Proof: Suppose $v = x_1f_1 + x_2f_2 + \cdots + x_nf_n$ and $v = y_1f_1 + y_2f_2 + \cdots + y_nf_n$ notice that

$$0 = v - v = (x_1 f_1 + x_2 f_2 + \dots + x_n f_n) - (y_1 f_1 + y_2 f_2 + \dots + y_n f_n)$$

= $(x_1 - y_1)f_1 + (x_2 - y_2)f_2 + \dots + (x_n - y_n)f_n$

then by linear independence of the basis vectors we find $x_i - y_i = 0$ for each *i*. Thus $x_i = y_i$ for all *i*. Notice that linear independence and spanning were both necessary for the idea of a coordinate to make sense. \Box

Example 4.5.7. Let $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ find the coordinates of v relative to B_1, B_2 and B_3 where $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1, e_1 + e_2\}$ and $B_3 = \{e_2, e_1 + e_2\}$. We'll begin with the standard basis, (I hope

you could see this without writing it)

$$v = \begin{bmatrix} 1\\3 \end{bmatrix} = 1 \begin{bmatrix} 1\\0 \end{bmatrix} + 3 \begin{bmatrix} 0\\1 \end{bmatrix} = 1e_1 + 3e_2$$

thus $[v]_{B_1} = [1 \ 3]^T$. Find coordinates relative to the other two bases is not quite as obvious. Begin with B_2 . We wish to find x, y such that

$$v = xe_1 + y(e_1 + e_2)$$

we can just use brute-force,

$$v = e_1 + 3e_2 = xe_1 + y(e_1 + e_2) = (x + y)e_1 + ye_2$$

using linear independence of the standard basis we find 1 = x + y and y = 3 thus x = 1 - 3 = -2and we see $v = -2e_1 + 3(e_1 + e_2)$ so $[v]_{B_2} = [-2 \ 3]^T$. This is interesting, the same vector can have different coordinate vectors relative to distinct bases. Finally, let's find coordinates relative to B_3 . I'll try to be more clever this time: we wish to find x, y such that

$$v = xe_2 + y(e_1 + e_2) \quad \Leftrightarrow \quad \begin{bmatrix} 1\\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1\\ 1 & 1 \end{bmatrix} \begin{bmatrix} x\\ y \end{bmatrix}$$

We can solve this via the augemented coefficient matrix

$$rref \begin{bmatrix} 0 & 1 & | & 1 \\ 1 & 1 & | & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix} \quad \Leftrightarrow \quad x = 2, \ y = 1.$$

Thus, $[v]_{B_3} = [2 \ 1]^T$. Notice this is precisely the rightmost column in the rref matrix. Perhaps my approach for B_3 is a little like squashing a fly with with a dumptruck. However, once we get to an example with 4-component vectors you may find the matric technique useful.

Example 4.5.8. Given that $B = \{b_1, b_2, b_3, b_4\} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4\}$ is a basis for $\mathbb{R}^{4 \times 1}$ find coordinates for $v = [1, 2, 3, 4]^T \in \mathbb{R}^{4 \times 1}$. Given the discussion in the preceding example it is clear we can find coordinates $[x_1, x_2, x_3, x_4]^T$ such that $v = \sum_i x_i b_i$ by calculating $rref[b_1|b_2|b_3|b_4|v]$ the rightmost column will be $[v]_B$.

$$rref \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 1 & 1 & 0 & 0 & | & 2 \\ 0 & 1 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} v \\ v \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

This calculation should be familar. We discussed it at length in the spanning section.

Remark 4.5.9.

Curvelinear coordinate systems from calculus III are in a certain sense more general than the idea of a coordinate system in linear algebra. If we focus our attention on a single point in space then a curvelinear coordinate system will produce three linearly independent vectors which are tangent to the coordinate curves. However, if we go to a different point then the curvelinear coordinate system will produce three different vectors in general. For example, in spherical coordinates the radial unit vector $e_{\rho} = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$ and you can see that different choices for the angles θ, ϕ make e_{ρ} point in different directions. In contrast, in this course we work with vector spaces. Our coordinate systems have the same basis vectors over the whole space. Vector spaces are examples of flat manifolds since they allow a single global coordinate system. Vector spaces also allow for curvelinear coordinates. However the converse is not true; spaces with nonzero curvature do not allow for global coordinates. I digress, we may have occassion to discuss these matters more cogently in our Advanced Calculus course (Math 332) offered in the Spring (join us)

Definition 4.5.10.

If a vector space V has a basis which consists of a finite number of vectors then we say that V is **finite-dimensional** vector space. Otherwise V is said to be **infinite-dimensional**

Example 4.5.11. $\mathbb{R}^{n \times 1}$, $\mathbb{R}^{m \times n}$, P_n are examples of finite-dimensional vector spaces. On the other hand, $\mathcal{F}(\mathbb{R})$, $C^0(\mathbb{R})$, $C^1(\mathbb{R})$, $C^{\infty}(\mathbb{R})$ are infinite-dimensional.

Example 4.5.12. We can prove that S from Example 4.3.16 is linearly independent, thus symmetric 2×2 matrices have a S as a basis

$$S = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$$

thus the dimension of the vector space of 2×2 symmetric matrices is 3. (notice \overline{S} from that example is not a basis because it is linearly dependent). While we're thinking about this let's find the coordinates of $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ with respect to S. Denote $[A]_S = [x, y, z]^T$. We calculate,

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies [A]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

4.5.1 how to calculate a basis for a span of row or column vectors

Given some subspace of $\mathbb{R}^{n\times 1}$ we would like to know how to find a basis for that space. In particular, if $V = span\{v_1, v_2, \ldots, v_k\}$ then what is a basis for W? Likewise, given some set of row vectors $W = \{w_1, w_2, \ldots, w_k\} \subset \mathbb{R}^{1\times n}$ how can we select a basis for span(W). We would like to find answers to these question since most subspaces are characterized either as spans or solution sets(see the next section on Null(A)). We already have the tools to answer these questions, we just need to apply them to the tasks at hand.

Proposition 4.5.13.

Let $W = span\{v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^{n \times 1}$ then a basis for W can be obtained by selecting the vectors that reside in the pivot columns of $[v_1|v_2|\cdots|v_k]$.

Proof: this is immediately obvious from Proposition 4.4.15. \Box

The proposition that follows is also follows immediately from Proposition 4.4.15.

Proposition 4.5.14.

Let $A \in \mathbb{R}^{m \times n}$ the pivot columns of A form a basis for Col(A).

Example 4.5.15. Suppose A is given as below: (I omit the details of the Gaussian elimination)

	1	2	3	4]		[1]	0	5/3	0]
A =	2	1	4	1	\Rightarrow	rref[A] =	0	1	2/3	0	.
	0	0	0	3			0	0	0	1	

Identify that columns 1,2 and 4 are pivot columns. Moreover,

$$Col(A) = span\{col_1(A), col_2(A), col_4(A)\}$$

In particular we can also read how the second column is a linear combination of the basis vectors.

$$col_3(A) = \frac{5}{3}col_1(A) + \frac{2}{3}col_2(A)$$

= $\frac{5}{3}[1, 2, 0]^T + \frac{2}{3}[2, 1, 0]^T$
= $[5/3, 10/3, 0]^T + [4/3, 2/3, 0]^T$
= $[3, 4, 0]^T$

What if we want a basis for Row(A) which consists of rows in A itself?

Proposition 4.5.16.

Let $W = span\{w_1, w_2, \dots, w_k\} \subset \mathbb{R}^{1 \times n}$ and construct A by concatenating the row vectors in W into a matrix A:

$$A = \begin{bmatrix} \frac{w_1}{w_2} \\ \vdots \\ w_k \end{bmatrix}$$

A basis for W is given by the transposes of the pivot columns for A^T .

Proof: this is immediately obvious from Proposition 4.4.19. \Box

The proposition that follows is also follows immediately from Proposition 4.4.19.

Proposition 4.5.17.

Let $A \in \mathbb{R}^{m \times n}$ the rows which are transposes of the pivot columns of A^T form a basis for Row(A).

Example 4.5.18.

$$A^{T} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 0 \\ 4 & 1 & 3 \end{bmatrix} \implies rref[A^{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that each column is a pivot column in A^T thus a basis for Row(A) is simply the set of all rows of A; $Row(A) = span\{[1, 2, 3, 4], [2, 1, 4, 1], [0, 0, 1, 0]\}$ and the spanning set is linearly independent.

Example 4.5.19.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 2 \end{bmatrix} \Rightarrow A^{T} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 0 & 2 \end{bmatrix} \Rightarrow rref[A^{T}] = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We deduce that rows 1 and 3 or A form a basis for Row(A). Notice that $row_2(A) = 2row_1(A)$ and $row_4(A) = row_3(A) + 2row_1(A)$. We can read linear dependendcies of the rows from the corresponding linear dependencies of the columns in the rref of the transpose.

The preceding examples are nice, but what should we do if we want to find both a basis for Col(A) and Row(A) for some given matrix? Let's pause to think about how elementary row operations modify the row and column space of a matrix. In particular, let A be a matrix and let A' be the result of performing an elementary row operation on A. It is fairly obvious that

$$Row(A) = Row(A').$$

Think about it. If we swap to rows that just switches the order of the vectors in the span that makes Row(A). On the other hand if we replace one row with a nontrivial linear combination of itself and other rows then that will not change the span either. Column space is not so easy though. Notice that elementary row operations can change the column space. For example,

$$A = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \quad \Rightarrow rref[A] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right]$$

has $Col(A) = span\{[1,1]^T\}$ whereas $Col(rref(A)) = span([1,0]^T)$. We cannot hope to use columns of ref(A) (or rref(A)) for a basis of Col(A). That's no big problem though because we already have the CCP-principle which helped us pick out a basis for Col(A). Let's collect our thoughts:

Proposition 4.5.20.

Let $A \in \mathbb{R}^{m \times n}$ then a basis for Col(A) is given by the pivot columns in A and a basis for Row(A) is given by the nonzero rows in ref(A).

This means we can find a basis for Col(A) and Row(A) by preforming the forward pass on A. We need only calculate the ref(A) as the pivot columns are manifest at the end of the forward pass.

Example 4.5.21.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = ref[A]$$

We deduce that $\{[1,1,1], [0,1,2]\}$ is a basis for Row(A) whereas $\{[1,1,1]^T, [1,1,2]^T\}$ is a basis for Col(A). Notice that if I wanted to reveal further linear dependencies of the non-pivot columns on the pivot columns of A it would be wise to calculate rref[A] by making the backwards pass on ref[A].

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = rref[A]$$

From which I can read $col_3(A) = 2col_2(A) - col_1(A)$, a fact which is easy to verify.

Example 4.5.22.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 8 & 10 \\ 1 & 2 & 4 & 11 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix} = ref[A]$$

We find that Row(A) has basis

$$\{[1,2,3,4], [0,1,5,6], [0,0,1,7]\}$$

and Col(A) has basis

$$\left\{ \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 2\\3\\2 \end{bmatrix}, \begin{bmatrix} 3\\8\\4 \end{bmatrix} \right\}$$

Proposition 4.5.20 was the guide for both examples above.

4.5.2 calculating basis of a solution set

Often a subspace is described as the solution set of some equation Ax = 0. How do we find a basis for Null(A)? If we can do that we find a basis for subspaces which are described by some equation.

Proposition 4.5.23.

Let $A \in \mathbb{R}^{m \times n}$ and define W = Null(A). A basis for W is obtained from the solution set of Ax = 0 by writing the solution as a linear combination where the free variables appear as coefficients in the vector-sum.

Proof: $x \in W$ implies Ax = 0. Denote $x = [x_1, x_2, \ldots, x_n]^T$. Suppose that rref[A] has r-pivot columns (we must have $0 \le r \le n$). There will be (m - r)-rows which are zero in rref(A) and (n - r)-columns which are not pivot columns. The non-pivot columns correspond to free-variables in the solution. Define p = n - r for convenience. Suppose that $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ are free whereas $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ are functions of the free variables: in particular they are linear combinations of the free variables as prescribed by rref[A]. There exist constants b_{ij} such that

$$\begin{array}{ll} x_{j_1} &= b_{11}x_{i_1} + b_{12}x_{i_2} + \dots + b_{1p}x_{i_p} \\ x_{j_2} &= b_{21}x_{i_1} + b_{22}x_{i_2} + \dots + b_{2p}x_{i_p} \\ \vdots & \vdots & \dots & \vdots \\ x_{j_r} &= b_{r1}x_{i_1} + b_{r2}x_{i_2} + \dots + b_{rp}x_{i_p} \end{array}$$

For convenience of notation assume that the free variables are put at the end of the list. We have

$$\begin{aligned} x_1 &= b_{11}x_{r+1} + b_{12}x_{r+2} + \dots + b_{1p}x_n \\ x_2 &= b_{21}x_{r+1} + b_{22}x_{r+2} + \dots + b_{2p}x_n \\ \vdots &\vdots & \ddots & \vdots \\ x_r &= b_{r1}x_{r+1} + b_{r2}x_{n-p+2} + \dots + b_{rp}x_n \end{aligned}$$

and $x_j = x_j$ for j = r + 1, r + 2, ..., r + p = n (those are free, we have no conditions on them, they can take any value). We find,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = x_{r+1} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2} \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{r2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{rp} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We define the vectors in the sum above as v_1, v_2, \ldots, v_p . If any of the vectors, say v_j , was linearly dependent on the others then we would find that the variable x_{r+j} was likewise dependent on the

other free variables. This would contradict the fact that the variable x_{r+j} was free. Consequently the vectors v_1, v_2, \ldots, v_p are linearly independent. Moreover, they span the null-space by virtue of their construction. \Box

Didn't follow the proof above? No problem. See the examples to follow here. These are just the proof in action for specific cases. We've done these sort of calculations in §1.3. We're just adding a little more insight here.

Example 4.5.24. Find a basis for the null space of A given below,

$$A = \left[\begin{array}{rrrr} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{array} \right]$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.7 for details of the Gaussian elimination)

$$rref \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 \end{bmatrix}$$

Denote $x = [x_1, x_2, x_3, x_4, x_5]^T$ in the equation Ax = 0 and identify from the calculation above that x_4 and x_5 are free thus solutions are of the form

$$x_{1} = -x_{4}$$

$$x_{2} = x_{4} - \frac{1}{2}x_{5}$$

$$x_{3} = \frac{1}{2}x_{5}$$

$$x_{4} = x_{4}$$

$$x_{5} = x_{5}$$

for all $x_4, x_5 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for Null(A),

$$x = \begin{bmatrix} -x_4 \\ x_4 - \frac{1}{2}x_5 \\ \frac{1}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for Null(A) is simply

$$\left\{ \begin{bmatrix} -1\\ 1\\ 0\\ 1\\ 0 \end{bmatrix}, \begin{bmatrix} 0\\ -\frac{1}{2}\\ \frac{1}{2}\\ 0\\ 1 \end{bmatrix} \right\}$$

Of course, you could multiply the second vector by 2 if you wish to avoid fractions. In fact there is a great deal of freedom in choosing a basis. We simply show one way to do it. **Example 4.5.25.** Find a basis for the null space of A given below,

Gaussian elimination on the augmented coefficient matrix reveals:

Denote $x = [x_1, x_2, x_3, x_4]^T$ in the equation Ax = 0 and identify from the calculation above that x_2, x_3 and x_4 are free thus solutions are of the form

$$\begin{aligned}
 x_1 &= -x_2 - x_3 - x_4 \\
 x_2 &= x_2 \\
 x_3 &= x_3 \\
 x_4 &= x_4
 \end{aligned}$$

for all $x_2, x_3, x_4 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for Null(A),

$$x = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for Null(A) is simply

$$\left\{ \begin{bmatrix} -1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\0\\1 \end{bmatrix} \right\}$$

4.5.3 what is dimension?

We prove a number of theorems in the section which show that dimension is a well-defined quantity for a finite dimensional vector space. Up to this point we have only used the phrase "finitedimensional" to mean that there exists one basis with finitely many vectors. In this section we prove that if that is the case then all other bases for the vector space must likewise have the same number of basis vectors. In addition we give several existence theorems which are of great theoretical importance. Finally, we discuss dimensions of column, row and null space of a matrix.

The proposition that follows is the baby version of Proposition 4.5.30. I include this proposition in the notes because the proof is fun.

Proposition 4.5.26.

Let V be a finite-dimensional vector space and suppose $B = \{b_1, b_2, \dots, b_n\}$ is any basis of V,

- 1. $B \cup \{v\}$ is linearly dependent
- 2. for any $1 \le k \le n$, $B \{b_k\}$ does not span V

Proof of (1.): Since B spans V it follows that v is a linear combination of vectors in B thus $B \cup \{v\}$ is linearly dependent.

Proof of (2.): We argue that $b_k \notin span(B - \{b_k\})$. Argue by contradiction. Suppose that $b_k \in span(B - \{b_k\})$ then there exist constants $c_1, c_2, \ldots, \widehat{c_k}, c_n$ such that

$$b_k = c_1 b_1 + c_2 b_2 + \dots + \widehat{c_k b_k} + \dots + c_n b_n$$

but this contradicts the linear independence of the basis as

$$c_1b_1 + c_2b_2 + \dots - b_k + \dots + c_nb_n = 0$$

does not imply all the coefficients are zero. Therefore, using proof by contradiction, $span(B - \{b_k\}) \neq V$. \Box

Proposition 4.5.27.

Let V be a finite-dimensional vector space and suppose $B = \{b_1, b_2, \ldots, b_n\}$ is any basis of V then any other basis for V also has n-elements.

Proof: Suppose $B = \{b_1, b_2, \ldots, b_n\}$ and $F = \{f_1, f_2, \ldots, f_p\}$ are both bases for a vector space V. Since F is a basis it follows $b_k \in span(F)$ for all k so there exist constants c_{ik} such that

$$b_k = c_{1k}f_1 + c_{2k}f_2 + \dots + c_{pk}f_p$$

for k = 1, 2, ..., n. Likewise, since $f_j \in span(B)$ there exist constants d_{lj} such that

$$f_j = d_{1j}b_1 + d_{2j}b_2 + \dots + d_{nj}b_n$$

for $j = 1, 2, \ldots, p$. Substituting we find

$$f_j = d_{1j}b_1 + d_{2j}b_2 + \dots + d_{nj}b_n$$

$$= d_{1j}(c_{11}f_1 + c_{21}f_2 + \dots + c_{p1}f_p) + + d_{2j}(c_{12}f_1 + c_{22}f_2 + \dots + c_{p2}f_p) + + \dots + d_{nj}(c_{1n}f_1 + c_{2n}f_2 + \dots + c_{pn}f_p)$$

$$= (d_{1j}c_{11} + d_{2j}c_{12} + \cdots + d_{nj}c_{1n})f_1$$

(d_{1j}c_{21} + d_{2j}c_{22} + \cdots + d_{nj}c_{2n})f_2 +
+ \cdots + (d_{1j}c_{p1} + d_{2j}c_{p2} + \cdots + d_{nj}c_{pn})f_p

Suppose j = 1. We deduce, by the linear independence of F, that

$$d_{11}c_{11} + d_{21}c_{12} + \cdots + d_{n1}c_{1n} = 1$$

from comparing coefficients of f_1 , whereas for f_2 we find,

$$d_{11}c_{21} + d_{21}c_{22} + \cdots + d_{n1}c_{2n} = 0$$

likewise, for f_q with $q \neq 1$,

$$d_{11}c_{q1} + d_{21}c_{q2} + \cdots + d_{n1}c_{qn} = 0$$

Notice we can rewrite all of these as

$$\delta_{q1} = c_{q1}d_{11} + c_{q2}d_{21} + \dots + c_{qn}d_{n1}$$

Similarly, for arbitrary j we'll find

$$\delta_{qj} = c_{q1}d_{1j} + c_{q2}d_{2j} + \cdots + c_{qn}d_{nj}$$

If we define $C = [c_{ij}] \in \mathbb{R}^{p \times n}$ and $D = [d_{ij}] \in \mathbb{R}^{n \times p}$ then we can translate the equation above into the matrix equation that follows:

 $CD = I_p.$

We can just as well argue that

$$DC = I_n$$

From your Problem Set, we learned that tr(AB) = tr(BA) if the product AB and BA are both defined. Moreover, you also proved $tr(I_p) = p$ and $tr(I_q) = q$. It follows that,

 $tr(CD) = tr(DC) \Rightarrow tr(I_p) = tr(I_q) \Rightarrow p = q.$

Since the bases were arbitrary this proves any pair have the same number of vectors. \Box

Given the preceding proposition the following definition is logical.

Definition 4.5.28.

If V is a finite-dimensional vector space then the **dimension** of V is the number of vectors in any basis of V and it is denoted dim(V).

Example 4.5.29. Let me state the dimensions which follow from the standard bases of $\mathbb{R}^{n \times 1}$ and $\mathbb{R}^{m \times n}$ respective,

$$\dim(\mathbb{R}^{n \times 1}) = n \qquad \dim(\mathbb{R}^{m \times n}) = mn$$

these results follow from counting.

Proposition 4.5.30.

Suppose V is a vector space with dim(V) = n.

- 1. If S is a set with more than n vectors then S is linearly dependent.
- 2. If S is a set with less than n vectors then S does not generate V.

Proof of (1.): Suppose $S = \{s_1, s_2, \ldots, s_m\}$ has *m* vectors and m > n. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis of *V*. Consider the corresponding set of coordinate vectors of the vectors in *S*, we denote

$$[S]_B = \{[s_1]_B, [s_2]_B, \dots, [s_m]_B\}.$$

The set $[S]_B$ has m vectors in $\mathbb{R}^{n \times 1}$ and m > n therefore by Proposition 4.4.10 we know $[S]_B$ is a linearly dependent set. Therefore at least one, say $[s_j]_B$, vector can be written as a linear combination of the other vectors in $[S]_B$ thus there exist constants c_i with (this is a vector equation)

$$[s_j]_B = c_1[s_1]_B + c_2[s_2]_B + \dots + \widehat{c_j[s_j]_B} + \dots + c_m[s_m]_B$$

Also notice that (introducing a new shorthand $B[s_j]$ which is not technically matrix multiplication since b_i are not column vectors generally, they could be chickens for all we know)

$$s_j = B[s_j] = s_{j1}b_1 + s_{j2}b_2 + \dots + s_{jn}b_n$$

We also know, using the notation $([s_i]_B)_k = s_{ik}$,

$$s_{jk} = c_1 s_{1k} + c_2 s_{2k} + \dots + \widehat{c_j s_{jk}} + \dots + c_m s_{mk}$$

for $k = 1, 2, \ldots, n$. Plug these into our s_j equation,

$$s_{j} = (c_{1}s_{11} + c_{2}s_{21} + \dots + \widehat{c_{j}s_{j1}} + \dots + c_{m}s_{m1})b_{1} + (c_{1}s_{12} + c_{2}s_{22} + \dots + \widehat{c_{j}s_{j2}} + \dots + c_{m}s_{m2})b_{2} + \dots + (c_{1}s_{1n} + c_{2}s_{2n} + \dots + \widehat{c_{j}s_{jn}} + \dots + c_{m}s_{mn})b_{n}$$

$$= c_{1}(s_{11}b_{1} + s_{12}b_{2} + \dots + s_{1n}b_{n}) + c_{2}(s_{21}b_{1} + s_{22}b_{2} + \dots + s_{2n}b_{n}) + \dots + c_{m}(s_{m1}b_{1} + s_{m2}b_{2} + \dots + s_{mn}b_{n}) : \text{ excluding } c_{j}(\dots)$$

$$= c_{1}s_{1} + c_{2}s_{2} + \dots + \widehat{c_{j}s_{j}} + \dots + c_{n}s_{n}.$$

Well this is a very nice result, the same linear combination transfers over to the abstract vectors. Clearly s_j linearly depends on the other vectors in S so S is linearly dependent. The heart of the proof was Proposition 4.4.10 and the rest was just battling notation.

Proof of (2.): Use the corresponding result for $\mathbb{R}^{n \times 1}$ which was given by Proposition 4.3.14. Given *m* abstract vectors if we concantenate their coordinate vectors we will find a matrix [S] in

 $\mathbb{R}^{n \times m}$ with m < n and as such there will be some choice of the vector b for which $[S]x \neq b$. The abstract vector corresponding to b will not be covered by the span of S. \Box

Anton calls the following proposition the "Plus/Minus" Theorem.

Proposition 4.5.31.

Let V be a vector space and suppose S is a nonempty set of vectors in V.

- 1. If S is linearly independent a nonzero vector $v \notin span(S)$ then $S \cup \{v\}$ is a linearly independent set.
- 2. If $v \in S$ is a linear combination of other vectors in S then $span(S \{v\}) = span(S)$.

Proof of (1.): Suppose $S = \{s_1, s_2, \ldots, s_k\}$ and consider,

$$c_1s_1 + c_2s_2 + \dots + c_ks_k + c_{k+1}v = 0$$

If $c_{k+1} \neq 0$ it follows that v is a linear combination of vectors in S but this is impossible so $c_{k+1} = 0$. Then since S is linear independent

$$c_1s_1 + c_2s_2 + \dots + c_ks_k = 0 \implies c_1 = c_2 = \dots = c_k = 0$$

thus $S \cup \{v\}$ is linearly independent.

Proof of (2.): Suppose $v = s_i$. We are given that there exist constants d_i such that

$$s_j = d_1 s_1 + \dots + d_j s_j + \dots + d_k s_k$$

Let $w \in span(S)$ so there exist constants c_i such that

$$w = c_1 s_1 + c_2 s_2 + \dots + c_j s_j + \dots + c_k s_k$$

Now we can substitute the linear combination with d_i -coefficients for s_i ,

$$w = c_1 s_1 + c_2 s_2 + \dots + c_j (d_1 s_1 + \dots + \widehat{d_j s_j} + \dots + d_k s_k) + \dots + c_k s_k$$
$$= (c_1 + c_j d_1) s_1 + (c_2 + c_j d_2) s_2 + \dots + \widehat{c_j d_j s_j} + \dots + (c_k + c_j d_k) s_k$$

thus w is a linear combination of vectors in S, but not $v = s_j$, thus $w \in span(S - \{v\})$ and we find $span(S) \subseteq span(S - \{v\})$.

Next, suppose $y \in span(S - \{v\})$ then y is a linear combination of vectors in $S - \{v\}$ hence y is a linear combination of vectors in S and we find $y \in span(S)$ so $span(S - \{v\}) \subseteq span(S)$. (this inclusion is generally true even if v is linearly independent from other vectors in S). We conclude that $span(S) = span(S - \{v\})$. \Box

Proposition 4.5.32.

Let V be an n-dimensional vector space. A set S with n-vectors is a basis for V if S is either linearly independent or if span(S) = V.

Proof: Assume S has n-vectors which are linearly independent in a vector space V with dimension n. Suppose towards a contradiction that S does not span V. Then there exists $v \in V$ such that $v \notin span(S)$. Then by Proposition 4.5.31 we find $V \cup \{v\}$ is linearly independent. But, Proposition 4.5.30 the set $V \cup \{v\}$ is linearly dependent. This is a contradiction, thus S spans V and we find D is a basis.

Assume S has n-vectors which span a vector space V with dimension n. Suppose towards a contradiction that S is not linearly independent V. This means there exists $v \in S$ which is a linear combination of other vectors in S. Therefore, by 4.5.30, S does not span V. This is a contradicts the assumption span(S) = V therefore S is linearly independent and consequently S is a basis. \Box

Remark 4.5.33.

Intuitively speaking, linear independence is like injectivity for functions whereas spanning is like the onto property for functions. Suppose A is a finite set. If a function $f: A \to A$ is 1-1 then it is onto. Also if the function is onto then it is 1-1. The finiteness of A is what blurs the concepts. For a vector space, we also have a sort of finiteness in play if dim(V) = n. When a set with dim(V)-vectors spans (like onto) V then it is automatically linearly independent. When a set with dim(V)-vectors is linearly independent (like 1-1) V then it automatically spans V.

Proposition 4.5.34.

Let S be a subset of a finite dimensional vector space V.

- 1. If span(S) = V but S is not a basis then S can be modified to make a basis by removing redundant vectors.
- 2. If S is linearly independent and $span(S) \neq V$ then S can be modified to make a basis by unioning vectors outside span(S).

Proof of (1.): If span(S) = V but S is not a basis we find S is linearly dependent. (if S is linearly independent then Proposition 4.5.32 says S is a basis which is a contradiction). Since S is linearly dependent we can write some $v \in S$ as a linear combination of other vectors in S. Furthermore, by Proposition 4.5.30 $span(S) = span(S - \{v\})$. If $S - \{v\}$ is linearly independent then $S - \{v\}$ is a basis. Otherwise $S - \{v\}$ is linearly dependent and we can remove another vector. Continue until the resulting set is linearly independent (we know this happens when there are just dim(V)-vectors in the set so this is not an endless loop)

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Proof of (2.): If S is linearly independent but $span(S) \neq V$ then there exists $v \in V$ but $v \notin span(S)$. Proposition 4.5.31 shows that $S \cup \{v\}$ is linearly independent. If $span(S \cup \{v\}) = V$ then $S \cup \{v\}$ is a basis. Otherwise there is still some vector outside $span(S \cup \{v\}) = V$ and we can repeat the argument for that vector and so forth until we generate a set which spans V. Again we know this is not an endless loop because V is finite dimensional and once the set is linearly independent and contains dim(V) vectors it must be a basis (see Proposition 4.5.32). \Box

Remark 4.5.35.

We already saw in the previous sections that we can implement part (1.) of the preceding proposition in $\mathbb{R}^{n \times 1}$ and $\mathbb{R}^{1 \times n}$ through matrix calculations. There are really nice results about row and column spaces which show us precisely which vectors we need to remove or add to obtain a basis. I don't believe we will tackel the question in the abstract in this course, but once you understand the $\mathbb{R}^{n \times 1}$ -case you can do the abstract case by lifting the arguments through the coordinate maps. We've already seen this "lifting" idea come into play in several proof of Proposition 4.5.30. Part (2.) involves making a choice. How do you choose a vector outside the span? I leave this question to the reader for the moment.

Proposition 4.5.36.

If V is a finite-dimensional vector space and $W \leq V$ then $dim(W) \leq dim(V)$. Moreover, if dim(V) = dim(W) then V = W.

Proof: Left to the reader, I don't want to be too greedy. Besides, I need something to put on the test⁵. \Box

These were defined before, I restate them here along with their dimensions for convenience.

Definition 4.5.37.

Let $A \in \mathbb{R}^{m \times n}$. We define

1. $Col(A) = span\{col_j(A) | j = 1, 2, ..., n\}$ and r = rank(A) = dim(Col(A))

2.
$$Row(A) = span\{row_i(A) | i = 1, 2, ..., m\}$$

3. $Null(A) = \{x \in \mathbb{R}^{n \times 1} | Ax = 0\}$ and $\nu = nullity(A) = dim(Null(A))$

Proposition 4.5.38.

Let $A \in \mathbb{R}^{m \times n}$ then dim(Row(A)) = dim(Col(A))

Proof: By Proposition 4.5.14 we know the number of vectors in the basis for Col(A) is the number of pivot columns in A. Likewise, Proposition 4.5.20 showed the number of vectors in the basis for

⁵I'm kidding

Row(A) was the number of nonzero rows in ref(A). But the number of pivot columns is precisely the number of nonzero rows in ref(A) therefore, dim(Row(A)) = dim(Col(A)). \Box

Proposition 4.5.39.

Let $A \in \mathbb{R}^{m \times n}$ then n = rank(A) + nullity(A).

Proof: The proof of Proposition 4.5.23 makes is clear that if a $m \times n$ matrix A has r-pivot columns then there will be n - r vectors in the basis of Null(A). It follows that

rank(A) + nullity(A) = r + (n - r) = n.

4.6 general theory of linear systems

Let $A \in \mathbb{R}^{m \times n}$ we should notice that $Null(A) \leq \mathbb{R}^{n \times 1}$ is only possible since homogeneous systems of the form Ax = 0 have the nice property that linear combinations of solutions is again a solution:

Proposition 4.6.1.

Let Ax = 0 denote a homogeneous linear system of *m*-equations and *n*-unknowns. If v_1 and v_2 are solutions then any linear combination $c_1v_1 + c_2v_2$ is also a solution of Ax = 0.

Proof: Suppose $Av_1 = 0$ and $Av_2 = 0$. Let $c_1, c_2 \in \mathbb{R}$ and recall Theorem 2.3.13 part 13,

$$A(c_1v_1 + c_2v_2) = c_1Av_1 + c_2Av_2 = c_10 + c_20 = 0.$$

Therefore, $c_1v_1 + c_2v_2 \in Sol_{[A|0]}$. \Box

We proved this before, but I thought it might help to see it again here.

Proposition 4.6.2.

Let $A \in \mathbb{R}^{m \times n}$. If v_1, v_2, \ldots, v_k are solutions of Av = 0 then $V = [v_1|v_2|\cdots|v_k]$ is a solution matrix of Av = 0 (V a solution matrix of Av = 0 iff AV = 0)

Proof: Let $A \in \mathbb{R}^{m \times n}$ and suppose $Av_i = 0$ for i = 1, 2, ..., k. Let $V = [v_1|v_2|\cdots|v_k]$ and use the solution concatenation Proposition 2.6.3,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k] = [0|0|\cdots|0] = 0.$$

. 🗆.

In simple terms, a solution matrix of a linear system is a matrix in which each column is itself a solution to the system.

Proposition 4.6.3.

Let $A \in \mathbb{R}^{m \times n}$. The system of equations Ax = b is consistent iff $b \in Col(A)$.

Proof: Observe,

$$Ax = b \iff \sum_{i,j} A_{ij} x_j e_i = b$$

$$\Leftrightarrow \sum_j x_j \sum_i A_{ij} e_i = b$$

$$\Leftrightarrow \sum_j x_j \operatorname{col}_j(A) = b$$

$$\Leftrightarrow b \in \operatorname{Col}(A)$$

Therefore, the existence of a solution to Ax = b is interchangeable with the statement $b \in Col(A)$. They both amount to saying that b is a linear combination of columns of A. \Box

Proposition 4.6.4.

Let $A \in \mathbb{R}^{m \times n}$ and suppose the system of equations Ax = b is consistent. We find $x \in \mathbb{R}^{n \times 1}$ is a solution of the system if and only if it can be written in the form

$$x = x_h + x_p = c_1 v_1 + c_2 v_2 + \dots + c_\nu v_\nu + x_p$$

where $Ax_h = 0$, $\{v_j\}_{j=1}^{\nu}$ are a basis for Null(A), and $Ax_p = b$. We call x_h the homogeneous solution and x_p is the nonhomogeneous solution.

Proof: Suppose Ax = b is consistent then $b \in Col(A)$ therefore there exists $x_p \in \mathbb{R}^{n \times 1}$ such that $Ax_p = b$. Let x be any solution. We have Ax = b thus observe

$$A(x - x_p) = Ax - Ax_p = Ax - b = 0 \quad \Rightarrow \quad (x - x_p) \in Null(A).$$

Define $x_h = x - x_p$ it follows that there exist constants c_i such that $x_h = c_1v_1 + c_2v_2 + \cdots + c_{\nu}v_{\nu}$ since the vectors v_i span the null space.

Conversely, suppose $x = x_p + x_h$ where $x_h = c_1v_1 + c_2v_2 + \cdots + c_\nu v_\nu \in Null(A)$ then it is clear that

$$Ax = A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$$

thus $x = x_p + x_h$ is a solution. \Box

Example 4.6.5. Consider the system of equations x + y + z = 1, x + z = 1. In matrix notation,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow rref[A|b] = rref \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

It follows that x = 1 - y - z is a solution for any choice of $y, z \in \mathbb{R}$.

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - y - z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We recognize that $v_p = [1, 0, 0]^T$ while $v_h = y[-1, 1, 0]^T + z[-1, 0, 1]^T$ and $\{[-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a basis for the null space of A. We call y, z parameters in the solution.

We will see that null spaces play a central part in the study of eigenvectors. In fact, about half of the calculation is finding a basis for the null space of a certain matrix. So, don't be too disappointed if I don't have too many examples here. You'll work dozens of them later.

I conclude this section with a proposition which summarizes what we just calculated:

Proposition 4.6.6.

Let $A \in \mathbb{R}^{m \times n}$. If the system of equations Ax = b is consistent then the general solution has as many parameters as the dim(Null(A)).

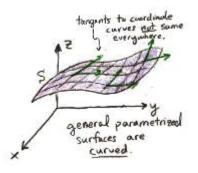
4.7 applications

Geometry is conveniently described by parametrizations. The number of parameters needed to map out some object is the dimension of the object. For example, the rule $t \mapsto \vec{r}(t)$ describes a curve in \mathbb{R}^n . Of course we have the most experience in the cases $\vec{r} = \langle x, y \rangle$ or $\vec{r} = \langle x, y, z \rangle$, these give so-called *planar curves* or *space curves* respectively. Generally, a mapping from $\gamma : \mathbb{R} \to S$ where S is some space⁶ is called a *path*. The point set $\gamma(S)$ can be identified as a sort of copy of \mathbb{R} which resides in S.

Next, we can consider mappings from \mathbb{R}^2 to some space S. In the case $S = \mathbb{R}^3$ we use $X(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ to parametrize a surface. For example,

 $X(\phi, \theta) = <\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) >$

parametrizes a sphere if we insist that the angles $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. We call ϕ and θ coordinates on the sphere, however, these are not coordinates in the technical sense we defined in this chapter. These are so-called *curvelinear coordinates*. Generally a surface in some space is sort-of a copy of \mathbb{R}^2 (well, to be more precise it resembles some subset of \mathbb{R}^2).



Past the case of a surface we can talk about volumes which are parametrized by three parameters. A volume would have to be embedded into some space which had at least 3 dimensions. For the same reason we can only place a surface in a space with at least 2 dimensions. Perhaps you'd be interested to learn that in relativity theory one considers the world-volume that a particle traces out through spacetime, this is a hyper-volume, it's a 4-dimensional subset of 4-dimensional spacetime.

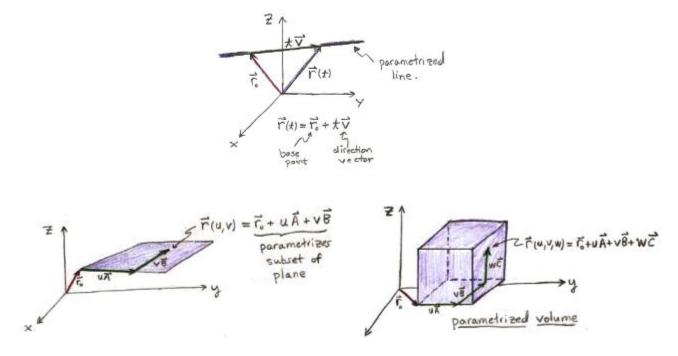
Let me be a little more technical, if the space we consider is to be a k-dimensional parametric subspace of S then that means there exists an **invertible** mapping $X : U \subseteq \mathbb{R}^k \to S \subseteq \mathbb{R}^n$. It turns out that for $S = \mathbb{R}^n$ where $n \ge k$ the condition that X be invertible means that the derivative $D_p X : T_p U \to T_{X(p)} S$ must be an invertible linear mapping at each point p in the parameter space U. This in turn means that the tangent-vectors to the coordinate curves must come together to form a linearly independent set. Linear independence is key.

⁶ here S could be a set of matrices or functions or an abstract manifold... the concept of a path is very general

Curvy surfaces and volumes and parametrizations that describe them analytically involve a fair amount of theory which I have only begun to sketch here. However, if we limit our discussion to **affine subspaces** of \mathbb{R}^n we can be explicit. Let me go ahead and write the general form for a line, surface, volume etc... in terms of linearly indpendent vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$

$$\vec{r}(u) = \vec{r}_o + u\vec{A}$$
$$X(u, v) = \vec{r}_o + u\vec{A} + v\vec{B}$$
$$X(u, v, w) = \vec{r}_o + u\vec{A} + v\vec{B} + w\vec{C}$$

I hope you you get the idea.



In each case the parameters give an invertible map only if the vectors are linearly independent. If there was some linear dependence then the dimension of the subspace would collapse. For example,

X(u, v) = <1, 1, 1 > +u < 1, 0, 1 > +v < 2, 0, 2 >

appears to give a plane, but upon further inspection you'll notice

$$X(u,v) = <1 + u + 2v, 1, 1 + u + 2v > = <1, 1, 1 > +(u + 2v) <1, 0, 1 >$$

which reveals this is just a line with direction-vector < 1, 0, 1 > and parameter u + 2v.

Finally, if \vec{r}_o we actually have that the parametrized space is a subspace and we can honestly call u, v, w, \dots the coordinates (in the sense of linear algebra) with respect to the basis $\vec{A}, \vec{B}, \vec{C}, \dots$

4.8 conclusions

We continue Theorem 3.8.1 from the previous chapter.

Theorem 4.8.1.

Let A be a real $n \times n$ matrix then the following are equivalent: (a.) A is invertible, (b.) rref[A|0] = [I|0] where $0 \in \mathbb{R}^{n \times 1}$, (c.) Ax = 0 iff x = 0, (d.) A is the product of elementary matrices, (e.) there exists $B \in \mathbb{R}^{n \times n}$ such that AB = I, (f.) there exists $B \in \mathbb{R}^{n \times n}$ such that BA = I, (g.) rref[A] = I, (h.) rref[A|b] = [I|x] for an $x \in \mathbb{R}^{n \times 1}$, (i.) Ax = b is consistent for every $b \in \mathbb{R}^{n \times 1}$, (j.) Ax = b has exactly one solution for every $b \in \mathbb{R}^{n \times 1}$, (k.) A^T is invertible, (1.) $det(A) \neq 0$, (m.) Kramer's rule yields solution of Ax = b for every $b \in \mathbb{R}^{n \times 1}$. (n.) $Col(A) = \mathbb{R}^{n \times 1}$, (o.) $Row(A) = \mathbb{R}^{1 \times n}$, (p.) rank(A) = n, (q.) $Null(A) = \{0\},\$ (r.) $\nu = 0$ for A where $\nu = dim(Null(A))$, (s.) the columns of A are linearly independent, (t.) the rows of A are linearly independent

The addition of the comments about row, column and null space are huge since these gives us easy concise tools to characterize subspaces in $\mathbb{R}^{n \times 1}$. As we've seen in this chapter we can test for linear independence and spanning all through solving particular systems. However, clever use of matrix

notations allows us to do these calculations even without explicitly writing those equations. Again, continuing Theorem 3.8.2 from the previous chapter:

Theorem 4.8.2.

Let A be a real n × n matrix then the following are equivalent:
(a.) A is not invertible,
(b.) Ax = 0 has at least one nontrivial solution.,
(c.) there exists b ∈ ℝ ^{n×1} such that Ax = b is inconsistent,
(d.) det(A) = 0,
(e.) Null(A) ≠ {0},
(f.) there are ν = dim(Null(A)) parameters in the general solution to Ax = 0,

Can you think of anything else to add here? Let me know if you think I missed something here. If it's sufficiently interesting it'll be worth some points.

Chapter 5

linear transformations

Functions which preserve the structure of a vector space are called linear transformations. Many important operations in calculus are linear transformations: definite integrals, differentiation even taking a limit. Many differential equations can be written as a linear transformation acting on a function space. Linear transformations which are 1-1 and onto are called isomorphisms. It turns out that all finite dimensional vector spaces of the same dimension are isomorphic. Coordinate maps are isomorphisms. In the finite dimensional case, we can always use coordinate maps to convert a linear transformation to matrix multiplication at the level of coordinate maps. We discuss how to find the matrix of a linear transformation from $\mathbb{R}^{n\times 1}$ to $\mathbb{R}^{m\times 1}$. Finally we study change of basis. It turns out the matrix of a linear transformation undergoes a similarity transformation as coordinates are changed.

5.1 examples of linear transformations

Definition 5.1.1.

Let V, W be vector spaces. If a mapping $L: V \to W$ satisfies

1. L(x+y) = L(x) + L(y) for all $x, y \in V$,

2. L(cx) = cL(x) for all $x \in V$ and $c \in \mathbb{R}$

then we say L is a linear transformation.

Example 5.1.2. Let L(x,y) = x + 2y. This is a mapping from \mathbb{R}^2 to \mathbb{R} . Notice

L((x,y) + (z,w)) = L(x + z, y + w) = x + z + 2(y + w) = (x + 2y) + (z + 2w) = L(x,y) + L(z,w)

and

$$L(c(x,y)) = L(cx, cy) = cx + 2(cy) = c(x + 2y) = cL(x,y)$$

for all $(x, y), (z, w) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Therefore, L is a linear transformation.

Example 5.1.3. Let $A \in \mathbb{R}^{m \times n}$ and define $L : \mathbb{R}^{n \times 1} \to \mathbb{R}^{m \times 1}$ by L(x) = Ax for each $x \in \mathbb{R}^{n \times 1}$. Let $x, y \in \mathbb{R}^{n \times 1}$ and $c \in \mathbb{R}$,

$$L(x + y) = A(x + y) = Ax + Ay = L(x) + L(y)$$

and

$$L(cx) = A(cx) = cAx = cL(x)$$

thus L is a linear transformation.

Example 5.1.4. Define $L : \mathbb{R}^{m \times n} \to \mathbb{R}^{n \times m}$ by $L(A) = A^T$. This is clearly a linear transformation since

$$L(cA + B) = (cA + B)^{T} = cA^{T} + B^{T} = cL(A) + L(B)$$

for all $A, B \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}$.

Example 5.1.5. Let V, W be a vector spaces and $L : V \to W$ defined by L(x) = 0 for all $x \in V$. This is a linear transformation known as the **trivial transformation**

$$L(x + y) = 0 = 0 + 0 = L(x) + L(y)$$

and

$$L(cx) = 0 = c0 = cL(x)$$

for all $c \in \mathbb{R}$ and $x, y \in V$.

Example 5.1.6. The identity function on a vector space is also a linear transformation. Let $Id: V \to V$ satisfy L(x) = x for each $x \in V$. Observe that

$$Id(x + cy) = x + cy = x + c(y) = Id(x) + cId(y)$$

for all $x, y \in V$ and $c \in \mathbb{R}$.

Example 5.1.7. Define $L: C^0(\mathbb{R}) \to \mathbb{R}$ by $L(f) = \int_0^1 f(x) dx$. Notice that L is well-defined since all continuous functions are integrable and the value of a definite integral is a number. Furthermore,

$$L(f+cg) = \int_0^1 (f+cg)(x)dx = \int_0^1 \left[f(x) + cg(x) \right] dx = \int_0^1 f(x)dx + c \int_0^1 g(x)dx = L(f) + cL(g)$$

for all $f, g \in C^0(\mathbb{R}(and \ c \in \mathbb{R}))$. The definite integral is a linear transformation.

Example 5.1.8. Let $L: C^1(\mathbb{R}) \to C^0(\mathbb{R})$ be defined by L(f)(x) = f'(x) for each $f \in P_3$. We know from calculus that

$$L(f+g)(x) = (f+g)'(x) = f'(x) + g'(x) = L(f)(x) + L(g)(x)$$

and

$$L(cf)(x) = (cf)'(x) = cf'(x) = cL(f)(x)$$

The equations above hold for all $x \in \mathbb{R}$ thus we find function equations L(f+g) = L(f) + L(g) and L(cf) = cL(f) for all $f, g \in C^1(\mathbb{R})$ and $c \in \mathbb{R}$.

Example 5.1.9. Let $a \in \mathbb{R}$. The evaluation mapping $\phi_a : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is defined by $\phi_a(f) = f(a)$. This is a linear transformation as (f + cg)(a) = f(a) + cg(a) by definition of function addition and scalar multiplication.

Example 5.1.10. Let $L(x,y) = x^2 + y^2$ define a mapping from \mathbb{R}^2 to \mathbb{R} . This is not a linear transformation since

$$L(c(x,y)) = L(cx,cy) = (cx)^{2} + (cy)^{2} = c^{2}(x^{2} + y^{2}) = c^{2}L(x,y).$$

We say L is a nonlinear transformation.

Example 5.1.11. Let $T : \mathbb{R}^{n \times n} \to \mathbb{R}$ be the determinant map; T(A) = det(A). Notice this is not a linear map since we found $det(A + B) \neq det(A) + det(B)$ in general. Also we know $det(cA) = c^n det(A)$. As I mentioned in the determinants chapter, the determinant is actually a multilinear transformation on the columns of the matrix.

Example 5.1.12. Let V be a finite dimensional vector space with basis $\beta = \{v_1, v_2, \dots, v_n\}$. The coordinate map $\Phi_\beta : V \to \mathbb{R}^{n \times 1}$ is defined by

$$\Phi_{\beta}(x_1v_1 + x_2v_2 + \dots + x_nv_n) = x_1e_1 + x_2e_2 + \dots + x_ne_n$$

for all $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n \in V$. This is a linear mapping, I leave the details to the reader.

Proposition 5.1.13.

Let $L: V \to W$ be a linear transformation, 1. L(0) = 02. $L(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{R}$.

Proof: to prove of (1.) let $x \in V$ and notice that x - x = 0 thus

$$L(0) = L(x - x) = L(x) + L(-1x) = L(x) - L(x) = 0.$$

To prove (2.) we use induction on n. Notice the proposition is true for n=1,2 by definition of linear transformation. Assume inductively $L(c_1v_1+c_2v_2+\cdots c_nv_n)=c_1L(v_1)+c_2L(v_2)+\cdots+c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{R}$ where $i=1,2,\ldots,n$. Let $v_1,v_2,\ldots,v_n,v_{n+1} \in V$ and $c_1,c_2,\ldots,c_n,c_{n+1} \in \mathbb{R}$ and consider, $L(c_1v_1+c_2v_2+\cdots c_nv_n+c_{n+1}v_{n+1}) =$

$$= L(c_1v_1 + c_2v_2 + \dots + c_nv_n) + c_{n+1}L(v_{n+1})$$
 by linearity of L
= $c_1L(v_1) + c_2L(v_2) + \dots + c_nL(v_n) + c_{n+1}L(v_{n+1})$ by the induction hypothesis.

Hence the proposition is true for n+1 and we conclude by the principle of mathematical induction that (2.) is true for all $n \in \mathbb{N}$. \Box

Remark 5.1.14.

I may have neglected an induction here or there earlier. Pretty much any time I state something for n there is likely an induction lurking about. Many times in mathematics if we can state a proposition for two objects then we can just as well extend the statement for n-objects. Extended linearity of limits, derivatives, integrals, etc... all follow from (2.) since those operations are linear operations.

5.2 matrix of a linear operator

We saw that if a map is defined as a matrix multiplication then it will be linear. A natural question to ask: is the converse true? Given a linear transformation from $\mathbb{R}^{n \times 1}$ to $\mathbb{R}^{m \times 1}$ can we write the transformation as multiplication by a matrix ?

Proposition 5.2.1.

 $L: \mathbb{R}^{n \times 1} \to \mathbb{R}^{m \times 1}$ is a linear transformation if and only if there exists $A \in \mathbb{R}^{m \times n}$ such that L(x) = Ax for all $x \in \mathbb{R}^{n \times 1}$.

Proof: (\Leftarrow) Assume there exists $A \in \mathbb{R}^{m \times n}$ such that L(x) = Ax for all $x \in \mathbb{R}^{n \times 1}$. As we argued before,

$$L(x + cy) = A(x + cy) = Ax + cAy = L(x) + cL(y)$$

for all $x, y \in \mathbb{R}^{n \times 1}$ and $c \in \mathbb{R}$ hence L is a linear transformation.

 (\Rightarrow) Assume $L : \mathbb{R}^{n \times 1} \to \mathbb{R}^{m \times 1}$ is a linear transformation. Let e_i denote the standard basis in $\mathbb{R}^{n \times 1}$ and let f_j denote the standard basis in $\mathbb{R}^{m \times 1}$. If $x \in \mathbb{R}^{n \times 1}$ then there exist constants x_i such that $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ and

$$L(x) = L(x_1e_1 + x_2e_2 + \dots + x_ne_n)$$

= $x_1L(e_1) + x_2L(e_2) + \dots + x_nL(e_n)$

where we made use of Propostion 5.1.13. Notice $L(e_i) \in \mathbb{R}^{m \times 1}$ thus there exist constants, say A_{ij} , such that

$$L(e_i) = A_{1i}f_1 + A_{2i}f_2 + \dots + A_{mi}f_m$$

for each $i = 1, 2, \ldots, n$. Let's put it all together,

$$L(x) = \sum_{i=1}^{n} x_i L(e_i)$$
$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} A_{ji} f_j$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ji} x_i f_j$$
$$= Ax.$$

Notice that $A \in \mathbb{R}^{m \times n}$ by its construction. \Box

Definition 5.2.2.

Let $L : \mathbb{R}^{n \times 1} \to \mathbb{R}^{m \times 1}$ be a linear transformation, the matrix $A \in \mathbb{R}^{m \times n}$ such that L(x) = Ax for all $x \in \mathbb{R}^{n \times 1}$ is called the **standard matrix** of L. We denote this by [L] = A or more compactly, $[L_A] = A$, we say that L_A is the linear transformation induced by A.

Example 5.2.3. Given that $L([x, y, z]^T) = [x + 2y, 3y + 4z, 5x + 6z]^T$ for $[x, y, z]^T \in \mathbb{R}^{3 \times 1}$ find the the standard matrix of L. We wish to find a 3×3 matrix such that L(v) = Av for all $v = [x, y, z]^T \in \mathbb{R}^{3 \times 1}$. Write L(v) then collect terms with each coordinate in the domain,

$$L\left(\left[\begin{array}{c}x\\y\\z\end{array}\right]\right) = \left[\begin{array}{c}x+2y\\3y+4z\\5x+6z\end{array}\right] = x\left[\begin{array}{c}1\\0\\5\end{array}\right] + y\left[\begin{array}{c}2\\3\\0\end{array}\right] + z\left[\begin{array}{c}0\\4\\6\end{array}\right]$$

It's not hard to see that,

$$L\left(\left[\begin{array}{c}x\\y\\z\end{array}\right]\right) = \left[\begin{array}{ccc}1&2&0\\0&3&4\\5&0&6\end{array}\right]\left[\begin{array}{c}x\\y\\z\end{array}\right] \quad \Rightarrow \quad A = [L] = \left[\begin{array}{ccc}1&2&0\\0&3&4\\5&0&6\end{array}\right]$$

Notice that the columns in A are just as you'd expect from the proof of Proposition 5.2.1. $[L] = [L(e_1)|L(e_2)|L(e_3)]$. In future examples I will exploit this observation to save writing.

Example 5.2.4. Suppose that $L([t, x, y, z]^T) = (t + x + y + z, z - x, 0, 3t - z)^T$, find [L].

$$\begin{split} & L(e_1) = L([1,0,0,0]^T) = (1,0,0,3)^T \\ & L(e_2) = L([0,1,0,0]^T) = (1,-1,0,0)^T \\ & L(e_3) = L([0,0,1,0]^T) = (1,0,0,0)^T \\ & L(e_4) = L([0,0,0,1]^T) = (1,1,0,-1)^T \end{split} \Rightarrow \ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}.$$

I invite the reader to check my answer here and see that L(v) = [L]v for all $v \in \mathbb{R}^{4 \times 1}$ as claimed.

5.3 composition of linear transformations

Linear transformations are functions so we already know how to compose them if they have suitably matched domains and ranges. However, linear transformations have very special structure so we can say more:

Proposition 5.3.1.

Let V_1, V_2, V_3 be vector spaces and suppose $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ are linear transformations then $L_2 \circ L_1 : V_1 \to V_3$ is a linear transformation and if V_1, V_2 are column spaces then $[L_2 \circ L_1] = [L_2][L_1]$.

Proof: Let $x, y \in V_1$ and $c \in \mathbb{R}$,

$$(L_2 \circ L_1)(x + cy) = L_2(L_1(x + cy))$$
defn. of composite
$$= L_2(L_1(x) + cL_1(y))$$
$$L_1$$
is linear trans.
$$= L_2(L_1(x)) + cL_2(L_1(y))$$
$$L_2$$
is linear trans.
$$= (L_2 \circ L_1)(x) + c(L_2 \circ L_1)(y)$$
defn. of composite

thus $L_2 \circ L_1$ is a linear transformation. To find the matrix of the composite we need only calculate its action on the standard basis:

$$(L_{2} \circ L_{1})(e_{i}) = L_{2}(L_{1}(e_{i}))$$

$$= L_{2}([L_{1}]e_{i})$$

$$= L_{2}(\sum_{j} [L_{1}]_{ji}e_{j})$$

$$= \sum_{j} [L_{1}]_{ji}L_{2}(e_{j})$$

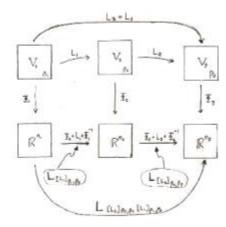
$$= \sum_{j} [L_{1}]_{ji}\sum_{k} [L_{2}]_{kj}e_{k}$$

$$= \sum_{j} \sum_{k} [L_{2}]_{kj}[L_{1}]_{ji}e_{k}$$

$$= \sum_{k} ([L_{2}][L_{1}])_{ki}e_{k}$$

$$= col_{i}([L_{2}][L_{1}])$$

We deduce that the *i*-th column of the matrix $[L_2 \circ L_1]$ is precisely the *i*-th column of the product $[L_2][L_1]$. Item (2.) follows. \Box



(this is a diagram of how to generalize the proposition for abstract vector spaces)

Remark 5.3.2.

The proposition above is the heart of the chain rule in multivariate calculus. The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $p \in \mathbb{R}^n$ gives the best linear approximation to f in the sense that

$$L_f(p+h) = f(p) + D_p f(h) \cong f(p+h)$$

if $h \in \mathbb{R}^n$ is close to the zero vector; the graph of L_f gives the tangent line or plane or hypersurface depending on the values of m, n. The so-called Frechet derivative is $D_p f$, it is a linear transformation from \mathbb{R}^n to \mathbb{R}^m . The simplest case is $f : \mathbb{R} \to \mathbb{R}$ where $D_p f(h) = f'(p)h$ and you should recognize $L_f(p+h) = f(p) + f'(p)h$ as the function whose graph is the tangent line, perhaps $L_f(x) = f(p) + f'(p)(x-p)$ is easier to see but it's the same just set p + h = x. Given two functions, say $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ then it can be shown that $D(g \circ f) = Dg \circ Df$. Then the proposition above generates all sorts of chain-rules by simply multiplying the matrices of Df and Dg. The matrix of the Frechet derivative is called the Jacobian matrix. The determinant of the Jacobian matrix plays an important role in changing variables for multiple integrals. It is likely we would cover this discussion in some depth in the Advanced Calculus course, while linear algebra is not a pre-req, it sure would be nice if you had it. Linear is truly foundational for most interesting math.

Example 5.3.3. Let $T : \mathbb{R}^{2 \times 1} \to \mathbb{R}^{2 \times 1}$ be defined by

$$T([x,y]^T) = [x+y, 2x-y]^T$$

for all $[x, y]^T \in \mathbb{R}^{2 \times 1}$. Also let $S : \mathbb{R}^{2 \times 1} \to \mathbb{R}^{3 \times 1}$ be defined by

$$S([x,y]^T) = [x, x, 3x + 4y]^2$$

for all $[x, y]^T \in \mathbb{R}^{2 \times 1}$. We calculate the composite as follows:

$$\begin{aligned} (S \circ T)([x, y]^T) &= S(T([x, y]^T)) \\ &= S([x + y, 2x - y]^T) \\ &= [x + y, x + y, 3(x + y) + 4(2x - y)]^T \\ &= [x + y, x + y, 11x - y]^T \end{aligned}$$

Notice we can write the formula above as a matrix multiplication,

$$(S \circ T)([x,y]^T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad [S \circ T] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix}.$$

Notice that the standard matrices of S and T are:

$$[S] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} \qquad [T] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

It's easy to see that $[S \circ T] = [S][T]$ (as we should expect since these are linear operators)

5.4 isomorphism of vector spaces

A one-one correspondence is a map which is 1-1 and onto. If we can find such a mapping between two sets then it shows those sets have the same cardnality. Cardnality is a crude idea of size. In linear algebra we will find it useful to have a way of characterizing when two vector spaces are essentially the same space. At a minimum we should expect they are in 1-1 correspondence. In addition we would like for linear combinations to be maintained. We've seen an example of this already, the coordinate mapping maintains linear combinations. This is absolutely crucial if coordinates of a vector are to completely describe it. ¹

Definition 5.4.1.

Let V, W be vector spaces then $\Phi : V \to W$ is an **isomorphism** if it is a 1-1 and onto mapping which is also a linear transformation. If there is an isomorphism between vector spaces V and W then we say those vector spaces are **isomorphic** and we denote this by $V \cong W$.

Other authors sometimes denote isomorphism by equality. But, I'll avoid that custom as I am reserving = to denote set equality.

Example 5.4.2. Let $V = \mathbb{R}^3$ and $W = P_2$. Define a mapping $\Phi : P_2 \to \mathbb{R}^3$ by

$$\Phi(ax^2 + bx + c) = (a, b, c)$$

for all $ax^2 + bx + c \in P_2$. To prove this function is onto we should chose an arbitrary element in the codomain, say $(c_1, c_2, c_3) \in \mathbb{R}^3$. I think it's pretty obvious that $c_1x^2 + c_2x + c_3$ maps to (c_1, c_2, c_3) under Φ hence Φ is onto. To prove Φ is one-one, assume $\Phi(a_1x^2 + a_2x + a_3) = \Phi(b_1x^2 + b_2x + b_3)$,

 $(a_1, a_2, a_3) = (b_1, b_2, b_3) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$

thus $a_1x^2 + a_2x + a_3 = b_1x^2 + b_2x + b_3$. Therefore, Φ is a bijection. Check for linearity,

$$\Phi(a_1x^2 + a_2x + a_3 + c(b_1x^2 + b_2x + b_3)) = \Phi((a_1 + cb_1)x^2 + (a_2 + cb_2)x + a_3 + cb_3)$$

= $(a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$
= $(a_1, a_2, a_3) + c(b_1, b_2, b_3)$
= $\Phi(a_1x^2 + a_2x + a_3) + c\Phi(b_1x^2 + b_2x + b_3)$

for all $a_1x^2 + a_2x + a_3$, $b_1x^2 + b_2x + b_3 \in P_2$ and $c \in \mathbb{R}$. Therefore, Φ is an isomorphism and $\mathbb{R}^3 \cong P_2$. As vector spaces, \mathbb{R}^3 and polynomials of up to quadratic order are the same.

Example 5.4.3. Let $\Phi : \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}$ be defined by

$$\Phi(\sum_{i,j} A_{ij} E_{ij}) = (A_{11}, \dots, A_{1n}, A_{21}, \dots, A_{2n}, \dots, A_{m1}, \dots, A_{mn})$$

This map simply takes the entries in the matrix and strings them out to a vector of length mn. I leave it to the reader to prove that Φ is an isomorphism.

¹I assume you know what the terms "onto", "1-1", "injective", "surjective", "1-1 correspondence", "bijection" mean. I also assume you know the basic theorems for compositions and inverses of functions. You can read my Math 200 notes if you're rusty

Example 5.4.4. Let S_2 be the set of 2×2 symmetric matrices. Let $\Psi : \mathbb{R}^3 \to S_2$ be defined by

$$\Psi(x,y,z) = \left[\begin{array}{cc} x & y \\ y & z \end{array} \right]$$

Notice

$$(\Psi(x,y,z))^T = \begin{bmatrix} x & y \\ y & z \end{bmatrix}^T = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

so the function Ψ is well-defined, it actually maps where I said it should. Also, if $A = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_2$ clearly $\Psi(x, y, z) = A$ thus Ψ is surjective. Injectivity is also easy to verify,

$$\Psi(x,y,z) = \Psi(a,b,c) \quad \Rightarrow \quad \left[\begin{array}{cc} x & y \\ y & z \end{array} \right] = \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \quad x = a, y = b, z = c$$

thus (x, y, z) = (a, b, c). Apparently 2×2 symmetric matrices are also the same as \mathbb{R}^3 as vector spaces.

Proposition 5.4.5.

Proof: The inverse of a bijection is a bijection. We need only demonstrate linearity, let $x, y \in V_2$ and $c \in \mathbb{R}$. Note by surjectivity of Ψ there exist $a, b \in V_1$ such that $\Psi(a) = x$ and $\Psi(b) = y$ which means $\Psi^{-1}(x) = a$ and $\Psi^{-1}(y) = b$. Consider then

$$\begin{split} \Phi^{-1}(x+cy) &= \Phi^{-1}(\Psi(a)+c\Psi(b)) \quad \Psi \text{ is onto} \\ &= \Phi^{-1}(\Psi(a+cb)) \qquad \Psi \text{ is linear} \\ &= a+cb \qquad \qquad \text{defn of inverse function} \\ &= \Psi^{-1}(x)+c\Psi^{-1}(y) \qquad \text{defn. of } a,b. \end{split}$$

this proves (1.). To prove (2.) recall that the composition of bijections is again a bijection. Moreover, Proposition 5.3.1 proves the composition of linear transformation is a linear transformation. \Box

Theorem 5.4.6.

Vector spaces with the same dimension are isomorphic.

Proof: Let V, W be a vector spaces of dimension n. It follows by definition of dimension there exists bases β_V of V and β_W of W. Moreover, we have coordinate maps with respect to each basis,

 $\Phi_{\beta_V}: V \to \mathbb{R}^{n \times 1} \qquad \Phi_{\beta_W}: W \to \mathbb{R}^{n \times 1}$

These are isomorphisms. By (1.) of Proposition 5.4.5 $\Phi_{\beta_W}^{-1}$ is an isomorphism. Observe that the map $\Phi_{\beta_W}^{-1} \circ \Phi_{\beta_V} : V \to W$ is the composition of isomorphisms and by (2.) (1.) of Proposition 5.4.5 this mapping provides an isomorphism of V and W. \Box

This theorem helps affirm my contention that coordinates encapsulate the linear structure of a vector space. The technique used in this proof is also used in many discussions of abstract vector spaces. We take some idea for matrices and lift it up to the abstract vector space by the coordinate isomorphism. Then we can define things like the trace or determinant of a linear operator. We simply define the trace or determinant of a linear operator to be the trace of determinant of the standard matrix of the operator. We need to understand coordinate change a little better before we make those definitions.

Remark 5.4.7.

There is much more to say about the theory of linear operators. There are interesting theorems about invariant operators and minimal polynomials. My intent in this chapter is just to alert you to the basics and a few technical results. In Problem 62 of Problem Set II I ask you to discover the proof of one of the more important theorems about linear operators. Another theorem that is missing from these notes is that $ker(T) = \{0\}$ iff T is one-one.

5.5 change we can believe in (no really, no joke)

A vector space does not come with a basis. We typically **choose** a basis and work out everything in terms of that particular choice. The coordinates prescribed from our choice are but one of many choices. It is natural to inquire how the coordinates of vectors and the matrices of linear transformations change form when we begin using a different basis. Coordinate change is an important idea in physics and engineering since a correct choice of coordinates will better reveal hidden symmetries and special properties of a given system. One of the great contributions of Einstein was to emphasize that coordinates were simply a picture of an underlying coordinate independent physics. That physical principle encouraged the study of **coordinate free** geometry which is at the heart of modern differential geometry. Linear algebra is perhaps the simplest implementation of these ideas. The linear transformations are coordinate free objects which have many different matrix representations in various coordinate systems. These various representations are related according to a **similarity transformation** as described later in this section.

Let me be a little more concrete about the task before us: The standard coordinates are just one of many choices of $\mathbb{R}^{n \times 1}$, we would like to be able to find matrix representations that are consistent with nonstandard bases. In addition, it would be nice if we could find a matrix representation of a linear operator on an abstract vector space. It can't be direct since abstract vectors do not form column vectors to multiply on a matrix. For example, $f(x) = x^2 + 3x + 2 \in P_2$ but we cannot write Df(x) = 2x + 3 as a matrix multiplication using just polynomials. We need to make use of coordinates allow us to switch $x^2 + 3x + 2$ to a corresponding column vector.

Definition 5.5.1.

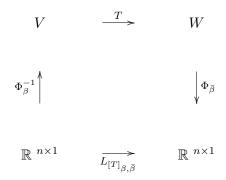
Let $T: V \to W$ be linear transformation bewteen vector spaces V and W and suppose $\Phi_{\beta}: V \to \mathbb{R}^{n \times 1}$ and $\Phi_{\bar{\beta}}: W \to \mathbb{R}^{m \times 1}$ are coordinate mappings with respect to the bases $\beta, \bar{\beta}$ for V, W respective. Given all the preceding data we define the matrix of T with respect to $\beta, \bar{\beta}$ to be $[T]_{\beta,\bar{\beta}} \in \mathbb{R}^{m \times n}$ which is defined implicitly through the equation

$$T = \Phi_{\bar{\beta}}^{-1} \circ L_{[T]_{\beta,\bar{\beta}}} \circ \Phi_{\beta}.$$

Or if you prefer, for each $x \in \mathbb{R}^{n \times 1}$

$$[T]_{\beta,\bar{\beta}} x = \Phi_{\bar{\beta}}(T(\Phi_{\beta}^{-1}(x)))$$

Let's walk through the formula above: we begin with a column vector x, then Φ_{β}^{-1} lifts the column vector up to the abstract vector $\Phi_{\beta}^{-1}(x)$ in V. Next we operate by T which moves us over to the vector $T(\Phi_{\beta}^{-1}(x))$ which is in W. Finally the coordinate map $\Phi_{\bar{\beta}}$ pushes the abstract vector in Wback to a column vector $\Phi_{\bar{\beta}}(T(\Phi_{\beta}^{-1}(x)))$ which is in $\mathbb{R}^{m\times 1}$. The same journey is accomplished by just multiplying² x by the $m \times n$ matrix $[T]_{\beta,\bar{\beta}}$.



Proposition 5.5.2.

Given the data in the preceding definition,

$$col_i([T]_{\beta,\bar{\beta}}) = \Phi_{\bar{\beta}}(T(\Phi_{\beta}^{-1}(e_i))).$$

Proof: Apply Theorem 2.8.16. \Box

Enough generalities, let's see how this definition is fleshed out in a concrete example.

²remember the notation $L_{[T]_{\beta,\bar{\beta}}}$ indicates the operation of left multiplication by the matrix $[T]_{\beta,\bar{\beta}}$; that is $L_{[T]_{\beta,\bar{\beta}}}(x) = [T]_{\beta,\bar{\beta}}x$ for all x.

Example 5.5.3. Let $\beta = \{1, x, x^2\}$ be the basis for P_2 and consider the derivative mapping $D : P_2 \rightarrow P_2$. Find the matrix of D assuming that P_2 has coordinates with respect to β on both copies of P_2 . Define and observe

$$\Phi(x^n) = e_{n+1}$$
 whereas $\Phi^{-1}(e_n) = x^{n-1}$

for n = 0, 1, 2. Recall $D(ax^2 + bx + c) = 2ax + bx$.

$$col_1([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_1))) = \Phi_\beta(D(1)) = \Phi_\beta(0) = 0$$

$$col_2([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_2))) = \Phi_\beta(D(x)) = \Phi_\beta(1) = e_1$$

$$col_3([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_3))) = \Phi_\beta(D(x^2)) = \Phi_\beta(2x) = 2e_2$$

Therefore we find,

$$[D]_{\beta,\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Calculate D^3 . Is this surprising?

5.5.1 change of basis for linear transformations on $\mathbb{R}^{n \times 1}$

We already know how to find the matrix of a linear transformation with respect to standard bases. Finding a matrix with respect to a nonstandard basis is not much harder. We consider the case where the domain and range are the same dimension since this is the case of most interest. Since both domain and range are column vectors of the same dimension we can express the coordinate mappings and their inverses directly by a matrix multiplication.

Proposition 5.5.4.

If
$$\mathbb{R}^{n \times 1}$$
 has basis $\beta = \{f_1, f_2, \dots, f_n\}$ and we denote $[\beta] = [f_1|f_2|\cdots|f_n]$ then
 $\Phi_\beta(x) = [\beta]^{-1}x \qquad \Phi_\beta^{-1}(y) = [\beta]y.$

Proof: Let $v \in \mathbb{R}^{n \times 1}$ then $v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n = [e_i][v]$ where e_i denote the standard basis and [v] is the standard coordinate vector of v (which is the same as v in this rather special case). We also may write $v = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n = [f_i][v]_\beta$ where $[w_i] = [v]_\beta$ are the coordinates of v with respect to the basis β . Since $\{f_1, f_2, \ldots, f_n\}$ form a basis the matrix $[f_i]$ is invertible and we can solve for $[v]_\beta = [f_i]^{-1}v$. Consider then,

$$\Phi_{\beta}(v) = [v]_{\beta} = [f_i]^{-1}v.$$

We find $\Phi_{\beta} = L_{[f_i]^{-1}}$. Since $\Phi_{\beta} \circ \Phi_{\beta}^{-1} = Id$ it follows that $[\Phi_{\beta}][\Phi_{\beta}^{-1}] = I$ thus $[\Phi_{\beta}]^{-1} = [\Phi_{\beta}^{-1}]$ and we conclude $\Phi_{\beta}^{-1} = L_{[f_i]}$ which means $\Phi_{\beta}^{-1}(y) = L_{[f_i]}(y) = [f_i]y$ for all $y \in \mathbb{R}^{n \times 1}$. \Box

Proposition 5.5.5. Coordinate change for vectors and linear operators on $\mathbb{R}^{n \times 1}$.

Let $\mathbb{R}^{n \times 1}$ have bases $\beta = \{f_1, f_2, \dots, f_n\}$ and $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ such that $[\beta]P = [\bar{\beta}]$ where I denoted $[\beta] = [f_1|f_2|\cdots|f_n]$ and $[\bar{\beta}] = [\bar{f}_1|\bar{f}_2|\cdots|\bar{f}_n]$. If $v = \sum_i v_i f_i$ and $v = \sum_j \bar{v}_j \bar{f}_j$ we denote $[v]_{\beta} = [v_i]$ and $[v]_{\bar{\beta}} = [\bar{v}_j]$ and the coordinate vectors of v are related by

$$[v]_{\bar{\beta}} = P^{-1}[v]_{\beta}$$

Moreover, if $T: \mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ is a linear operator then

$$[T]_{\bar{\beta},\bar{\beta}} = P^{-1}[T]_{\beta,\beta}P$$

Proof: Given the data above, note we can write $\sum_i v_i f_i = [\beta][v]_\beta$ and $\sum_j \bar{v}_j \bar{f}_j = [\bar{\beta}][v]_{\bar{\beta}}$ (we can do this since we are in $\mathbb{R}^{n \times 1}$)

$$v = [\beta][v]_{\beta} = [\beta]PP^{-1}[v]_{\beta} = [\bar{\beta}]P^{-1}[v]_{\beta}$$

However, we also have $v = [\bar{\beta}][v]_{\bar{\beta}}$. But $[\bar{\beta}]$ is an invertible matrix thus $[\bar{\beta}][v]_{\bar{\beta}} = [\bar{\beta}]P^{-1}[v]_{\beta}$ implies $[v]_{\bar{\beta}} = P^{-1}[v]_{\beta}$.

We defined $[T]_{\bar{\beta},\bar{\beta}}$ implicitly through the equation $T = \Phi_{\bar{\beta}}^{-1} \circ L_{[T]_{\bar{\beta},\bar{\beta}}} \circ \Phi_{\bar{\beta}}$. In this special case the coordinate maps and their inverses are matrix multiplication as described by Proposition 5.5.4 and we calculate

$$T = L_{\bar{\beta}} \circ L_{[T]_{\bar{\beta},\bar{\beta}}} \circ L_{\bar{\beta}^{-1}}$$

But the matrix of a composite of linear transformations is the product the matrices of those transformations, thus

$$T = L_{[\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1}}$$

Therefore, the standard matrix of T is $[T] = [\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1}$. By the same argument we find $[T] = [\beta][T]_{\beta,\beta}[\beta]^{-1}$. Thus,

$$[T] = [\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1} = [\beta][T]_{\beta,\beta}[\beta]^{-1} \quad \Rightarrow \quad [T]_{\bar{\beta},\bar{\beta}} = [\bar{\beta}]^{-1}[\beta][T]_{\beta,\beta}[\beta]^{-1}[\bar{\beta}]$$

However, we defined P to be the matrix which satisfies $[\beta]P = [\bar{\beta}]$ thus $P = [\beta]^{-1}[\bar{\beta}]$ and $P^{-1} = [\bar{\beta}]^{-1}[\beta]$. \Box .

Example 5.5.6. Let $\beta = \{[1,1]^T, [1,-1]^T\}$ and $\gamma = \{[1,0]^T, [1,1]^T\}$ be bases for $\mathbb{R}^{2\times 1}$. Find $[v]_{\beta}$ and $[v]_{\gamma}$ if $v = [2,4]^T$. Let me frame the problem, we wish to solve:

$$v = [\beta][v]_{eta}$$
 and $v = [\gamma][v]_{\gamma}$

where I'm using the basis in brackets to denote the matrix formed by concatenating the basis into a single matrix,

$$[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad and \quad [\gamma] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is the 2×2 case so we can calculate the inverse from our handy-dandy formula:

$$[\beta]^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad and \quad [\gamma]^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

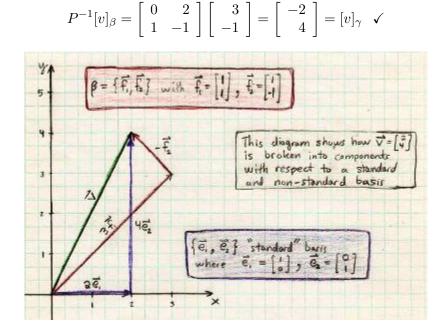
Then multiplication by inverse yields $[v]_{\beta} = [\beta]^{-1}v$ and $[v]_{\gamma} = [\gamma]^{-1}v$ thus:

$$[v]_{\beta} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad and \quad [v]_{\gamma} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Let's verify the relation of $[v]_{\gamma}$ and $[v]_{\beta}$ relative to the change of basis matrix we denoted by P in the prop; we hope to find $[v]_{\gamma} = P^{-1}[v]_{\beta}$ (note γ is playing the role of $\overline{\beta}$ in the statement of the prop.)

$$[\beta]P = [\gamma] \quad \Rightarrow \quad P^{-1} = [\gamma]^{-1}[\beta] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

Consider then (as a check on our calculations and also the proposition)



Now that we've seen an example, let's find $[v]_{\beta}$ for an arbitrary $v = [x, y]^T$,

$$[v]_{\beta} = \frac{1}{2} \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix} \begin{bmatrix} x\\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+y)\\ \frac{1}{2}(x-y) \end{bmatrix}$$

If we denote $[v]_{\beta} = [\bar{x}, \bar{y}]^T$ then we can understand the calculation above as the relation between the barred and standard coordinates:

$$\bar{x} = \frac{1}{2}(x+y)$$
 $\bar{y} = \frac{1}{2}(x-y)$

Conversely, we can solve these for x, y to find the inverse transformations:

$$x = \bar{x} + \bar{y} \qquad y = \bar{x} - \bar{y}$$

Similar calculations are possible with respect to the γ -basis.

Example 5.5.7. Let $\bar{\beta} = \{[1,0,1]^T, [0,1,1]^T, [4,3,1]^T\}$. Furthermore, define a linear transformation $T : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ by the rule $T([x,y,z]^T) = [2x - 2y + 2z, x - z, 2x - 3y + 2z]^T$. Find the matrix of T with respect to the basis β . Note first that the standard basis is read from the rule:

$$T\left(\begin{bmatrix} x\\ y\\ z \end{bmatrix} \right) = \begin{bmatrix} 2x - 2y + 2z\\ x - z\\ 2x - 3y + 2z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2\\ 1 & 0 & -1\\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix}$$

Next, use the proposition, take $\beta = \{e_1, e_2, e_3\}$ thus $[\beta] = I_3$ and then P satisfies $I_3P = [\bar{\beta}]$. The change of basis matrix for changing from the standard basis to a nonstandard basis is just the matrix of the nonstandard matrix; $P = [\bar{\beta}]$ Consider then (omitting the details of calculating P^{-1})

$$P^{-1}[T]P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix} \begin{bmatrix} 4 & 0 & 4 \\ 0 & -1 & 3 \\ 4 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, in the $\bar{\beta}$ -coordinates the linear operator T takes on a particularly simple form:

$$\begin{bmatrix} T\left(\begin{array}{c} \bar{x}\\ \bar{y}\\ \bar{z} \end{bmatrix} \right) \end{bmatrix}_{\bar{\beta}} = \begin{bmatrix} 4\bar{x}\\ -\bar{y}\\ \bar{z} \end{bmatrix}$$

In other words, if $\bar{\beta} = \{f_1, f_2, f_3\}$ then

$$T([\bar{x}, \bar{y}, \bar{z}]^T) = 4\bar{x}f_1 - \bar{y}f_2 + \bar{z}f_3$$

This linear transformation acts in a special way in the f_1, f_2 and f_3 directions. The basis we considered here is actually what is known as a an eigenbasis for T.

We uncovered a useful formula in the proof preceding the example. Let me state it clearly for future reference

Corollary 5.5.8.

Given a linear transformation $T : \mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ and a basis β for $\mathbb{R}^{n \times 1}$ the matrix of T with respect to β is related to the standard matrix by

$$[T] = [\beta][T]_{\beta,\beta}[\beta]^{-1}$$

which is equivalent to stating,

 $[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]$

If two matrices are related as [T] and $[T]_{\beta,\beta}$ are related above then the matrices are similar.

5.5.2 similar matrices and invariant quantities for abstract linear operators

Definition 5.5.9.

Let $A, B \in \mathbb{R}^{n \times n}$ we say A and B are **similar matrices** if there exists an invertible matrix P such that $A = P^{-1}BP$. Moreover, we say that A is obtained from B by a **similarity transformation** if A and B are similar.

There are many nice properties for similar matrices. I'll state a few of those here but I'm leaving the proof for you. They're not too hard to prove so I figure they'll make nice Problem Set problems.

Proposition 5.5.10.

Let $A, B, C \in \mathbb{R}^{n \times n}$.

- 1. A is similar to A.
- 2. If A is similar to B then B is similar to A.
- 3. If A is similar to B and B is similar to C then A similar to C.
- 4. If A and B are similar then det(A) = det(B)
- 5. If A and B are similar then tr(A) = tr(B)
- 6. If A and B are similar then Col(A) = Col(B) and Null(A) = Null(B)

Given the proposition above we can make the following definitions without ambiguity.

Definition 5.5.11.

Let $T: V \to V$ be a linear transformation on a finite-dimensional vector space V and let β be any basis of V,

1. $det(T) = det([T]_{\beta,\beta}).$ 2. $tr(T) = tr([T]_{\beta,\beta}).$

$$2. \ \iota r(I) = \iota r([I]_{\beta,\beta})$$

3. $rank(T) = rank([T]_{\beta,\beta}).$

Finally, we can define analogs of the null space and column space of a matrix in the abstract. These are called the kernel and range of the linear transformation.

Definition 5.5.12.

Let $T: V \to W$ be a linear transformation then 1. $ker(T) = T^{-1}(\{0\}).$ 2. range(T) = T(V) It is easy to show $ker(T) \leq V$ and $range(T) \leq W$. Incidentally, it can be argued that every subspace is the kernel of some linear transformation. One method for arguing a given set is a subspace is simply to find a mapping which is clearly linear and happens to have the subspace in question as the kernel. The idea I describe here is an abstract algebraic idea, I have not made much effort to bring these sort of ideas to the forefront. If you would like to read a book on linear algebra from the abstract algebra perspective I highly reccommend the lucid text *Abstract Linear Algebra* by Morton L. Curtis

Proposition 5.5.13.

Let $T: V \to W$ be a linear transformation then V/ker(T) is a vector space and $V/ker(T) \cong range(T)$.

The space V/ker(T) is called a *quotient space*. It is formed from the set of cosets of ker(T). These cosets have the form x + ker(T) where $x \notin ker(T)$. If ker(T) is a plane through the origin then the cosets are planes with the same normal but shifted off the origin. Of course, ker(T) need not be a plane, it could be the origin, a line, a volume, a hypervolume. In fact, $ker(T) \subset V$ so ascribing geometry to ker(T) is likely misguided if you want to be literal. Remember, V could be a solution set to a DEqn or a set of polynomials, it hardly makes sense to talk about a plane of polynomials. However, we could talk about a subset of polynomials whose coordinates fill out some plane in $\mathbb{R}^{n \times 1}$. Coordinates take abstract vector spaces and convert them back to $\mathbb{R}^{n \times 1}$. Anyhow, this proposition is an example of the Fundmental Theorem of Homomorphisms from abstract algebra. In the language of Math 421, a vector space is an abelian group with respect to vector addition with an \mathbb{R} -module structure given by the scalar multiplication. A linear transformation is just a mapping which preserves addition, it is a homomorphism in abstract-algebra-speak. You might recall from Math 200 that there is a more basic theorem, any mapping induces a bijection from its fiber quotient to its range. That is at the base of all other such fundamental theorems. Category theory is the study of such generalities in mathematics. If you intend to study pure mathematics make sure to look into category theory at some point. The basic idea is that different things in math form categories. Each category permits some family of morphisms. One can then look for theorems which hold for all categories. Or, if you see a theorem in one branch of mathematics you can use category theoretic intuition to propose analogus theorems in another branch of math. This is abstraction at its core. We try to extend patterns that we know to more general patterns. The only trouble is sometimes we can get so abstract we forget where we are. This may be an argument for women in mathematics.

5.6 applications

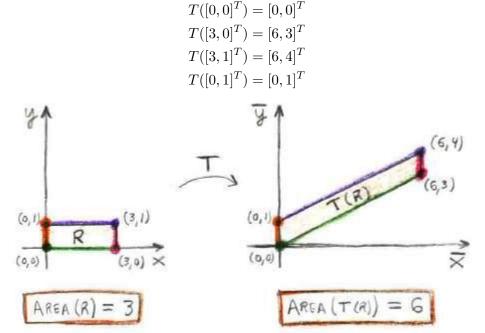
There are many applications in general. We'll simply take a few moments to examine how linear transformations transform lines and then rectangles.

Example 5.6.1. Let $A \in \mathbb{R}^{frm-e}$ and define $T([x, y]^T) = A[x, y]^T$. This makes T a linear transformation on \mathbb{R}^2 . A line in the plane can be described parametrically by $\vec{r}(t) = \vec{r_o} + t\vec{v}$. Consider then that

$$T(\vec{r}(t)) = A\vec{r}(t) = A(\vec{r}_o + t\vec{v}) = A\vec{r}_o + tA\vec{v}$$

Therefore, a linear transformation transforms a line with direction \vec{v} to a new line with direction $A\vec{v}$. There is one exception to this rule, if $\vec{v} \in Null(A)$ then the image of the line is just a point. In the problem set Problem 58 I assume that A is invertible to avoid this problem. Remember that if A is invertible then $Null(A) = \{0\}$.

Example 5.6.2. Let us consider a linear transformation $T([x, y]^T) = [2x, x + y]^T$. Furthermore, let's see how a rectangle R with corners (0,0), (3,0), (3,1), (0,1). Since this linear transformation is invertible (I invite you to prove that) it follows that the image of a line is again a line. Therefore, if we find the image of the corners under the mapping T then we can just connect the dots in the image to see what T(R) resembles. Our goal here is to see what a linear transformation does to a rectangle.



As you can see from the picture we have a paralellogram with base 6 and height 1 thus Area(T(R)) = 6. In constrast, Area(R) = 3. You can calculate that det(T) = 2. Curious, Area(T(R)) = det(T)Area(R). I wonder if this holds in general?³

³ok, actually I don't wonder, I just make homework problems about this

5.7 conclusions

Continuing Theorem 4.8.1 from the previous chapter,

Theorem 5.7.1.

Let A be a real $n \times n$ matrix then the following are equivalent: (a.) A is invertible, (b.) rref[A|0] = [I|0] where $0 \in \mathbb{R}^{n \times 1}$, (c.) Ax = 0 iff x = 0, (d.) A is the product of elementary matrices, (e.) there exists $B \in \mathbb{R}^{n \times n}$ such that AB = I, (f.) there exists $B \in \mathbb{R}^{n \times n}$ such that BA = I, (g.) rref[A] = I, (h.) rref[A|b] = [I|x] for an $x \in \mathbb{R}^{n \times 1}$, (i.) Ax = b is consistent for every $b \in \mathbb{R}^{n \times 1}$, (j.) Ax = b has exactly one solution for every $b \in \mathbb{R}^{n \times 1}$, (k.) A^T is invertible, (1.) $det(A) \neq 0$, (m.) Kramer's rule yields solution of Ax = b for every $b \in \mathbb{R}^{n \times 1}$. (n.) $Col(A) = \mathbb{R}^{n \times 1}$, (0.) $Row(A) = \mathbb{R}^{1 \times n}$, (p.) rank(A) = n, (q.) $Null(A) = \{0\},\$ (r.) $\nu = 0$ for A where $\nu = dim(Null(A))$, (s.) the columns of A are linearly independent, (t.) the rows of A are linearly independent,

Let A be a real $n \times n$ matrix then the following are equivalent:

- (u.) the induced linear operator L_A is onto; $L_A(\mathbb{R}^{n \times 1}) = \mathbb{R}^{n \times 1}$.
- (v.) the induced linear operator L_A is 1-1
- (w.) the induced linear operator L_A is an isomorphism.
- (x.) the kernel of the induced linear operator is trivial; $ker(L_A) = \{0\}$.

Again, we should pay special attention to the fact that the above comments hold only for a square matrix. If we consider a rectangular matrix then the connection between the concepts in the theorem are governed by the dimension formulas we discovered in the previous chapter.

Next, continuing Theorem 4.8.2 from the previous chapter:

Theorem 5.7.2.

Let A be a real $n \times n$ matrix then the following are equivalent: (a.) A is not invertible, (b.) Ax = 0 has at least one nontrivial solution., (c.) there exists $b \in \mathbb{R}^{n \times 1}$ such that Ax = b is inconsistent, (d.) det(A) = 0, (e.) $Null(A) \neq \{0\}$, (f.) there are $\nu = dim(Null(A))$ parameters in the general solution to Ax = 0, (g.) the induced linear operator L_A is **not** onto; $L_A(\mathbb{R}^{n \times 1}) \neq \mathbb{R}^{n \times 1}$. (h.) the induced linear operator L_A is **not** 1-1 (i.) the induced linear operator L_A is **not** an isomorphism. (j.) the kernel of the induced linear operator is **non**trivial; $ker(L_A) \neq \{0\}$.

Can you think of anything else to add here? Let me know if you think I missed something here. If it's sufficiently interesting it'll be worth some points.

Chapter 6

linear geometry

The concept of a geometry is very old. Philosophers in the nineteenth century failed miserably in their analysis of geometry and the physical world. They became mired in the popular misconception that mathematics must be physical. They argued that because 3 dimensional Eulcidean geometry was the only geometry familar to everyday experience it must surely follow that a geometry which differs from Euclidean geometry must be nonsensical. Why should physical intuition factor into the argument? Geometry is a mathematical construct, not a physical one. There are many possible geometries. On the other hand, it would seem the geometry of space and time probably takes just one form. We are tempted by this misconception every time we ask "but what is this math really". That question is usually wrong-headed. A better question is "is this math logically consistent" and if so what physical systems is it known to model.

The modern view of geometry is stated in the langauge of manifolds, fiber bundles, algebraic geometry and perhaps even more fantastic structures. There is currently great debate as to how we should model the true intrinsic geometry of the universe. Branes, strings, quivers, noncommutative geometry, twistors, ... this list is endless. However, at the base of all these things we must begin by understanding what the geometry of a flat space entails.

Vector spaces are flat manifolds. They possess a global coordinate system once a basis is chosen. Up to this point we have only cared about algebraic conditions of linear independence and spanning. There is more structure we can assume. We can ask what is the length of a vector? Or, given two vectors we might want to know what is the angle bewtween those vectors? Or when are two vectors orthogonal?

If we desire we can also insist that the basis consist of vectors which are *orthogonal* which means "perpendicular" in a generalized sense. A geometry is a vector space plus an idea of orthogonality and length. The concepts of orthogonality and length are encoded by an inner-product. Inner-products are symmetric, positive definite, bilinear forms, they're like a dot-product. Once we have a particular geometry in mind then we often restrict the choice of bases to only those bases which preserve the length of vectors.

The mathematics of orthogonality is exhibited by the dot-products and vectors in calculus III. However, it turns out the concept of an *inner-product* allows us to extend the idea or perpendicular to abstract vectors such as functions. This means we can even ask interesting questions such as "how close is one function to another" or "what is the closest function to a set of functions". Least-squares curve fitting is based on this geometry.

This chapter begins by defining dot-products and the norm (a.k.a. length) of a vector in $\mathbb{R}^{n \times 1}$. Then we discuss orthogonality, the Gram Schmidt algorithm, orthogonal complements and finally the application to the problem of least square analysis. The chapter concludes with a consideration of the similar, but abstract, concept of an inner product space. We look at how least squares generalizes to that context and we see how Fourier analysis naturally flows from our finite dimensional discussions of orthogonality. ¹

Let me digress from linear algebra for a little while. In physics it is customary to only allow coordinates which fit the physics. In classical mechanics one often works with intertial frames which are related by a rigid motion. Certain quantities are the same in all intertial frames, notably force. This means Newtons laws have the same form in all intertial frames. The geometry of special relativity is 4 dimensional. In special relativity, one considers coordinates which preserve Einstein's three axioms. Allowed coordinates are related to other coordinates by Lorentz transformations. These Lorentz transformations include rotations and velocity boosts. These transformations are designed to make the speed of a light ray invariant in all frames. For a linear algebraist the vector space is the starting point and then coordinates are something we add on later. Physics, in contrast, tends to start with coordinates and if the author is kind he might warn you which transformations are allowed.

What coordinate transformations are allowed actually tells you what kind of physics you are dealing with. This is an interesting and nearly universal feature of modern physics. The allowed transformations form what is known to physicsists as a "group" (however, strictly speaking these groups do not always have the strict structure that mathematicians insist upon for a group). In special relativity the group of interest is the Poincaire group. In quantum mechanics you use unitary groups because unitary transformations preserve probabilities. In supersymmetric physics you use the super Poincaire group because it is the group of transformations on superspace which preserves supersymmetry. In general relativity you allow general coordinate transformations which are locally lorentzian because all coordinate systems are physical provided they respect special relativity in a certain approximation. In solid state physics there is something called the renormilzation group which plays a central role in physical predictions of field-theoretic models. My point? Transformations of coordinates are important if you care about physics. We study the basic case of vector spaces in this course. If you are interested in the more sophisticated topics just ask, I can show you where to start reading.

 $^{^{1}}$ we ignore analytical issues of convergence since we have only in mind a Fourier approximation, not the infinite series

6.1 Euclidean geometry of \mathbb{R}^n

The dot-product is a mapping from $\mathbb{R}^{n \times 1} \times \mathbb{R}^{n \times 1}$ to \mathbb{R} . We take in a pair of vectors and output a real number.

Definition 6.1.1.

Let $x, y \in \mathbb{R}$ $n \times 1$ we define $x \cdot y \in \mathbb{R}$ by

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

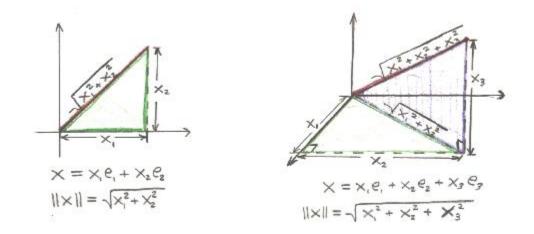
Example 6.1.2. Let $v = [1, 2, 3, 4, 5]^T$ and $w = [6, 7, 8, 9, 10]^T$

$$v \cdot w = 6 + 14 + 24 + 36 + 50 = 130$$

The dot-product can be used to define the length or norm of a vector and the angle between two vectors.

Definition 6.1.3.

The length or norm of $x \in \mathbb{R}^{n \times 1}$ is a real number which is defined by $||x|| = \sqrt{x \cdot x}$. Furthermore, let x, y be nonzero vectors in $\mathbb{R}^{n \times 1}$ we define the **angle** θ between x and y by $\cos^{-1}\left[\frac{x \cdot y}{||x|| ||y||}\right]$. \mathbb{R} together with these definitions of length and angle forms a Euclidean Geometry.



Technically, before we make this definition we should make sure that the formulas given above even make sense. I have not shown that $x \cdot x$ is nonnegative and how do we know that the inverse cosine is well-defined? The first proposition below shows the norm of x is well-defined and establishes several foundational properties of the dot-product.

Proposition 6.1.4.

Suppose $x, y, z \in \mathbb{R}^{n \times 1}$ and $c \in \mathbb{R}$ then 1. $x \cdot y = y \cdot x$ 2. $x \cdot (y + z) = x \cdot y + x \cdot z$ 3. $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$ 4. $x \cdot x \ge 0$ and $x \cdot x = 0$ iff x = 0

Proof: the proof of (1.) is easy, $x \cdot y = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = y \cdot x$. Likewise,

$$x \cdot (y+z) = \sum_{i=1}^{n} x_i (y+z)_i = \sum_{i=1}^{n} (x_i y_i + x_i z_i) = \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i z_i = x \cdot y + x \cdot z$$

proves (2.) and since

$$c\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} c x_i y_i = \sum_{i=1}^{n} (cx)_i y_i = \sum_{i=1}^{n} x_i (cy)_i$$

we find $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$. Continuting to (4.) notice that $x \cdot x = x_1^2 + x_2^2 + \dots + x_n^2$ thus $x \cdot x$ is the sum of squares and it must be nonnegative. Suppose x = 0 then $x \cdot x = x^T x = 0^T 0 = 0$. Conversely, suppose $x \cdot x = 0$. Suppose $x \neq 0$ then we find a contradiction since it would have a nonzero component which implies $x_1^2 + x_2^2 + \dots + x_n^2 \neq 0$. This completes the proof of (4.). \Box

The formula $\cos^{-1}\left[\frac{x \cdot y}{||x|| \cdot ||y||}\right]$ is harder to justify. The inequality that we need for it to be reasonable is $\left|\frac{x \cdot y}{||x|| \cdot ||y||}\right| \leq 1$, otherwise we would not have a number in the $dom(\cos^{-1}) = range(\cos) = [-1, 1]$. An equivalent inequality is $|x \cdot y| \leq ||x|| \cdot ||y||$ which is known as the **Cauchy-Schwarz** inequality.

Proposition 6.1.5.

$$|\text{If } x, y \in \mathbb{R}^{n \times 1} \text{ then } |x \cdot y| \le ||x|| ||y||$$

Proof: I've looked in a few linear algebra texts and I must say the proof given in Spence, Insel and Friedberg is probably the most efficient and clear. Other texts typically run up against a quadratic inequality in some part of their proof (for example the linear algebra texts by Apostle, Larson& Edwards, Anton & Rorres to name a few). That is somehow hidden in the proof that follows: let $x, y \in \mathbb{R}^{n \times 1}$. If either x = 0 or y = 0 then the inequality is clearly true. Suppose then that both x and y are nonzero vectors. It follows that $||x||, ||y|| \neq 0$ and we can define vectors of unit-length; $\hat{x} = \frac{x}{||x||}$ and $\hat{y} = \frac{y}{||y||}$. Notice that $\hat{x} \cdot \hat{x} = \frac{x}{||x||} \cdot \frac{x}{||x||} = \frac{1}{||x||^2} \hat{x} \cdot x = \frac{x \cdot x}{x \cdot x} = 1$ and likewise $\hat{y} \cdot \hat{y} = 1$.

Consider,

$$0 \le ||\hat{x} \pm \hat{y}||^2 = (\hat{x} \pm \hat{y}) \cdot (\hat{x} \pm \hat{y})$$

$$= \hat{x} \cdot \hat{x} \pm 2(\hat{x} \cdot \hat{y}) + \hat{y} \cdot \hat{y}$$

$$= 2 \pm 2(\hat{x} \cdot \hat{y})$$

$$\Rightarrow -2 \le \pm 2(\hat{x} \cdot \hat{y})$$

$$\Rightarrow \pm \hat{x} \cdot \hat{y} \le 1$$

$$\Rightarrow |\hat{x} \cdot \hat{y}| \le 1$$

Therefore, noting that $x = ||x||\hat{x}$ and $y = ||y||\hat{y}$,

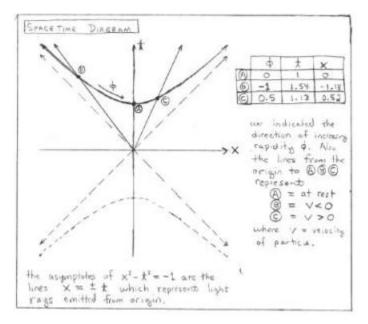
$$|x \cdot y| = |\,||x||\hat{x} \cdot ||y||\hat{y}\,| = ||x||\,||y||\,|\hat{x} \cdot \hat{y}| \le ||x||\,||y||$$

The use of unit vectors is what distinguishes this proof from the others I've found. \Box

Remark 6.1.6.

The dot-product is but one of many geometries for \mathbb{R}^n . We will explore generalizations of the dot-product in a later section. However, in this section we will work exclusively with the standard dot-product on \mathbb{R}^n . Generally, unless explicitly indicated otherwise, we assume Euclidean geometry for \mathbb{R}^n .

Just for fun here's a picture of a circle in the hyperbolic geometry of special relativity, technically it's not a geometry since we have nonzero-vectors with zero length (so-called null-vectors). Perhaps we will offer a course in special relativity some time and we could draw these pictures with understanding in that course.



Example 6.1.7. Let $v = [1, 2, 3, 4, 5]^T$ and $w = [6, 7, 8, 9, 10]^T$ find the angle between these vectors and calculate the unit vectors in the same directions as v and w. Recall that, $v \cdot w = 6 + 14 + 24 + 36 + 50 = 130$. Furthermore,

$$||v|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2} = \sqrt{1 + 4 + 9 + 16 + 25} = \sqrt{55}$$
$$||w|| = \sqrt{6^2 + 7^2 + 8^2 + 9^2 + 10^2} = \sqrt{36 + 49 + 64 + 81 + 100} = \sqrt{330}$$

We find unit vectors via the standard trick, you just take the given vector and multiply it by the reciprocal of its length. This is called **normalizing** the vector,

$$\hat{v} = \frac{1}{\sqrt{55}} [1, 2, 3, 4, 5]^T$$
 $\hat{w} = \frac{1}{\sqrt{330}} [6, 7, 8, 9, 10]^T$

The angle is calculated from the definition of angle,

$$\theta = \cos^{-1} \left(\frac{130}{\sqrt{55}\sqrt{330}} \right) = 15.21^{\circ}$$

It's good we have this definition, 5-dimensional protractors are very expensive.

Proposition 6.1.8.

Let $x, y \in \mathbb{R}^{n \times 1}$ and suppose $c \in \mathbb{R}$ then 1. ||cx|| = |c| ||x||2. $||x + y|| \le ||x|| + ||y||$

Proof: let $x \in \mathbb{R}^{n \times 1}$ and $c \in \mathbb{R}$ then calculate,

$$||cx||^2 = (cx) \cdot (cx) = c^2 x \cdot x = c^2 ||x||^2$$

Since $||cx|| \ge 0$ the squareroot yields $||cx|| = \sqrt{c^2} ||x||$ and $\sqrt{c^2} = |c|$ thus ||cx|| = |c|||x||. Item (2.) is called the **triangle inequality** for reasons that will be clear when we later discuss the distance function. Let $x, y \in \mathbb{R}^{n \times 1}$,

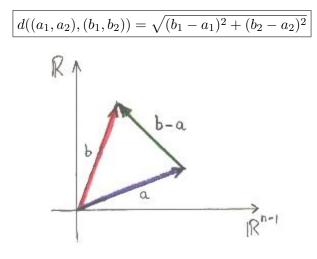
$ x+y ^{2} = (x+y) \cdot (x+y) $	defn. of norm
$= x \cdot (x+y) + y \cdot (x+y) $	prop. of dot-product
$= x \cdot x + x \cdot y + y \cdot x + y \cdot y $	prop. of dot-product
$= x ^2 + 2x \cdot y + y ^2 $	prop. of dot-product
$\leq x ^2 + 2 x \cdot y + y ^2$	triangle in eq. for $\mathbb R$
$\leq x ^2 + 2 x y + y ^2$	Cauchy-Schwarz ineq.
$\leq (x + y)^2$	algebra

Notice that both ||x + y|| and ||x|| + ||y|| are nonnegative by (4.) of Proposition 6.1.4 hence the inequality above yields $||x + y|| \le ||x|| + ||y||$. \Box

Definition 6.1.9.

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The distance between a \in \mathbb{R}^{n \times 1} and b \in \mathbb{R}^{n \times 1} is defined to be d(a, b) \equiv ||b - a||.
```

If we draw a picture this definition is very natural. Here we are thinking of the points a, b as vectors from the origin then b - a is the vector which points from a to b (this is algebraically clear since a + (b - a) = b). Then the distance between the points is the length of the vector that points from one point to the other. If you plug in two dimensional vectors you should recognize the distance formula from middle school math:



Proposition 6.1.10.

Let $d : \mathbb{R}^{n \times 1} \times \mathbb{R}^{n \times 1} \to \mathbb{R}$ be the distance function then 1. d(x, y) = d(y, x)2. $d(x, y) \ge 0$ 3. d(x, x) = 04. $d(x, y) + d(y, z) \ge d(x, z)$

Proof: I leave the proof of (1.), (2.) and (3.) to the reader. Item (4.) is also known as the **triangle inequality**. Think of the points x, y, z as being the vertices of a triangle, this inequality says the sum of the lengths of two sides cannot be smaller than the length of the remaining side. Let $x, y, z \in \mathbb{R}^{n \times 1}$ and note by the triangle inequality for $|| \cdot ||$,

$$d(x,z) = ||z - x|| = ||z - y + y - x|| \le ||z - y|| + ||y - x|| = d(y,z) + d(x,y). \quad \Box$$

We study the 2 and 3 dimensional case in some depth in calculus III. I would recommend you take that course, even if it's not "on your sheet".

6.2 orthogonality in $\mathbb{R}^{n \times 1}$

Two vectors are orthogonal if the vectors point in mutually exclusive directions. We saw in calculus III the dot-product allowed us to pick apart vectors into pieces. The same is true in n-dimensions: we can take a vector and isassemble it into component vectors which are orthogonal.

Definition 6.2.1.

Let $v, w \in \mathbb{R}^{n \times 1}$ then we say v and w are **orthogonal** iff $v \cdot w = 0$.

Example 6.2.2. Let $v = [1,2,3]^T$ describe the set of all vectors which are orthogonal to v. Let $r = [x, y, z]^T$ be an arbitrary vector and consider the orthogonality condition:

 $0 = v \cdot r = [1, 2, 3][x, y, z]^T = x + 2y + 3z = 0.$

If you've studied 3 dimensional Cartesian geometry you should recognize this as the equation of a plane through the origin with normal vector < 1, 2, 3 >.

Proposition 6.2.3. Pythagorean Theorem in n-dimensions

If $x, y \in \mathbb{R}^{n \times 1}$ are orthogonal vectors then $||x||^2 + ||y||^2 = ||x+y||^2$. Moreover, if x_1, x_2, \ldots, x_k are orthogonal then

 $||x_1||^2 + ||x_2||^2 + \dots + ||x_k||^2 = ||x_1 + x_2 + \dots + x_k||^2$

Proof: Calculuate $||x + y||^2$ from the dot-product,

 $||x+y||^2 = (x+y) \cdot (x+y) = x \cdot x + x \cdot y + y \cdot x + y \cdot y = ||x||^2 + ||y||^2. \quad \Box$

Proposition 6.2.4.

The zero vector is orthogonal to all other vectors in $\mathbb{R}^{n \times 1}$.

Proof: let $x \in \mathbb{R}^{n \times 1}$ note 2(0) = 0 thus $0 \cdot x = 2(0) \cdot x = 2(0 \cdot x)$ which implies $0 \cdot x = 0$. \Box

Definition 6.2.5.

A set S of vectors in $\mathbb{R}^{n \times 1}$ is **orthogonal** iff every pair of vectors in the set is orthogonal. If S is orthogonal and all vectors in S have length one then we say S is **orthonormal**.

Example 6.2.6. Let u = [1, 1, 0], v = [1, -1, 0] and w = [0, 0, 1]. We calculate

$$u \cdot v = 0, \quad u \cdot w, \quad v \cdot w = 0$$

thus $S = \{u, v, w\}$ is an orthogonal set. However, it is not orthonormal since $||u|| = \sqrt{2}$. It is easy to create an orthonormal set, we just normalize the vectors; $T = \{\hat{u}, \hat{v}, \hat{w}\}$ meaning,

$$T = \left\{ \frac{1}{\sqrt{2}} [1, 1, 0], \frac{1}{\sqrt{2}} [1, -1, 0], [0, 0, 1] \right\}$$

Proposition 6.2.7. Extended Pythagorean Theorem in n-dimensions

If
$$x_1, x_2, \dots x_k$$
 are orthogonal then
 $||x_1||^2 + ||x_2||^2 + \dots + ||x_k||^2 = ||x_1 + x_2 + \dots + x_k||^2$

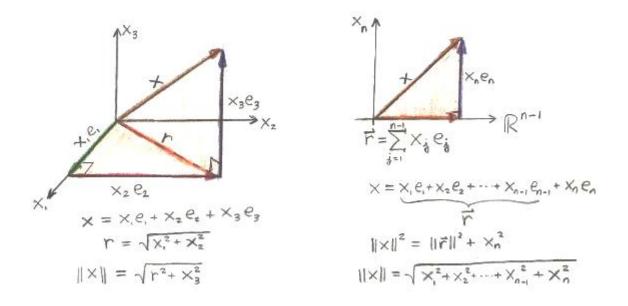
Proof: we can prove the second statement by applying the Pythagorean Theorem for two vectors repeatedly, starting with

$$||x_1 + (x_2 + \dots + x_k)||^2 = ||x_1||^2 + ||x_2 + \dots + x_k||^2$$

but then we can apply the Pythagorean Theorem to the rightmost term

$$||x_2 + (x_3 + \dots + x_k)||^2 = ||x_2||^2 + ||x_3 + \dots + x_k||^2.$$

Continuing in this fashion until we obtain the Pythagorean Theorem for k-orthogonal vectors. \Box



I have illustrated the proof above in the case of three dimensions and k-dimensions, however my k-dimensional diagram takes a little imagination. Another thing to think about: given $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n$ if e_i are orthonormal then $||v||^2 = v_1^2 + v_2^2 + \cdots + v_n^2$. Therefore, if we use a basis which is orthonormal then we obtain the standard formula for length of a vector with respect to the coordinates. If we were to use a basis of vectors which were not orthogonal or normalizes then the formula for the length of a vector in terms of the coordinates could look quite different.

Example 6.2.8. Use the basis $\{v_1 = [1,1]^T, v_2 = [2,0]^T\}$ for $\mathbb{R}^{2\times 1}$. Notice that $\{v_1, v_2\}$ is not orthogonal or normal. Given $x, y \in \mathbb{R}$ we wish to find $a, b \in \mathbb{R}$ such that $r = [x, y]^T = av_1 + bv_2$,

this amounts to the matrix calculation:

$$rref[v_1|v_2|r] = rref \left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & 0 & y \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 1 & \frac{1}{2}(x-y) \end{array} \right]$$

Thus a = y and $b = \frac{1}{2}(x - y)$. Let's check my answer,

$$av_1 + bv_2 = y[1,1]^T + \frac{1}{2}(x-y)[2,0]^T = [y+x-y,y+0]^T = [x,y]^T.$$

Furthermore, solving for x, y in terms of a, b yields x = 2b + a and y = a. Therefore, $||[x, y]^T||^2 = x^2 + y^2$ is modified to

$$||av_1 + bv_2||^2 = (2b+a)^2 + a^2 \neq ||av_1||^2 + ||bv_2||^2.$$

If we use a basis which is not orthonormal then we should take care not to assume formulas given for the standard basis equally well apply. However, if we trade the standard basis for a new basis which is orthogonal then we have less to worry about. The Pythagorean Theorem only applies in the orthogonal case. For two normalized, but possibly non-orthogonal, vectors we can replace the Pythagorean Theorem with a generalization of the Law of Cosines in $\mathbb{R}^{n \times 1}$.

$$||av_1 + bv_2||^2 = a^2 + b^2 + 2ab\cos\theta$$

where $v_1 \cdot v_2 = \cos \theta$. (I leave the proof to the reader)

Proposition 6.2.9.

If $S = \{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^{n \times 1}$ is an orthogonal set of nonzero vectors then S is linearly independent.

Proof: suppose $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

Take the dot-product of both sides with respect to $v_i \in S$,

$$c_1v_1 \cdot v_j + c_2v_2 \cdot v_j + \dots + c_kv_k \cdot v_j = 0 \cdot v_j = 0$$

Notice all terms in the sum above vanish by orthogonality except for one term and we are left with $c_j v_j \cdot v_j = 0$. However, $v_j \neq 0$ thus $v_j \cdot v_j \neq 0$ and it follows we can divide by the nonzero scalar $v_j \cdot v_j$ leaving $c_j = 0$. But j was arbitrary hence $c_1 = c_2 = \cdots = c_k = 0$ and hence S is linearly independent. \Box

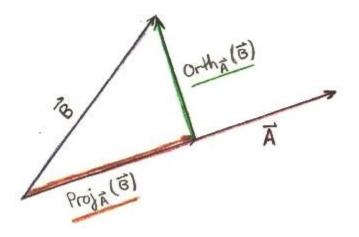
The converse of the proposition above is false. Given a linearly indepdent set of vectors it is not necessarily true that set is also orthogonal. However, we can modify any linearly independent set of vectors to obtain a linearly indepedent set. The procedure for this modification is known as the *Gram-Schmidt orthogonalization*. It is based on a generalization of the idea the vector projection from calculus III. Let me remind you: we found the projection operator to be a useful construction in calculus III. The projection operation allowed us to select the vector component of one vector that pointed in the direction of another given vector. We used this to find the distance from a point to a plane.

Definition 6.2.10.

Let $\vec{A} \neq 0, \vec{B}$ be vectors then we define

$$Proj_{\vec{A}}(\vec{B}) = (\vec{B} \cdot \hat{A})\hat{A}$$

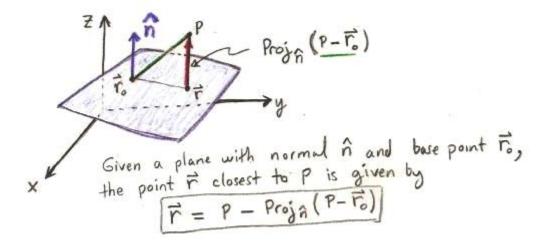
where $\hat{A} = \frac{1}{||A||}A$. Moreover, the length of $Proj_{\vec{A}}(\vec{B})$ is called the component of \vec{B} in the \vec{A} -direction and is denoted $Comp_{\vec{A}}(\vec{B}) = ||Proj_{\vec{A}}(\vec{B})||$. Finally, the **orthogonal complement** is defined by $Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B})$.



Example 6.2.11. Suppose $\vec{A} = <2,2,1>$ and $\vec{B} = <2,4,6>$ notice that we can also express the projection opertation by $\operatorname{Proj}_{\vec{A}}(\vec{B}) = (\vec{B} \cdot \hat{A})\hat{A} = \frac{1}{||\vec{A}||^2}(\vec{B} \cdot \vec{A})\vec{A}$ thus

$$Proj_{\vec{A}}(\vec{B}) = \frac{1}{9}(<2,4,6>\cdot<2,2,1>) < 2,2,1> = \frac{4+8+6}{9} < 2,2,1> = <4,4,2>$$

The length of the projection vector gives $\operatorname{Comp}_{\vec{A}}(\vec{B}) = \sqrt{16 + 16 + 4} = 6$. One application of this algebra is to calculate the distance from the plane 2x + 2y + z = 0 to the point (2, 4, 6). The "distance" from a plane to a point is defined to be the shortest distance. It's geometrically clear that the shortest path from the plane is found along the normal to the plane. If you draw a picture its not hard to see that $(2, 4, 6) - \operatorname{Proj}_{\vec{A}}(\vec{B}) = <2, 4, 6 > - <4, 4, 2 >= (-2, 0, 4)$ is the closest point to (2, 4, 6) that lies on the plane 2x + 2y + z = 0. Moreover the distance from the plane to the point is just 6.



Example 6.2.12. We studied $\vec{A} = < 2, 2, 1 > and \vec{B} = < 2, 4, 6 > in the preceding example. We found that notice that <math>\operatorname{Proj}_{\vec{A}}(\vec{B}) = < 4, 4, 2 >$. The projection of \vec{B} onto \vec{A} is the part of \vec{B} which points in the direction of \vec{A} . It stands to reason that if we subtract away the projection then we will be left with the part of \vec{B} which does not point in the direction of \vec{A} , it should be orthogonal.

$$Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B}) = <2,4,6> - <4,4,2> = <-2,0,4>$$

Let's verify $Orth_{\vec{A}}(\vec{B})$ is indeed orthogonal to \vec{A} ,

$$Orth_{\vec{A}}(\vec{B}) \cdot \vec{A} = <-2, 0, 4 > \cdot < 2, 2, 1 > = -4 + 4 = 0.$$

Notice that the projection operator has given us the following orthogonal decomposition of \vec{B} :

$$<2,4,6>=\vec{B}=Proj_{\vec{A}}(\vec{B})+Orth_{\vec{A}}(\vec{B})=<4,4,2>+<-2,0,4>$$

If \vec{A}, \vec{B} are any two nonzero vectors it is probably clear that we can perform the decomposition outlined in the example above. It would not be hard to show that if $S = \{\vec{A}, \vec{B}\}$ is linearly independent then $S' = \{\vec{A}, Orth_{\vec{A}}(\vec{B})\}$ is an orthogonal set, moreover they have the same span. This is a partial answer to the converse of Proposition 6.2.9. But, what if we had three vectors instead of two? How would we orthogonalize a set of three linearly independent vectors?

Remark 6.2.13.

I hope you can forgive me for reverting to calculus III notation in the last page or two. It should be clear enough to the reader that the orthogonalization and projection operations can be implemented on either rows or columns. I return to our usual custom of thinking primarily about column vectors at this point. We've already seen the definition from Calculus III, now we turn to the *n*-dimensional case in matrix notation.

Definition 6.2.14.

Suppose $a \neq 0 \in \mathbb{R}^{n \times 1}$, define the **projection of** b **onto** a to be the mapping $Proj_a$: $\mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ such that $Proj_a(b) = \frac{1}{a^T a}(a^T b)a$. Moreover, we define $Orth_a : \mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ by $Orth_a(b) = b - Proj_a(b) = b - \frac{1}{a^T a}(a^T b)a$ for all $b \in \mathbb{R}^{n \times 1}$.

Proposition 6.2.15.

- If $a \neq 0 \in \mathbb{R}^{n \times 1}$ then $Proj_a$ and $Orth_a$ are linear transformations.
 - 1. $Orth_a(b) \cdot a = 0$ for all $b \in \mathbb{R}^{n \times 1}$,
 - 2. $Orth_a(b) \cdot Proj_a(y) = 0$ for all $b, y \in \mathbb{R}^{n \times 1}$,
 - 3. the projection is idempotent; $Proj_a \circ Proj_a = Proj_a$.

I leave the proof of linearity as an exercise. Begin with (1.): let $a \neq 0 \in \mathbb{R}^{n \times 1}$ and let $b \in \mathbb{R}^{n \times 1}$,

$$a \cdot Orth_a(b) = a^T (b - \frac{1}{a^T a} (a^T b)a)$$
$$= a^T b - a^T (\frac{1}{a^T a} (a^T b)a)$$
$$= a^T b - \frac{1}{a^T a} (a^T b) a^T a$$
$$= a^T b - a^T b = 0.$$

notice I used the fact that $a^T b, a^T a$ were scalars to commute the a^T to the end of the expression. Notice that (2.) follows since $Proj_a(y) = ka$ for some constant k. Next, let $b \in \mathbb{R}^{n \times 1}$ and consider:

$$(Proj_a \circ Proj_a)(b) = Proj_a(Proj_a(b))$$
$$= Proj_a(\frac{1}{a^T a}(a^T b)a)$$
$$= \frac{1}{a^T a}(a^T[\frac{1}{a^T a}(a^T b)a])a$$
$$= \frac{1}{a^T a}(\frac{a^T b}{a^T a}a^T a)a$$
$$= \frac{1}{a^T a}(a^T b)a$$
$$= Proj_a(b)$$

since the above holds for all $b \in \mathbb{R}^{n \times 1}$ we find $Proj_a \circ Proj_a = Proj_a$. This can also be denoted $Proj_a^2 = Proj_a$. \Box

Proposition 6.2.16.

If $S = \{a, b, c\}$ be a linearly independent set of vectors in $\mathbb{R}^{n \times 1}$ then $S' = \{a', b', c'\}$ is an orthogonal set of vectors in $\mathbb{R}^{n \times 1}$ if we define a', b', c' as follows:

$$a' = a,$$
 $b' = Orth_{a'}(b),$ $c' = Orth_{a'}(Orth_{b'}(c))$

Proof: to prove S' orthogonal we must show that $a' \cdot b' = 0$, $a' \cdot c' = 0$ and $b' \cdot c' = 0$. We already proved $a' \cdot b' = 0$ in the Proposition 6.2.15. Likewise, $a' \cdot c' = 0$ since $Orth_{a'}(x)$ is orthogonal to a' for any x. Consider:

$$b' \cdot c' = b' \cdot Orth_{a'}(Orth_{b'}(c))$$

= $b' \cdot \left[Orth_{b'}(c) - Proj_{a'}(Orth_{b'}(c)) \right]$
= $b' \cdot Orth_{b'}(c) - Orth_{a}(b) \cdot Proj_{a}(Orth_{b'}(c))$
= 0

Where we again used (1.) and (2.) of Proposition 6.2.15 in the critical last step. The logic of the formulas is very natural. To construct b' we simply remove the part of b which points in the direction of a'. Then to construct c' we first remove the part of c in the b' direction and then the part in the a' direction. This means no part of c' will point in the a' or b' directions. In principle, one might worry we would subtract away so much that nothing is left, but the linear independence of the vectors insures that is not possible. If it were that would imply a linear dependence of the original set of vectors. \Box

For convenience let me work out the formulas we just discovered in terms of an explicit formula with dot-products. We can also perform the same process for a set of 4 or 5 or more vectors. I'll state the process for arbitrary order, you'll forgive me if I skip the proof this time. There is a careful proof on page 379 of Spence, Insel and Friedberg. The connection between my *Orth* operator approach and the formulas in the proposition that follows is *just algebra*:

$$\begin{split} v_3' &= Orth_{v_1'}(Orth_{v_2'}(v_3)) \\ &= Orth_{v_2'}(v_3) - Proj_{v_1'}(Orth_{v_2'}(v_3)) \\ &= v_3 - Proj_{v_2'}(v_3) - Proj_{v_1'}(v_3 - Proj_{v_2'}(v_3)) \\ &= v_3 - Proj_{v_2'}(v_3) - Proj_{v_1'}(v_3) - Proj_{v_1'}(Proj_{v_2'}(v_3)) \\ &= v_3 - \frac{v_3 \cdot v_2'}{v_2' \cdot v_2'}v_2' - \frac{v_3 \cdot v_1'}{v_1' \cdot v_1'}v_1' \end{split}$$

The last term vanished because $v_1' \cdot v_2' = 0$ and the projections are just scalar multiples of those vectors.

Proposition 6.2.17. The Gram-Schmidt Process

If $S = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^{n \times 1}$ then $S' = \{v'_1, v'_2, \dots, v'_k\}$ is an orthogonal set of vectors in $\mathbb{R}^{n \times 1}$ if we define v'_i as follows: $v'_1 = v_1$ $v'_2 = v_2 - \frac{v_2 \cdot v'_1}{v'_1 \cdot v'_1} v'_1$ $v'_3 = v_3 - \frac{v_3 \cdot v'_2}{v'_2 \cdot v'_2} v'_2 - \frac{v_3 \cdot v'_1}{v'_1 \cdot v'_1} v'_1$ $v'_k = v_k - \frac{v_k \cdot v'_{k-1}}{v'_{k-1} \cdot v'_{k-1}} v'_{k-1} - \frac{v_k \cdot v'_{k-2}}{v'_{k-2} \cdot v'_{k-2}} v'_{k-2} - \dots - \frac{v_k \cdot v'_1}{v'_1 \cdot v'_1} v'_1.$

Example 6.2.18. Suppose $v_1 = [1, 0, 0, 0]^T$, $v_2 = [3, 1, 0, 0]^T$, $v_3 = [3, 2, 0, 3]^T$. Let's use the Gram-Schmidt Process to orthogonalize these vectors: let $v'_1 = v_1 = [1, 0, 0, 0]^T$ and calculate:

$$v'_{2} = v_{2} - \frac{v_{2} \cdot v_{1}}{v_{1} \cdot v_{1}} v_{1} = [3, 1, 0, 0]^{T} - 3[1, 0, 0, 0]^{T} = [0, 1, 0, 0]^{T}.$$

Next,

$$v_3' = v_3 - \frac{v_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' - \frac{v_3 \cdot v_1'}{v_1' \cdot v_1'} v_1' = [3, 2, 0, 3]^T - 2[0, 1, 0, 0] - 3[1, 0, 0, 0]^T = [0, 0, 0, 3]^T$$

We find the orthogonal set of vectors $\{e_1, e_2, e_4\}$. It just so happens this is also an orthonormal set of vectors.

Proposition 6.2.19. Normalization

If $S' = \{v'_1, v'_2, \dots, v'_k\}$ is an orthogonal subset of $\mathbb{R}^{n \times 1}$ then $S'' = \{v''_1, v''_2, \dots, v''_k\}$ is an orthonormal set if we define $v''_i = \widehat{v'_i} = \frac{1}{||v'_i||}v'_i$ for each $i = 1, 2, \dots, k$.

Example 6.2.20. Suppose $v_1 = [1, 1, 1]^T$, $v_2 = [1, 2, 3]^T$, $v_3 = [0, 0, 3]^T$ find an orthonormal set of vectors that spans $span\{v_1, v_2, v_3\}$. We can use Gram-Schmidt followed by a normalization, let $v'_1 = [1, 1, 1]^T$ then calculate

$$v'_2 = [1, 2, 3]^T - \left(\frac{1+2+3}{3}\right)[1, 1, 1]^T = [1, 2, 3]^T - [2, 2, 2]^T = [-1, 0, 1]^T.$$

as a quick check on my arthimetic note $v'_1 \cdot v'_2 = 0$ (good). Next,

$$v'_{3} = [0,0,3]^{T} - \left(\frac{0(-1) + 0(0) + 3(1)}{2}\right)[-1,0,1]^{T} - \left(\frac{0(1) + 0(1) + 3(1)}{3}\right)[1,1,1]^{T}$$

$$\Rightarrow v'_3 = [0, 0, 3]^T + [\frac{3}{2}, 0, -\frac{3}{2}]^T - [1, 1, 1]^T = [\frac{1}{2}, -1, \frac{1}{2}]^T$$

again it's good to check that $v'_2 \cdot v'_3 = 0$ and $v'_1 \cdot v'_3 = 0$ as we desire. Finally, note that $||v'_1|| = \sqrt{3}$, $||v'_2|| = \sqrt{2}$ and $||v'_3|| = \sqrt{3/2}$ hence

$$v_1'' = \frac{1}{\sqrt{3}}[1,1,1]^T, \quad v_2'' = \frac{1}{\sqrt{2}}[-1,0,1]^T, \quad v_3'' = \sqrt{\frac{2}{3}}[\frac{1}{2},-1,\frac{1}{2}]^T$$

are orthonormal vectors.

Definition 6.2.21.

A basis for a subspace W of $\mathbb{R}^{n \times 1}$ is an **orthogonal** basis for W iff it is an orthogonal set of vectors which is a basis for W. Likewise, an **orthonormal** basis for W is a basis which is **orthonormal**.

Proposition 6.2.22. Existence of Orthonormal Basis

If $W \leq \mathbb{R}^{n \times 1}$ then there exists an orthonormal basis of W

Proof: since W is a subspace it has a basis. Apply Gram-Schmidt to that basis then normalize the vectors to obtain an orthnormal basis. \Box

Example 6.2.23. Let $W = span\{[1,0,0,0]^T, [3,1,0,0]^T, [3,2,0,3]^T\}$. Find an orthonormal basis for $W \leq \mathbb{R}^{4\times 1}$. Recall from Example 6.2.18 we applied Gram-Schmidt and found the orthonormal set of vectors $\{e_1, e_2, e_4\}$. That is an orthonormal basis for W.

Example 6.2.24. In Example 6.2.20 we found $\{v_1'', v_2'', v_3''\}$ is an orthonormal set of vectors. Since orthogonality implies linear independence it follows that this set is in fact a basis for $\mathbb{R}^{3\times 1}$. It is an **orthonormal basis**. Of course there are other bases which are orthogonal. For example, the standard basis is orthonormal.

Example 6.2.25. Let us define $S = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^{4 \times 1}$ as follows:

$v_1 =$, $v_2 =$	1 1 1 1	, $v_3 =$	$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $v_4 =$	$\begin{bmatrix} 3\\2\\0\\3 \end{bmatrix}$
	_ 1 _		[1]		[3]		[3]

It is easy to verify that S defined below is a linearly independent set vectors basis for $span(S) \leq \mathbb{R}^{4\times 1}$. Let's see how to find an orthonormal basis for span(S). The procedure is simple: apply the

Gram-Schmidt algorithm then normalize the vectors.

$$\begin{aligned} v_1' &= v_1 = \begin{bmatrix} 1\\0\\1\\1 \end{bmatrix} \\ v_2' &= v_2 - \left(\frac{v_2 \cdot v_1'}{v_1' \cdot v_1'}\right) v_1' = \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1\\0\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} \\ - \frac{0}{1} \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 1\\0\\1\\1\\1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -5\\0\\1\\4 \end{bmatrix} \\ v_3' &= v_4 - \left(\frac{v_4 \cdot v_3'}{v_3' \cdot v_3'}\right) v_2' - \left(\frac{v_3 \cdot v_1'}{v_1' \cdot v_1'}\right) v_1' = \begin{bmatrix} 3\\2\\0\\3 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} -5\\0\\1\\4 \end{bmatrix} - \begin{bmatrix} 2\\0\\0\\0 \end{bmatrix} - \begin{bmatrix} 2\\0\\0\\0 \end{bmatrix} - \begin{bmatrix} 2\\0\\0\\0 \end{bmatrix} - \begin{bmatrix} 2\\0\\2\\2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 9\\0\\-27\\18 \end{bmatrix} \end{aligned}$$

Then normalize to obtain the orthonormal basis for Span(S) below:

$$\beta = \frac{1}{\sqrt{3}} \begin{bmatrix} 1\\0\\1\\1\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} -5\\0\\1\\4\\4 \end{bmatrix}, \frac{1}{\frac{1}{9\sqrt{14}}} \begin{bmatrix} 9\\0\\-27\\18\\\end{bmatrix} \right\}$$

Proposition 6.2.26. Coordinates with respect to an Orthonormal Basis

If $W \leq \mathbb{R}^{n \times 1}$ has an orthonormal basis $\{v_1, v_2, \dots, v_k\}$ and if $w = \sum_{i=1}^k w_i v_i$ then $w_i = w \cdot v_i$ for all $i = 1, 2, \dots, k$. In other words, each vector $w \in W$ may be expressed as $w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \dots + (w \cdots v_k)v_k$

Proof: Let $w = w_1v_1 + w_2v_2 + \cdots + w_kv_k$ and take the dot-product with v_i ,

$$w \cdot v_j = (w_1v_1 + w_2v_2 + \dots + w_kv_k) \cdot v_j = w_1(v_1 \cdot v_j) + w_2(v_2 \cdot v_j) + \dots + w_k(v_k \cdot v_j)$$

Orthonormality of the basis is compactly expressed by the Kronecker Delta; $v_i \cdot v_j = \delta_{ij}$ this is zero if $i \neq j$ and it is 1 if they are equal. The whole sum collapses except for the *j*-th term which yields: $w \cdot v_j = w_j$. But, *j* was arbitrary hence the proposition follows. \Box .

The proposition above reveals the real reason we like to work with orthonormal coordinates. It's easy to figure out the coordinates, we simply take dot-products. This technique was employed with great success in (you guessed it) Calculus III. The standard $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthonormal basis and one of the first things we discuss is that if $\vec{v} = \langle A, B, C \rangle$ then $A = \vec{v} \cdot \hat{i}, B = \vec{v} \cdot \hat{j}$ and $C = \vec{v} \cdot \hat{k}$.

Example 6.2.27. For the record, the standard basis of $\mathbb{R}^{n \times 1}$ is an orthonormal basis and

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + \dots + (v \cdot e_n)e_n$$

for any vector v in $\mathbb{R}^{n \times 1}$.

Example 6.2.28. Let v = [1, 2, 3, 4]. Find the coordinates of v with respect to the orthonormal basis β found in Example 6.2.25.

$$\beta = \{f_1, f_2, f_3, f_4\} = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1\\0\\1\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} -5\\0\\1\\4 \end{bmatrix}, \frac{1}{9\sqrt{14}} \begin{bmatrix} 9\\0\\-27\\18 \end{bmatrix} \right\}$$

Let us denote the coordinates vector $[v]_{\beta} = [w_1, w_2, w_3, w_4]$ we know we can calculate these by taking the dot-products with the vectors in the orthonormal basis β :

$$w_1 = v \cdot f_1 = \frac{1}{\sqrt{3}} [1, 2, 3, 4] [1, 0, 1, 1]^T = \frac{8}{\sqrt{3}}$$

 $w_2 = v \cdot f_2 = [1, 2, 3, 4][0, 1, 0, 0]^T = 2$

$$w_3 = v \cdot f_3 = \frac{1}{\sqrt{42}} [1, 2, 3, 4] [-5, 0, 1, 4]^T = \frac{14}{\sqrt{42}}$$

$$w_4 = v \cdot f_4 = \frac{1}{9\sqrt{14}} [1, 2, 3, 4] [9, 0, -27, 18]^T = \frac{0}{9\sqrt{14}} = 0$$

Therefore, $[v]_{\beta} = [\frac{8}{\sqrt{3}}, 2, \frac{14}{\sqrt{42}}, 0]$. Now, let's check our answer. What should this mean if it is

correct? We should be able verify $v = w_1f_1 + w_2f_2 + w_3f_3 + w_4f_4$:

$$w_{1}f_{1} + w_{2}f_{2} + w_{3}f_{3} + w_{4}f_{4} = \frac{8}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1\\0\\1\\1\\1 \end{bmatrix} + 2\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} + \frac{14}{\sqrt{42}} \frac{1}{\sqrt{42}} \begin{bmatrix} -5\\0\\1\\4 \end{bmatrix}$$
$$= \frac{8}{3} \begin{bmatrix} 1\\0\\1\\1\\1 \end{bmatrix} + 2\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -5\\0\\1\\4 \end{bmatrix}$$
$$= \begin{bmatrix} 8/3 - 5/3\\2\\8/3 + 1/3\\8/3 + 4/3 \end{bmatrix}$$
$$= \begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}$$

Well, that's a relief.

6.3 orthogonal complements and projections

Upto now we have discussed projections with respect to one vector at a time, however we can just as well discuss the projection onto some subspace of $\mathbb{R}^{n \times 1}$. We need a few definitions to clarify and motivate the projection.

Definition 6.3.1.

Suppose $W_1, W_2 \subseteq \mathbb{R}^{n \times 1}$ then we say W_1 is **orthogonal** to W_2 iff $w_1 \cdot w_2 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. We denote orthogonality by writing $W_1 \perp W_2$.

Example 6.3.2. Let $W_1 = span\{e_1, e_2\}$ and $W_2 = span\{e_3\}$ then $W_1, W_2 \leq \mathbb{R}^{n \times 1}$. Let $w_1 = ae_1 + be_2 \in W_1$ and $w_2 = ce_3 \in W_2$ calculate,

$$w_1 \cdot w_2 = (ae_1 + be_2) \cdot (ce_3) = ace_1 \cdot e_3 + bce_2 \cdot e_3 = 0$$

Hence $W_1 \perp W_2$. Geometrically, we have shown the xy-plane is orthogonal to the z-axis.

We notice that orthogonality relative to the basis will naturally extend to the span of the basis since the dot-product has nice linearity properties.

Proposition 6.3.3.

Suppose $W_1, W_2 \leq \mathbb{R}^{n \times 1}$ the subspace W_1 is **orthogonal** to the subspace W_2 iff $w_i \cdot v_j = 0$ for all i, j relative to a pair of bases $\{w_i\}$ for W_1 and $\{v_j\}$ for W_2 .

Proof: Suppose $\{w_i\}_{i=1}^r$ is a basis for $W_1 \leq \mathbb{R}^{n \times 1}$ and $\{v_j\}_{j=1}^s$ for $W_2 \leq \mathbb{R}^{n \times 1}$. If $W_1 \perp W_2$ then clearly $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$. Conversely, suppose $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$ then let $x \in W_1$ and $y \in W_2$:

$$x \cdot y = \left(\sum_{i=1}^r x_i w_i\right) \cdot \left(\sum_{i=1}^s y_j w_j\right) = \sum_{i=1}^r \sum_{j=1}^s x_i y_j (w_i \cdot v_j) = 0. \qquad \Box$$

Given a subspace W which lives in $\mathbb{R}^{n \times 1}$ we might wonder what is the largest subspace which is orthogonal to W? In $\mathbb{R}^{3 \times 1}$ it is clear that the xy-plane is the largest subspace which is orthogonal to the z-axis, however, if the xy-plane was viewed as a subset of $\mathbb{R}^{4 \times 1}$ we could actually find a volume which was orthogonal to the z-axis (in particular $span\{e_1, e_2, e_4\} \perp span\{e_3\}$).

Definition 6.3.4.

Let
$$W \subseteq \mathbb{R}^{n \times 1}$$
 then W^{\perp} is defined as follows:
$$W^{\perp} = \{ v \in \mathbb{R}^{n \times 1} | v \cdot w = 0 \text{ for all } w \in W \}$$

It is clear that W^{\perp} is the largest subset in $\mathbb{R}^{n \times 1}$ which is orthogonal to W. Better than just that, it's the largest subspace orthogonal to W.

Proposition 6.3.5.

Let $S \subset \mathbb{R}^{n \times 1}$ then $S^{\perp} \leq \mathbb{R}^{n \times 1}$.

Proof: Let $x, y \in S^{\perp}$ and let $c \in \mathbb{R}$. Furthermore, suppose $s \in S$ and note

$$(x + cy) \cdot s = x \cdot s + c(y \cdot s) = 0 + c(0) = 0$$

Thus an aribitrary linear combination of elements of S^{\perp} are again in S^{\perp} which is nonempty as $0 \in S^{\perp}$ hence by the subspace test $S^{\perp} \leq \mathbb{R}^{n \times 1}$. It is interesting that S need not be a subspace for this argument to hold. \Box

Example 6.3.6. Find the orthogonal complement to $W = span\{v_1 = [1, 1, 0, 0]^T, v_2 = [0, 1, 0, 2]^T\}$. Let's treat this as a matrix problem. We wish to describe a typical vector in W^{\perp} . Towards that goal, let $r = [x, y, z, w]^T \in W^{\perp}$ then the conditions that r must satisfy are $v_1 \cdot r = v_1^T r = 0$ and $v_2 \cdot r = v_2^T r = 0$. But this is equivalent to the single matrix equation below:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies r = \begin{bmatrix} 2w \\ -2w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $W^{\perp} = span\{[0, 0, 1, 0]^T, [2, -2, 0, 1]^T\}.$

If you study the preceding example it becomes clear that finding the orthogonal complement of a set of vectors is equivalent to calculating the null space of a particular matrix. We have considerable experience in such calculations so this is a welcome observation.

Proposition 6.3.7.

If
$$S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^{n \times 1}$$
 and $A = [v_1|v_2|\cdots|v_k]$ then $S^{\perp} = Null(A^T)$

Proof: Denote $A = [v_1|v_2|\cdots|v_k] \in \mathbb{R}^{n \times k}$ and $x = [x_1, x_2, \dots, x_k]^T$. Observe that:

$$\begin{aligned} x \in Null(A^T) \Leftrightarrow A^T x &= 0 \\ \Leftrightarrow row_1(A^T)x + row_2(A^T)x + \dots + row_k(A^T)x &= 0 \\ \Leftrightarrow (col_1(A))^T x + (col_2(A))^T x + \dots + (col_k(A))^T x &= 0 \\ \Leftrightarrow v_1^T x \cdot + v_2^T x + \dots + v_k^T x &= 0 \\ \Leftrightarrow v_1 \cdot x + v_2 \cdot x + \dots + v_k \cdot x &= 0 \\ \Leftrightarrow x \in S^{\perp} \end{aligned}$$

Therefore, $Null(A^T) = S^{\perp}$. \Box

Given the correspondence above we should be interested in statements which can be made about the row and column space of a matrix. It turns out there are two simple statements to be made in general:

Proposition 6.3.8.

Let $A \in \mathbb{R}^{m \times n}$ then 1. $Null(A^T) \perp Col(A)$. 2. $Null(A) \perp Row(A)$.

Proof: Let $S = \{col_1(A), col_2(A), \dots, col_n(A)\}$ and use Proposition 6.3.7 to deduce $S^{\perp} = Null(A^T)$. Therefore, each column of A is orthogonal to all vectors in $Null(A^T)$, in particular each column is orthogonal to the basis for $Null(A^T)$. Since the pivot columns are a basis for Col(A) we can use Proposition 6.3.3 to conclude $Null(A^T) \perp Col(A)$.

To prove of (2.) apply (1.) to $B = A^T$ to deduce $Null(B^T) \perp Col(B)$. Hence, $Null((A^T)^T) \perp Col(A^T)$ and we find $Null(A) \perp Col(A^T)$. But, $Col(A^T) = Row(A)$ thus $Null(A) \perp Row(A)$. \Box

The proof above makes ample use of previous work. I encourage the reader to try to prove this proposition from scratch. I don't think it's that hard and you might learn something. Just take an arbitrary element of each subspace and argue why the dot-product is zero.

Proposition 6.3.9.

Let $W_1, W_2 \leq \mathbb{R}^{n \times 1}$, if $W_1 \perp W_2$ then $W_1 \cap W_2 = \{0\}$

Proof: let $z \in W_1 \cap W_2$ then $z \in W_1$ and $z \in W_2$ and since $W_1 \perp W_2$ it follows $z \cdot z = 0$ hence z = 0 and $W_1 \cap W_2 \subseteq \{0\}$. The reverse inclusion $\{0\} \subseteq W_1 \cap W_2$ is clearly true since 0 is in every subspace. Therefore, $W_1 \cap W_2 = \{0\} \square$

Definition 6.3.10.

Let V be a vector space and $W_1, W_2 \leq V$. If every $v \in V$ can be written as $v = w_1 + w_2$ for a unique pair of $w_1 \in W_1$ and $w_2 \in W_2$ then we say that V is the **direct sum** of W_1 and W_2 . Moreover, we denote the statement "V is a direct sum of W_1 and W_2 " by $V = W_1 \oplus W_2$.

Proposition 6.3.11.

Let $W \leq \mathbb{R}^{n \times 1}$ then

- 1. $\mathbb{R}^{n \times 1} = W \oplus W^{\perp}$.
- 2. $dim(W) + dim(W^{\perp}) = n$,
- 3. $(W^{\perp})^{\perp} = W$,

Proof: Let $W \leq \mathbb{R}^{n \times 1}$ and choose an orthonormal basis $\beta = \{v_1, v_2, \dots, v_k\}$ for S. Let $z \in \mathbb{R}^{n \times 1}$ and define

$$Proj_W(z) = \sum_{i=1}^{\kappa} (z \cdot v_i) v_i$$
 and $Orth_W(z) = z - Proj_W(z).$

Observe that $z = Proj_W(z) + Orth_W(z)$ and clearly $Proj_W(z) \in S$. We now seek to argue that $Orth_W(z) \in S^{\perp}$. Let $v_j \in \beta$ then

$$v_{j} \cdot Orth_{W}(z) = v_{j} \cdot (z - Proj_{W}(z))$$
$$= v_{j} \cdot z - v_{j} \cdot \left(\sum_{i=1}^{k} (z \cdot v_{i})v_{i}\right)$$
$$= v_{j} \cdot z - \sum_{i=1}^{k} (z \cdot v_{i})(v_{j} \cdot v_{i})$$
$$= v_{j} \cdot z - \sum_{i=1}^{k} (z \cdot v_{i})\delta_{ij}$$
$$= v_{j} \cdot z - z \cdot v_{j}$$
$$= 0$$

Therefore, $\mathbb{R}^{n\times 1} = W \oplus W^{\perp}$. To prove (2.) notice we know by Proposition 6.3.5 that $W^{\perp} \leq \mathbb{R}^{n\times 1}$ and consequently there exists an orthonormal basis $\Gamma = \{w_1, w_2, \ldots, w_l\}$ for W^{\perp} . Furthermore, by Proposition 6.3.9 we find $\beta \cap \Gamma = \emptyset$ since 0 is not in either basis. We argue that $\beta \cup \Gamma$ is a basis for $\mathbb{R}^{n\times 1}$. Observe that $\beta \cup \Gamma$ clearly spans $\mathbb{R}^{n\times 1}$ since $z = \operatorname{Proj}_W(z) + \operatorname{Orth}_W(z)$ for each $z \in \mathbb{R}^{n\times 1}$ and $\operatorname{Proj}_W(z) \in \operatorname{span}(\beta)$ while $\operatorname{Orth}_W(z) \in \operatorname{span}(\Gamma)$. Furthermore, I argue that $\beta \cup \Gamma$ is an orthonormal set. By construction β and Γ are orthonormal, so all we need prove is that the dot-product of vectors from β and Γ is zero, but that is immediate from the construction of Γ . We learned in Proposition 6.2.9 that orthogonality for set of nonzero vectors implies linearly independence. Hence, $\beta \cup \Gamma$ is a linearly independent spanning set for $\mathbb{R}^{n\times 1}$. By the dimension theorem we deduce that there must be *n*-vectors in $\beta \cup \Gamma$ since it must have the same number of vectors as any other basis for $\mathbb{R}^{n\times 1}$ (the standard basis obviously has *n*-vectors). Therefore,

$$\dim(W) + \dim(W^{\perp}) = n.$$

in particular, we count $dim(W^{\perp}) = n - k$ in my current notation. Now turn to ponder the proof of (3.). Let $z \in (W^{\perp})^{\perp}$ and expand z in the basis $\beta \cup \Gamma$ to gain further insight, $z = z_1v_1 + z_2v_2 + \cdots + z_kv_k + z_{k+1}w_1 + z_{k+2}w_2 + \cdots + z_nw_{n-k}$. Since $z \in (W^{\perp})^{\perp}$ then $z \cdot w_{\perp} = 0$ for all $w_{\perp} \in W^{\perp}$, in particular $z \cdot w_j = 0$ for all $j = 1, 2, \ldots, n-k$. But, this implies $z_{k+1} = z_{k+2} = \cdots = z_n = 0$ since Proposition 6.2.26 showed the coordinates w.r.t. an orthonormal basis are given by dot-products. Therefore, $z \in span(\beta) = W$ and we have shown $(W^{\perp})^{\perp} \subseteq W$. In invite the reader to prove the reverse inclusion to complete this proof. \Box

Two items I defined for the purposes of the proof above have application far beyond the proof. Let's state them again for future reference. I give two equivalent definitions, technically we should prove that the second basis dependent statement follows from the first basis-independent statement. Primary definitions are as a point of mathematical elegance stated in a coordinate free langauge in as much as possible, however the second statement is far more useful.

Definition 6.3.12.

Let $W \leq \mathbb{R}^{n \times 1}$ if $z \in \mathbb{R}^{n \times 1}$ and z = u + w for some $u \in W$ and $w \in W^{\perp}$ then we define $u = Proj_W(z)$ and $w = Orth_W(z)$. Equivalently, choose an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_k\}$ for W then if $z \in \mathbb{R}^{n \times 1}$ we define

 $Proj_W(z) = \sum_{i=1}^k (z \cdot v_i)v_i \quad \text{and} \quad Orth_W(z) = z - Proj_W(z).$

Example 6.3.13. Let $W = span\{e_1+e_2, e_3\}$ and $x = [1, 2, 3]^T$ calculate $Proj_W(x)$. To begin I note that the given spanning set is orthogonal and hence linear indpendent. We need only orthonormalize to obtain an orthonormal basis β for W

$$\beta = \{v_1, v_2\}$$
 with $v_1 = \frac{1}{\sqrt{2}}[1, 1, 0]^T, v_2 = [0, 0, 1]^T$

Calculate, $v_1 \cdot x = \frac{3}{\sqrt{2}}$ and $v_2 \cdot x = 3$. Thus,

$$Proj_W([1,2,3]^T) = (v_1 \cdot x)v_1 + (v_2 \cdot x)v_2 = \frac{3}{\sqrt{2}}v_1 + 3v_2 = [\frac{3}{2}, \frac{3}{2}, 3]^T$$

Then it's easy to calculate the orthogonal part,

$$Orth_W([1,2,3]^T) = [1,2,3]^T - [\frac{3}{2},\frac{3}{2},3]^T = [-\frac{1}{2},\frac{1}{2},0]^T$$

As a check on the calculation note that $Proj_W(x) + Orth_W(x) = x$ and $Proj_W(x) \cdot Orth_W(x) = 0$. Example 6.3.14. Let $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^{4 \times 1}$ where

$$u_1 = \begin{bmatrix} 2\\1\\2\\0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0\\-2\\1\\1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1\\2\\0\\-1 \end{bmatrix}$$

calculate $Proj_W([0, 6, 0, 6]^T)^2$. Notice that the given spanning set appears to be linearly independent but it is not orthogonal. Apply Gram-Schmidt to fix it:

$$\begin{aligned} v_1 &= u_1 = [2, 1, 2, 0]^T \\ v_2 &= u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = u_2 = [0, -2, 1, 1]^T \\ v_3 &= u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = u_3 + \frac{5}{6} v_2 = [-1, 2, 0, -1]^T + [0, -\frac{10}{6}, \frac{5}{6}, \frac{5}{6}]^T \end{aligned}$$

We calculate,

$$v_3 = \begin{bmatrix} -1, \ 2 - \frac{5}{3}, \ \frac{5}{6}, \ -1 + \frac{5}{6} \end{bmatrix}^T = \begin{bmatrix} -1, \ \frac{1}{3}, \ \frac{5}{6}, \ -\frac{1}{6} \end{bmatrix}^T = \frac{1}{6} \begin{bmatrix} -6, 2, 5, -1 \end{bmatrix}^T$$

The normalized basis follows easily,

$$v_1' = \frac{1}{3}[2, 1, 2, 0]^T$$
 $v_2' = \frac{1}{\sqrt{6}}[0, -2, 1, 1]^T$ $v_3' = \frac{1}{\sqrt{66}}[-6, 2, 5, -1]^T$

Calculate dot-products in preparation for the projection calculation,

$$v'_1 \cdot x = \frac{1}{3} [2, 1, 2, 0] [0, 6, 0, 6]^T = 2$$
$$v'_2 \cdot x = \frac{1}{\sqrt{6}} [0, -2, 1, 1] [0, 6, 0, 6]^T = \frac{1}{\sqrt{6}} (-12 + 6) = -\sqrt{6}$$
$$v'_3 \cdot x = \frac{1}{\sqrt{66}} [-6, 2, 5, -1] [0, 6, 0, 6]^T = \frac{1}{\sqrt{66}} (12 - 6) = \frac{6}{\sqrt{66}}$$

Now we calculate the projection of $x = [0, 6, 0, 6]^T$ onto W with ease:

$$\begin{aligned} Proj_W(x) &= (x \cdot v_1')v_1' + (x \cdot v_2')v_2' + (x \cdot v_3')v_3' \\ &= (2)\frac{1}{3}[2, 1, 2, 0]^T - (\sqrt{6})\frac{1}{\sqrt{6}}[0, -2, 1, 1]^T + (\frac{6}{\sqrt{66}})\frac{1}{\sqrt{66}}[-6, 2, 5, -1]^T \\ &= [\frac{4}{3}, \frac{2}{3}, \frac{4}{3}, 0]^T + [0, 2, -1, -1]^T + [\frac{-6}{11}, \frac{2}{11}, \frac{5}{11}, \frac{-1}{11}]^T \\ &= [\frac{26}{33}, \frac{94}{33}, \frac{26}{33}, \frac{-36}{33}]^T \end{aligned}$$

and,

$$Orth_W(x) = \begin{bmatrix} -26\\ 33 \end{bmatrix}, \ \frac{104}{33}, \ \frac{-26}{33}, \ \frac{234}{33} \end{bmatrix}^T$$

²this problem is inspired from Anton & Rorres' §6.4 homework problem 3 part d.

6.4 the closest vector problem

Suppose we are given a subspace and a vector not in the subspace, which vector in the subspace is closest to the external vector ? Naturally the projection answers this question. The projection of the external vector onto the subspace will be closest. Let me be a bit more precise:

Proposition 6.4.1. Closest vector inequality.

If $S \leq \mathbb{R}^{n \times 1}$ and $b \in \mathbb{R}^{n \times 1}$ such that $b \notin S$ then for all $u \in S$ with $u \neq Proj_S(b)$, $||b - Proj_S(b)|| < ||b - u||.$

This means $Proj_S(b)$ is the closest vector to b in S.

Proof: Noice that $b - u = b - Proj_S(b) + Proj_S(b) - u$. Furthermore note that $b - Proj_S(b) = Orth_S(b) \in S^{\perp}$ whereas $Proj_S(b) - u \in S$ hence these are orthogonal vectors and we can apply the Pythagorean Theorem,

$$||b - u||^{2} = ||b - Proj_{S}(b)||^{2} + ||Proj_{S}(b) - u||^{2}$$

Notice that $u \neq Proj_S(b)$ implies $Proj_S(b) - u \neq 0$ hence $||Proj_S(b) - u||^2 > 0$. It follows that $||b - Proj_S(b)||^2 < ||b - u||^2$. And as the $|| \cdot ||$ is nonnegative³ we can take the squareroot to obtain $||b - Proj_S(b)|| < ||b - u||$. \Box

Remark 6.4.2.

In calculus III I show at least three distinct methods to find the point off a plane which is closest to the plane. We can minimize the distance function via the 2nd derivative test for two variables, or use Lagrange Multipliers or use the geometric solution which invokes the projection operator. It's nice that we have an explicit proof that the geometric solution is valid. We had argued on the basis of geometric intuition that $Orth_S(b)$ is the shortest vector from the plane S to the point b off the plane⁴ Now we have proof. Better yet, our proof equally well applies to subspaces of $\mathbb{R}^{n\times 1}$. In fact, this discussion extends to the context of inner product spaces.

Example 6.4.3. Consider $\mathbb{R}^{2 \times 1}$ let $S = span\{[1,1]\}$. Find the point on the line S closest to the point $[4,0]^T$.

$$Proj_{S}([4,0]^{T}) = \frac{1}{2}([1,1] \cdot [4,0])[1,1]^{T} = [2,2]^{T}$$

Thus, $[2,2]^T \in S$ is the closest point to $[4,0]^T$. Geometrically, this is something you should have been able to derive for a few years now. The points (2,2) and (4,0) are on the perpendicular bisector of y = x (the set S is nothing more than the line y = x making the usual identification of points and vectors)

³notice $a^2 < b^2$ need not imply a < b in general. For example, $(5)^2 < (-7)^2$ yet $5 \not< -7$. Generally, $a^2 < b^2$ together with the added condition a, b > 0 implies a < b.

Example 6.4.4. In Example 6.3.14 we found that $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^{4 \times 1}$ where

$$u_{1} = \begin{bmatrix} 2\\1\\2\\0 \end{bmatrix} \qquad u_{2} = \begin{bmatrix} 0\\-2\\1\\1 \end{bmatrix} \qquad u_{3} = \begin{bmatrix} -1\\2\\0\\-1 \end{bmatrix}$$

has $Proj_W([0, 6, 0, 6]^T) = \begin{bmatrix} \frac{26}{33}, & \frac{94}{33}, & \frac{26}{33}, & \frac{-36}{33} \end{bmatrix}^T$. We can calculate that

$$rref \begin{bmatrix} 2 & 0 & -1 & | & 0 \\ 1 & -2 & 2 & | & 6 \\ 2 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & | & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

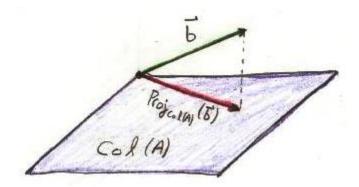
This means that $[0, 6, 0, 6]^T \notin W$. However, we learned in Proposition 6.4.1 that $\operatorname{Proj}_W([0, 6, 0, 6]^T)$ is the vector in W which is closest to $[0, 6, 0, 6]^T$. Notice that we can deduce that the orthogonal basis from Example 6.3.14 unioned with $\operatorname{Orth}_W([0, 6, 0, 6]^T)$ will form an orthogonal basis for $\mathbb{R}^{4 \times 1}$.

Example 6.4.5. Example 6.3.13 shows that $W = span\{e_1 + e_2, e_3\}$ and $x = [1, 2, 3]^T$ yields $Proj_W(x) = [\frac{3}{2}, \frac{3}{2}, 3]^T$. Again we can argue that $x \notin Col[e_1 + e_2|e_3] = W$ but $Proj_W(x)$ is in fact in W. Moreover, $Proj_W(x)$ is the closest vector to x which is in W. In this case, the geometry is that $Orth_W(x) = [-\frac{1}{2}, \frac{1}{2}, 0]^T$ is the precisely the normal vector to the plane W.

The examples above are somewhat special in that the subspaces considered have only one dimension less than the total vector space. This means that the orthogonal projection of any vector outside the subspace will return the same vector modulo a nonzero constant. In other words, the orthogonal complement is selecting the normal vector to our subspace. In general if we had a subspace which was two or more dimensions smaller than the total vector space then there would be more variety in the output of the orthogonal projection with respect to the subspace. For example, if we consider a plane inside $\mathbb{R}^{4\times 1}$ then there is more than just one direction which is orthogonal to the plane, the orthogonal projection would itself fill out a plane in $\mathbb{R}^{4\times 1}$.

6.5 inconsistent equations

We've spent considerable time solving systems of equations which were *consistent*. What if a system of equations Ax = b is inconsistent? What if anything can we say? Let $A \in \mathbb{R}^{m \times n}$ then we found in Proposition 4.6.3 Ax = b is consistent iff $b \in Col(A)$. In other words, the system has a solution iff there is some linear combination of the columns of A such that we obtain b. Here the columns of A and b are both m-dimensional vectors. If rank(A) = dim(Col(A)) = m then the system is consistent no matter which choice for b is made. However, if rank(A) < m then there are some vectors in $\mathbb{R}^{m \times 1}$ which are not in the column space of A and if $b \notin Col(A)$ then there will be no $x \in \mathbb{R}^{n \times 1}$ such that Ax = b. We can picture it as follows: the Col(A) is a subspace of $\mathbb{R}^{m \times 1}$ and $b \in \mathbb{R}^{m \times 1}$ is a vector pointing out of the subspace. The shadow of b onto the subspace Col(A) is given by $Proj_{Col(A)}(b)$.



Notice that $Proj_{Col(A)}(b) \in Col(A)$ thus the system $Ax = Proj_{Col(A)}(b)$ has a solution for any $b \in \mathbb{R}^{m \times 1}$. In fact, we can argue that x which solves $Ax = Proj_{Col(A)}(b)$ is the solution which comes closest to solving Ax = b. Closest in the sense that $||Ax - b||^2$ is minimized. We call such x the least squares solution to Ax = b (which is kind-of funny terminology since x is not actually a solution, perhaps we should really call it the "least squares approximation").

Proposition 6.5.1.

If Ax = b is inconsistent then the solution of $Au = Proj_{col(A)}(b)$ minimizes $||Ax - b||^2$.

Proof: We can break-up the vector b into a vector $Proj_{Col(A)}(b) \in Col(A)$ and $Orth_{col(A)}(b) \in Col(A)^{\perp}$ where

$$b = Proj_{Col(A)}(b) + Orth_{Col(A)}(b).$$

Since Ax = b is inconsistent it follows that $b \notin Col(A)$ thus $Orth_{Col(A)}(b) \neq 0$. Observe that:

$$||Ax - b||^{2} = ||Ax - Proj_{Col(A)}(b) - Orth_{Col(A)}(b)||^{2}$$
$$= ||Ax - Proj_{Col(A)}(b)||^{2} + ||Orth_{Col(A)}(b)||^{2}$$

Therefore, the solution of $Ax = Proj_{Col(A)}(b)$ minimizes $||Ax - b||^2$ since any other vector will make $||Ax - Proj_{Col(A)}(b)||^2 > 0$. \Box

Admittably, there could be more than one solution of $Ax = Proj_{Col(A)}(b)$, however it is usually the case that this system has a unique solution. Especially for experiminally determined data sets.

We already have a technique to calculate projections and of course we can solve systems but it is exceedingly tedious to use the proposition above from scratch. Fortunately there is no need:

Proposition 6.5.2.

If Ax = b is inconsistent then the solution(s) of $Au = Proj_{Col(A)}(b)$ are solutions of the so-called **normal equations** $A^T Au = A^T b$.

Proof: Observe that,

$$Au = Proj_{Col(A)}(b) \iff b - Au = b - Proj_{Col(A)}(b) = Orth_{Col(A)}(b)$$
$$\Leftrightarrow b - Au \in Col(A)^{\perp}$$
$$\Leftrightarrow b - Au \in Null(A^{T})$$
$$\Leftrightarrow A^{T}(b - Au) = 0$$
$$\Leftrightarrow A^{T}Au = A^{T}b,$$

where we used Proposition 6.3.8 in the third step. \Box

The proposition below follows immediately from the preceding proposition.

Proposition 6.5.3.

If $det(A^T A) \neq 0$ then there is a unique solution of $Au = Proj_{Col(A)}(b)$.

6.6 least squares analysis

In experimental studies we often have some model with coefficients which appear linearly. We perform an experiment, collect data, then our goal is to find coefficients which make the model fit the collected data. Usually the data will be inconsistent with the model, however we'll be able to use the idea of the last section to find the so-called *best-fit* curve. I'll begin with a simple linear model. This linear example contains all the essential features of the least-squares analysis.

6.6.1 linear least squares problem

Problem: find values of c_1, c_2 such that $y = c_1x + c_2$ most closely models a given data set: $\{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$

Solution: Plug the data into the model and see what equations result:

$$y_1 = c_1 x_1 + c_2, \quad y_2 = c_1 x_2 + c_2, \quad \dots \quad y_k = c_1 x_k + c_2$$

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_k & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \vec{y} = M\vec{v}$$

where $\vec{y} = [y_1, y_2, \dots, y_k]^T$, $v = [c_1, c_2]^T$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will⁵ make the results bounce above and below the true solution. We can solve the normal equations $M^T \vec{y} = M^T M \vec{v}$ to find c_1, c_2 which give the best-fit curve⁶.

Example 6.6.1. Find the best fit line through the points (0,2), (1,1), (2,4), (3,3). Our model is $y = c_1 + c_2 x$. Assemble M and \vec{y} as in the discussion preceding this example:

$$\vec{y} = \begin{bmatrix} 2\\1\\4\\3 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 1\\1 & 1\\2 & 1\\3 & 1 \end{bmatrix} \quad \Rightarrow \quad M^T M = \begin{bmatrix} 0 & 1 & 2 & 3\\1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1\\1 & 1\\2 & 1\\3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6\\6 & 4 \end{bmatrix}$$

and we calculate:
$$M^T y = \begin{bmatrix} 0 & 1 & 2 & 3\\1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2\\1\\4\\3 \end{bmatrix} = \begin{bmatrix} 18\\10 \end{bmatrix}$$

The normal equations⁷ are $M^T M \vec{v} = M^T \vec{y}$. Note that $(M^T M)^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$ thus the solution of the normal equations is simply,

$$\vec{v} = (M^T M)^{-1} M^T \vec{y} = \frac{1}{20} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 18 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, y = 0.6x + 1.6 is the best-fit line. This solution minimizes the vertical distances squared between the data and the model.

It's really nice that the order of the normal equations is only as large as the number of coefficients in the model. If the order depended on the size of the data set this could be much less fun for

⁵almost always

⁶notice that if x_i are not all the same then it is possible to show $det(M^T M) \neq 0$ and then the solution to the normal equations is unique

⁷notice my choice to solve this system of 2 equations and 2 unknowns is just a choice, You can solve it a dozen different ways, you do it the way which works best for you.

real-world examples. Let me set-up the linear least squares problem for 3-coefficients and data from \mathbb{R}^3 , the set-up for more coefficients and higher-dimensional data is similar. We already proved this in general in the last section, the proposition simply applies mathematics we already derived. I state it for your convenience.

Proposition 6.6.2.

Given data $\{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n\} \subset \mathbb{R}^3$, with $\vec{r}_k = [x_k, y_k, z_k]^T$, the best-fit of the linear model $z = c_1 x + c_2 y + c_3$ is obtained by solving the normal equations $M^T M \vec{v} = M^T \vec{z}$ where $\vec{z} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad M = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$

Example 6.6.3. Find the plane which is closest to the points (0,0,0), (1,2,3), (4,0,1), (0,3,0), (1,1,1). An arbitrary⁸ plane has the form $z = c_1x + c_2y + c_3$. Work on the normal equations,

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad M^T M = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 3 & 6 \\ 3 & 14 & 6 \\ 6 & 6 & 5 \end{bmatrix}$$
$$also, \quad M^T \vec{z} = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

We solve $M^T M \vec{v} = M^T \vec{z}$ by row operations, after some calculation we find:

$$rref[M^{T}M|M^{T}\vec{z}] = \begin{bmatrix} 1 & 0 & 1 & 89/279 \\ 0 & 1 & 1 & 32/93 \\ 0 & 0 & 1 & 19/93 \end{bmatrix} \Rightarrow \begin{array}{c} c_{1} = 89/279 \\ c_{2} = 32/93 \\ c_{3} = 19/93 \end{array}$$

Therefore, $z = \frac{89}{293}x + \frac{32}{93}y + \frac{19}{93}$ is the plane which is "closest" to the given points. Technically, I'm not certain that is is the absolute closest. We used the vertical distance squared as a measure of distance from the point. Distance from a point to the plane is measured along the normal direction, so there is no garauntee this is really the absolute "best" fit. For the purposes of this course we will ignore this subtle and annoying point. When I say "best-fit" I mean the least squares fit of the model.

⁸technically, the general form for a plane is ax + by + cz = d, if c = 0 for the best solution then our model misses it. In such a case we could let x or y play the role that z plays in our set-up.

6.6.2 nonlinear least squares

Problem: find values of c_1, c_2 such that $y = c_1 f_1(x)x + c_2 f_2(x) + \cdots + c_n f_n(x)$ most closely models a given data set: $\{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$. We assume the coefficients c_1, c_2 appear linearly on (possibly nonlinear) functions f_1, f_2, \dots, f_n .

Solution: Plug the data into the model and see what equations result:

$$y_1 = c_1 f_1(x_1) + c_2 f_2(x_1) + \dots + c_n f_n(x_1),$$

$$y_2 = c_1 f_1(x_2) + c_2 f_2(x_2) + \dots + c_n f_n(x_2),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_k = c_1 f_1(x_k) + c_2 f_2(x_k) + \dots + c_n f_n(x_k)$$

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_k) & f_2(x_k) & \cdots & f_n(x_k) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow \vec{y} = M\vec{v}$$

where $\vec{y} = [y_1, y_2, \dots, y_k]^T$, $v = [c_1, c_2, \dots, c_n]^T$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will⁹ make the results bounce above and below the true solution. We can solve the normal equations $M^T\vec{y} = M^T M\vec{v}$ to find c_1, c_2, \dots, c_n which give the best-fit curve¹⁰.

Remark 6.6.4.

Nonlinear least squares includes the linear case as a subcase, take $f_1(x) = x$ and $f_2(x) = 1$ and we return to the linear least squares examples. We will use data sets from \mathbb{R}^2 in this subsection. These techniques do extend to data sets with more variables as I demonstrated in the simple case of a plane.

Example 6.6.5. Find the best-fit parabola through the data (0,0), (1,3), (4,4), (3,6), (2,2). Our model has the form $y = c_1x^2 + c_2x + c_3$. Identify that $f_1(x) = x^2, f_2(x) = x$ and $f_3(x) = 1$ thus we should study the normal equations: $M^T M \vec{v} = M^T \vec{y}$ where:

	$\int f_1(0)$	$f_2(0)$	$f_3(0)$		0	0	1			[0]	
	$f_1(1)$	$f_2(1)$	$f_{3}(1)$		1	1	1			3	
M =	$f_1(4)$	$f_2(4)$	$f_3(4)$	=	16	4	1	and	$\vec{y} =$	4	.
	$f_1(3)$	$f_{2}(3)$	$f_{3}(3)$		9	3	1		$\vec{y} =$	6	
	$f_1(2)$	$f_2(2)$	$f_3(2)$		4	2	1			2	

⁹almost always

¹⁰notice that if $f_j(x_i)$ are not all the same then it is possible to show $det(M^T M) \neq 0$ and then the solution to the normal equations is unique

 $Hence,\ calculate$

$$M^{T}M = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{bmatrix}$$

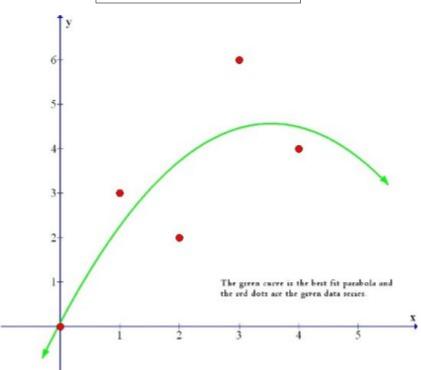
and,

$$M^{T}\vec{y} = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \\ 15 \end{bmatrix}$$

After a few row operations we can deduce,

$$rref[M^{T}M|M^{T}\vec{y}] = \begin{bmatrix} 1 & 0 & 1 & -5/14 \\ 0 & 1 & 1 & 177/70 \\ 0 & 0 & 1 & 3/35 \end{bmatrix} \Rightarrow \begin{array}{c} c_{1} = -5/14 \approx -0.357 \\ c_{2} = 177/70 \approx 2.529 \\ c_{3} = 3/35 = 0.086 \end{array}$$

We find the best-fit parabola is $y = -0.357x^2 + 2.529x + 0.086$



Yes..., but what's this for?

Example 6.6.6. Suppose you land on a mysterious planet. You find that if you throw a ball it's height above the ground y at time t is measured at times t = 0, 1, 2, 3, 4 seconds to be y = 0, 2, 3, 6, 4 meters respective. Assume that Newton's Law of gravity holds and determine the gravitational acceleration from the data. We already did the math in the last example. Newton's law approximated for heights near the surface of the planet simply says y'' = -g which integrates twice to yield $y(t) = -gt^2/2 + v_0t + y_0$ where v_0 is the initial velocity in the vertical direction. We find the best-fit parabola through the data set $\{(0,0), (1,3), (4,4), (3,6), (2,2)\}$ by the math in the last example,

$$y(t) = -0.357t^2 + 2.529 + 0.086$$

we deduce that $g = 2(0.357)m/s^2 = 0.714m/s^2$. Apparently the planet is smaller than Earth's moon (which has $g_{moon} \approx \frac{1}{6}9.8m/s^2 = 1.63m/s^2$.

Remark 6.6.7.

If I know for certain that the ball is at y = 0 at t = 0 would it be equally reasonable to assume y_o in our model? If we do it simplifies the math. The normal equations would only be order 2 in that case.

Example 6.6.8. Find the best-fit parabola that passes through the origin and the points (1,3), (4,4), (3,6), (2,2). To begin we should state our model: since the parabola goes through the origin we know the y-intercept is zero hence $y = c_1x^2 + c_2x$. Identify $f_1(x) = x^2$ and $f_2(x) = x$. As usual set-up the M and \vec{y} ,

$$M = \begin{bmatrix} f_1(1) & f_2(1) \\ f_1(4) & f_2(4) \\ f_1(3) & f_2(3) \\ f_1(2) & f_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} \quad and \quad \vec{y} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}.$$

Calculate,

$$M^{T}M = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 354 & 100 \\ 100 & 30 \end{bmatrix} \Rightarrow (M^{T}M)^{-1} = \frac{1}{620} \begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix}$$

and,

$$M^{T}\vec{y} = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \end{bmatrix}$$

We solve $M^T M \vec{v} = M^T \vec{y}$ by multiplying both sides by $(M^T M)^{-1}$ which yields,

$$\vec{v} = (M^T M)^{-1} M^T \vec{y} = \frac{1}{620} \begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix} \begin{bmatrix} 129 \\ 41 \end{bmatrix} = \begin{bmatrix} -23/62 \\ 807/310 \end{bmatrix} \Rightarrow \begin{array}{c} c_1 = -23/62 \approx -0.371 \\ c_2 = 807/310 \approx 2.603 \end{array}$$

Thus the best-fit parabola through the origin is $y = -0.371x^2 + 2.603x$

Sometimes an application may not allow for direct implementation of the least squares method, however a rewrite of the equations makes the unknown coefficients appear linearly in the model.

Example 6.6.9. Newton's Law of Cooling states that an object changes temperature T at a rate proportional to the difference between T and the room-temperature. Suppose room temperature is known to be 70° then dT/dt = -k(T-70) = -kT + 70k. Calculus reveals solutions have the form $T(t) = c_0 e^{-kt} + 70$. Notice this is very intuitive since $T(t) \rightarrow 70$ for t >> 0. Suppose we measure the temperature at successive times and we wish to find the best model for the temperature at time t. In particular we measure: T(0) = 100, T(1) = 90, T(2) = 85, T(3) = 83, T(4) = 82. One unknown coefficient is k and the other is c_1 . Clearly k does not appear linearly. We can remedy this by working out the model for the natural log of T - 70. Properties of logarithms will give us a model with linearly appearing unknowns:

$$\ln(T(t) - 70) = \ln(c_0 e^{-kt}) = \ln(c_0) + \ln(e^{-kt}) = \ln(c_0) - kt$$

Let $c_1 = \ln(c_0)$, $c_2 = -k$ then identify $f_1(t) = 1$ while $f_2(t) = t$ and $y = \ln(T(t) - 70)$. Our model is $y = c_1 f_1(t) + c_2 f_2(t)$ and the data can be generated from the given data for T(t):

$$t_1 = 0: y_1 = \ln(T(0) - 70) = \ln(100 - 70) = \ln(30)$$

$$t_2 = 1: y_2 = \ln(T(1) - 90) = \ln(90 - 70) = \ln(20)$$

$$t_3 = 2: y_3 = \ln(T(2) - 85) = \ln(85 - 70) = \ln(15)$$

$$t_4 = 3: y_4 = \ln(T(2) - 83) = \ln(83 - 70) = \ln(13)$$

$$t_5 = 4: y_5 = \ln(T(2) - 82) = \ln(82 - 70) = \ln(12)$$

Our data for (t, y) is $(0, \ln 30), (1, \ln 20), (2, \ln 15), (3, \ln 13), (4, \ln 12)$. We should solve normal equations $M^T M \vec{v} = M^T \vec{y}$ where

$$M = \begin{bmatrix} f_1(0) & f_2(0) \\ f_1(1) & f_2(1) \\ f_1(2) & f_2(2) \\ f_1(3) & f_2(3) \\ f_1(4) & f_2(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad and \quad \vec{y} = \begin{bmatrix} \ln 30 \\ \ln 20 \\ \ln 15 \\ \ln 13 \\ \ln 12 \end{bmatrix}.$$

We can calculate $M^T M = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}$ and $M^T \vec{y} \approx \begin{bmatrix} 14.15 \\ 26.05 \end{bmatrix}$. Solve $M^T M \vec{v} = M^T \vec{y}$ by multiplication by inverse of $M^T M$:

$$\vec{y} = (M^T M)^{-1} M^T \vec{y} = \begin{bmatrix} 3.284 \\ -0.2263 \end{bmatrix} \Rightarrow \begin{array}{c} c_1 \approx 3.284 \\ c_2 \approx -0.2263 \end{array}$$

Therefore, $y(t) = \ln(T(t) - 70) = 3.284 - 0.2263$ we identify that k = 0.2263 and $\ln(c_0) = 3.284$ which yields $c_0 = e^{3.284} = 26.68$. We find the best-fit temperature function is

$$T(t) = 26.68e^{-0.2263t} + 70.$$

Now we could give good estimates for the temperature T(t) for other times. If Newton's Law of cooling is an accurate model and our data was collected carefully then we ought to be able to make accurate predictions with our model.

Remark 6.6.10.

The accurate analysis of data is more involved than my silly examples reveal here. Each experimental fact comes with an error which must be accounted for. A real experimentalist never gives just a number as the answer. Rather, one must give a number and an uncertainty or error. There are ways of accounting for the error of various data. Our approach here takes all data as equally valid. There are weighted best-fits which minimize a weighted least squares. Technically, this takes us into the realm of math of inner-product spaces. Finite dimensional inner-product spaces also allows for least-norm analysis. The same philosophy guides the analysis: the square of the norm measures the sum of the squares of the errors in the data. The collected data usually does not precisely fit the model, thus the equations are inconsistent. However, we project the data onto the plane representative of model solutions and this gives us the best model for our data. Generally we would like to minimize χ^2 , this is the notation for the sum of the squares of the error often used in applications. In statistics finding the best-fit line is called doing "linear regression".

6.7 orthogonal transformations

If we begin with an orthogonal subset of $\mathbb{R}^{n \times 1}$ and we preform a linear transformation then will the image of the set still be orthogonal? We would like to characterize linear transformations which maintain orthogonality. These transformations should take an orthogonal basis to a new basis which is still orthogonal.

Definition 6.7.1.

If $T : \mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ is a linear transformation such that $T(x) \cdot T(y) = x \cdot y$ for all $x, y \in \mathbb{R}^{n \times 1}$ then we say that T is an **orthogonal transformation**

Example 6.7.2. Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^{2 \times 1}$ and let $R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ be a rotation of the coordinates by angle θ in the clockwise direction,

 $\begin{bmatrix} x'\\y'\end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta\\\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x\\y\end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta\\ -x\sin\theta + y\cos\theta \end{bmatrix}$

As a check on my sign conventions, consider rotating $[1,0]^T$ by $R(\pi/2)$, we obtain $[x',y']^T = [0,1]$. See the picture for how to derive these transformations from trigonometry. Intuitively, a rotation should not change the length of a vector, let's check the math: let $v, w \in \mathbb{R}^{2 \times 1}$,

$$R(\theta)v \cdot R(\theta)w = [R(\theta)v]^T R(\theta)w$$
$$= v^T R(\theta)^T R(\theta)w$$

Now calculate $R(\theta)^T R(\theta)$,

$$R(\theta)^T R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

Therefore, $R(\theta)v \cdot R(\theta) = v^T I w = v^T w = v \cdot w$ for all $v, w \in \mathbb{R}^{2 \times 1}$ and we find $L_{R(\theta)}$ is an orthogonal transformation.

This shows the matrix of a rotation L_R satisfies $R^T R = I$. Is this always true or was this just a special formula for rotations? Or is this just a two-dimensional thing? What if we look at orthhogonal transformations on $\mathbb{R}^{n \times 1}$ what general condition is there on the matrix of the transformation?

Definition 6.7.3.

Let $A \in \mathbb{R}^{n \times n}$ then we say that A is an **orthogonal matrix** iff $A^T A = I$. Moreover, we say A is a **reflection matrix** if A is orthogonal and det(A) = -1 whereas we say A is a **rotation matrix** if A is orthogonal with det(A) = 1. The set of all orthogonal $n \times n$ matrices is denoted O(n) and the set of all $n \times n$ rotation matrices is denoted SO(n).

Proposition 6.7.4. matrix of an orthogonal transformation is orthogonal

If A is the matrix of an orthogonal transformation on $\mathbb{R}^{n \times 1}$ then $A^T A = I$ and either A is a rotation matrix or A is a reflection matrix.

Proof: Suppose L(x) = Ax and L is an orthogonal transformation on $\mathbb{R}^{n \times 1}$. Notice that

$$L(e_i) \cdot L(e_j) = [Ae_i]^T Ae_j = e_i^T [A^T A]e_j$$

and

$$e_i \cdot e_j = e_i^T e_j = e_i^T I e_j$$

hence $e_i^T [A^T A - I] e_j = 0$ for all i, j thus $A^T A - I = 0$ by Example 2.8.15 and we find $A^T A = I$. Following a homework you did earlier in the course,

$$det(A^T A) = det(I) \quad \Leftrightarrow \quad det(A)det(A) = 1 \quad \Leftrightarrow det(A) = \pm 1$$

Thus $A \in SO(n)$ or A is a reflection matrix. \Box

The proposition below is immediate from the definitions of length, angle and linear transformation.

Proposition 6.7.5. orthogonal transformations preserve lengths and angles

If $v, w \in \mathbb{R}^{n \times 1}$ and L is an orthogonal transformation such that v' = L(v) and w' = L(w) then the angle between v' and w' is the same as the angle between v and w, in addition the length of v' is the same as v.

Remark 6.7.6.

Reflections, unlike rotations, will spoil the "handedness" of a coordinate system. If we take a right-handed coordinate system and perform a reflection we will obtain a new coordinate system which is left-handed. If you'd like to know more just ask me sometime.

If orthogonal transformations preserve the geometry of $\mathbb{R}^{n \times 1}$ you might wonder if there are other non-linear transformations which also preserve distance and angle. The answer is yes, but we need to be careful to distinguish between the length of a vector and the distance bewtween points. It turns out that the translation defined below will preserve the distance, but not the norm or length of a vector.

Definition 6.7.7.

Fix $b \in \mathbb{R}^{n \times 1}$ then a translation by b is the mapping $T_b(x) = x + b$ for all $x \in \mathbb{R}^{n \times 1}$.

This is known as an **affine transformation**, it is not linear since $T(0) = b \neq 0$ in general. (if b = 0 then the translation is both affine and linear). Anyhow, affine transformations should be familar to you: y = mx + b is an affine transformation on \mathbb{R} .

Proposition 6.7.8. translations preserve geometry

Suppose $T_b : \mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$ is a translation then

- 1. If $\angle (xyz)$ denotes the angle formed by line segments \bar{xy}, \bar{yz} which have endpoints x, yand y, z respectively then $\angle (T_b(x)T_b(y)T_b(z)) = \angle (xyz)$
- 2. The distance from x to y is the equal to the distance from $T_b(x)$ to $T_b(y)$.

Proof: I'll begin with (2.) since it's easy:

$$d(T_b(x), T_b(y)) = ||T_b(y) - T_b(x)|| = ||y + b - (x + b)|| = ||y - x|| = d(x, y).$$

Next, the angle $\angle (xyz)$ is the angle between x - y and z - y. Likewise the angle $\angle T_b(x)T_b(y)T_b(z)$ is the angle between $T_b(x) - T_b(y)$ and $T_b(z) - T_b(y)$. But, these are the same vectors since $T_b(x) - T_b(y) = x + b - (y + b) = x - y$ and $T_b(z) - T_b(y) = z + b - (y + b) = z - y$. \Box

Definition 6.7.9.

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Suppose T(x) = Ax + b where A \in SO(n) and b \in \mathbb{R}^{n \times 1} for all x \in \mathbb{R}^{n \times 1} then we say T is a rigid motion.
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In high-school geometry you studied the concept of *congruence*. To objects were congruent if they had the same size and shape. From the viewpoint of analytic geometry we can say two objects are congruent iff one is the image of the other with respect to some rigid motion. We leave further discussion of such matters to the modern geometry course where you study these concepts in depth.

Remark 6.7.10.

In Chapter 6 of my *Mathematical Models in Physics* notes I describe how Euclidean geometry is implicit and foundational in classical Newtonian Mechanics. The concept of a rigid motion is used to define what is meant by an *intertial frame*. I have these notes posted on my website, ask if your interested. Chapter 7 of the same notes describes how Special Relativity has hyperbolic geometry as its core. The dot-product is replaced with a Minkowski-product which yields all manner of curious results like time-dilation, length contraction, and the constant speed of light. If your interested in hearing a lecture or two on the geometry of Special Relativity please ask and I'll try to find a time and a place.

6.8 inner products

We follow Chapter 6 of Anton & Rorres' *Elementary Linear Algebra*, this material is also § 7.5 of Spence, Insel & Friedberg's *Elementary Linear Algebra*, a Matrix Approach. The definition of an inner product is based on the idea of the dot product. Proposition 6.1.4 summarized the most important properties. We take these as the definition for an inner product. If you examine proofs in § 6.1 you'll notice most of what I argued was based on using these 4 simple facts for the dot-product.

WARNING: the next couple pages is dense. It's a reiteration of the main theoretical accomplishments of this chapter in the context of inner product spaces. If you need to see examples first then skip ahead as needed.

Definition 6.8.1.

Let V be a vector space over \mathbb{R} . If there is a function <, $>: V \times V \to \mathbb{R}$ such that for all $x, y, z \in V$ and $c \in \mathbb{R}$,

- 1. $\langle x, y \rangle = \langle y, x \rangle$ (symmetric),
- 2. < x + y, z > = < x, z > + < y, z >,
- 3. < cx, y > = c < x, y >,
- 4. $\langle x, x \rangle \ge 0$ and $\langle x, x \rangle = 0$ iff x = 0,

then we say \langle , \rangle is an **inner product** on V. In this case we say V with $\langle \rangle$ is an inner product space. Items (1.), (2.) and (3.) together allow us to call \langle , \rangle a real-valued **symmetric-bilinear-form** on V. We may find it useful to use the notation $g(x,y) = \langle x, y \rangle$ for some later arguments, one should keep in mind the notation \langle , \rangle is not the only choice.

Technically, items (2.) and (3.) give us "linearity in the first slot". To obtain bilinearity we need to have linearity in the second slot as well. This means $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ and $\langle x, cy \rangle = c \langle x, y \rangle$ for all $x, y, z \in V$ and $c \in \mathbb{R}$. Fortunately, the symmetry property will transfer the linearity to the second slot. I leave that as an exercise for the reader.

Example 6.8.2. Obviously $\mathbb{R}^{n \times 1}$ together with the dot-product forms an inner product space. Moreover, the dot-product is an inner product.

Once we have an inner product for a vector space then we also have natural definitions for the length of a vector and the distance between two points.

Definition 6.8.3.

Let V be an inner product vector space with inner product \langle , \rangle . The **norm** or **length** of a vector is defined by $||x|| = \sqrt{\langle x, x \rangle}$ for each $x \in V$. Likewise the **distance** between $a, b \in V$ is defined by $d(a, b) = \sqrt{\langle b - a, b - a \rangle} = ||b - a||$ for all $a, b \in V$. We say these are the length and distance functions induced by \langle , \rangle . Likewise the **angle** between two nonzero vectors is defined implicitly by $\langle v, w \rangle = ||v||||w|| \cos(\theta)$.

As before the definition above is only logical if certain properties hold for the inner product, norm and distance function. Happily we find all the same general properties for the inner product and its induced norm and distance function.

Proposition 6.8.4.

If V is an inner product space with induced norm $|| \cdot ||$ and $x, y \in V$ then $| \langle x, y \rangle | \leq ||x|| ||y||$.

Proof: since $||x|| = \sqrt{\langle x, x \rangle}$ the proof we gave for the case of the dot-product equally well applies here. You'll notice in retrospect I only used those 4 properties which we take as the defining axioms for the inner product. \Box

In fact, all the propositions from $\S6.1$ apply equally well to an arbitrary finite-dimensional inner product space. The proof of the proposition below is similar to those I gave in $\S6.1$

Proposition 6.8.5. Properties for induced norm and distance function on an inner product space.

If V is an inner product space with inner product \langle , \rangle and norm $||x|| = \sqrt{x, x}$ and distance function d(x, y) = ||y - x|| then for all $x, y, z \in V$ and $c \in \mathbb{R}$ (i.) $||x|| \ge 0$ (v.) $d(x, y) \ge 0$ (ii.) $||x|| = 0 \Leftrightarrow x = 0$ (vi.) $d(x, y) = 0 \Leftrightarrow x = y$ (iii.) ||cx|| = |c|||x|| (vii.) d(x, y) = d(y, x)(iv.) $||x + y|| \le ||x|| + ||y||$ (viii.) $d(x, z) \le d(x, y) + d(y, z)$

An norm is simply an operation which satisfies (i.) - (iv.). If we are given a vector space with a norm then that is called a normed linear space. If in addition all Cauchy sequences converge in the space it is said to be a complete normed linear space. A **Banach Space** is defined to be a complete normed linear space. A distance function is simply an operation which satisfies (v.) - (viii.). A set with a distance function is called a **metric space**. I'll let you ponder all these things in some

other course, I mention them here merely for breadth. These topics are more interesting infinitedimensional case.

What is truly interesting is that the orthogonal complement theorems and closest vector theory transfer over to the case of an inner product space.

Definition 6.8.6.

Let V be an inner product space with inner product \langle , \rangle . Let $x, y \in V$ then we say x is **orthogonal** to y iff $\langle x, y \rangle = 0$. A set S is said to be orthogonal iff every pair of vectors in S is orthogonal. If $W \leq V$ then the **orthogonal complement** of W is defined to be $W^{\perp} = \{v \in V \mid v \cdot w = 0 \ \forall w \in W\}.$

Proposition 6.8.7. Orthogonality results for inner product space.

If V is an inner product space with inner product < , > and norm $||x|| = \sqrt{x, x}$ then for all $x, y, z \in V$ and $W \leq V$,

 $\begin{array}{ll} (i.) &< x, y >= 0 \Rightarrow ||x + y||^2 = ||x||^2 + ||y||^2\\ (ii.) \text{ if } S \subset V \text{ is orthogonal } \Rightarrow S \text{ is linearly independent}\\ (iii.) S \subset V \Rightarrow S^{\perp} \leq V\\ (iv.) W^{\perp} \cap W = \{0\}\\ (v.) V = W \oplus W^{\perp} \end{array}$

Definition 6.8.8.

Let V be an inner product space with inner product \langle , \rangle . A basis of \langle , \rangle -orthogonal vectors is an **orthogonal basis**. Likewise, if every vector in an orthogonal basis has length one then we call it an **orthonormal basis**.

Every finite dimensional inner product space permits a choice of an orthonormal basis. Examine my proof in the case of the dot-product. You'll find I made all arguments on the basis of the axioms for an inner-product. The Gram-Schmidt process works equally well for inner product spaces, we just need to exchange dot-products for inner-products as appropriate.

Proposition 6.8.9. Orthonormal coordinates and projection results.

If V is an inner product space with inner product \langle , \rangle and $\beta = \{v_1, v_2, \ldots, v_k\}$ is a orthonormal basis for a subspace W then

 $\begin{array}{l} (i.) \ w = < w, v_1 > v_1 + < w, v_2 > v_2 + \dots + < w, v_k > v_k \ \text{for each } w \in W, \\ (ii.) \ Proj_W(x) \equiv < x, v_1 > v_1 + < x, v_2 > v_2 + \dots + < x, v_k > v_k \in W \ \text{for each } x \in V, \\ (iii.) \ Orth_W(x) \equiv x - Proj_W(x) \in W^{\perp} \ \text{for each } x \in V, \\ (iv.) \ x = Proj_W(x) + Orth_W(x) \ \text{and} \ < Proj_W(x), Orth_W(x) >= 0 \ \text{for each } x \in V, \\ (v.) \ ||x - Proj_W(x)|| < ||x - y|| \ \text{for all } y \notin W. \end{array}$

Notice that we can use the Gram-Schmidt idea to implement the least squares analysis in the context of an inner-product space. However, we cannot multiply abstract vectors by matrices so the short-cut normal equations may not make sense in this context. We have to implement the closest vector idea without the help of those normal equations. I'll demonstrate this idea in the Fourier analysis section.

6.8.1 examples of inner-products

The dot-product is just one of many inner products. We examine an assortment of other innerproducts for various finite dimensional vector spaces.

Example 6.8.10. Let $V = \mathbb{R}^{2 \times 1}$ and define $\langle v, w \rangle = v_1 w_1 + 3v_2 w_2$ for all $v = [v_1, v_2]^T$, $w = [w_1, w_2]^T \in V$. Let $u, v, w \in V$ and $c \in \mathbb{R}$,

1. symmetric property,

$$\langle v, w \rangle = v_1 w_1 + 3v_2 w_2 = w_1 v_1 + 3w_2 v_2 = \langle w, v \rangle$$

2. additive property:

$$\langle u + v, w \rangle = (u + v)_1 w_1 + 3(u + v)_2 w_2 = (u_1 + v_1) w_1 + 3(u_2 + v_2) w_2 = u_1 w_1 + v_1 w_1 + 3u_2 w_2 + 3v_2 w_2 = \langle u, w \rangle + \langle v, w \rangle$$

3. homogeneous property:

$$< cv, w > = cv_1w_1 + 3cv_2w_2$$

= $c(v_1w_1 + 3v_2w_2)$
= $c < v, w >$

4. positive definite property:

$$v < v, v > = v_1^2 + 3v_2^2 \ge 0$$
 and $v < v, v > = 0 \Leftrightarrow v = 0$.

Notice $e_1 = [1,0]^T$ is an orthonormalized vector with respect to \langle , \rangle but $e_2 = [0,1]^T$ not unitlength. Instead, $\langle e_2, e_2 \rangle = 3$ thus $||e_2|| = \sqrt{3}$ so the unit-vector in the e_2 -direction is $u = \frac{1}{\sqrt{3}}[0,1]^T$ and with respect to \langle , \rangle we have an orthonormal basis $\{e_1, u\}$.

Example 6.8.11. Let $V = \mathbb{R}^{m \times n}$ we define the Frobenious inner-product as follows:

$$< A, B > = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}.$$

It is clear that $\langle A, A \rangle \ge 0$ since it is the sum of squares and it is also clear that $\langle A, A \rangle = 0$ iff A = 0. Symmetry follows from the calculation

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} B_{ij} A_{ij} = \langle B, A \rangle$$

where we can commute B_{ij} and A_{ij} for each pair i, j since the components are just real numbers. Linearity and homogeneity follow from:

$$<\lambda A + B, C > = \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda A + B)_{ij} C_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda A_{ij} + B_{ij}) C_{ij}$$
$$= \lambda \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} C_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} B_{ij} C_{ij} = \lambda < A, C > + < B, C >$$

Therefore. the Frobenius inner-product is in fact an inner product. The Frobenious norm of a matrix is induced as usual:

$$||A|| = \sqrt{\langle A, A \rangle}$$

as a consequence of the theory in this chapter we already know a few interesting properties form the matrix-norm, in particular $|| < A, B > || \le ||A||||B||$. The particular case of square matrices allows further comments. If $A, B \in \mathbb{R}^{n \times n}$ then notice

$$\langle A,B \rangle = \sum_{i,j} A_{ij}B_{ij} = \sum_i \sum_j A_i j(B^T)_{ji} = trace(AB^T) \quad \Rightarrow \quad ||A|| = trace(AA^T)$$

We find an interesting identity for any square matrix $|trace(AB^T)| \leq \sqrt{trace(AA^T)trace(BB^T)}$.

Example 6.8.12. Let C[a, b] denote the set of functions which are continuous on [a, b]. This is an infinite dimensional vector space. We can define an inner-product via the definite integral of the product of two functions: let $f, g \in C[a, b]$ define

$$\langle f,g \rangle = \int_{a}^{b} f(x)g(x)dx.$$

We can prove this is an inner-product. I'll just show additivity,

$$< f + g, h > = \int_{a}^{b} (f(x) + g(x))(x)h(x)dx$$

=
$$\int_{a}^{b} f(x)h(x)dx + \int_{a}^{b} g(x)h(x)dx = < f, h > + < g, h > .$$

I leave the proof of the other properties to the reader.

Example 6.8.13. Consider the inner-product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$ for $f, g \in C[-1, 1]$. Let's calculate the length squared of the standard basis:

$$<1,1>=\int_{-1}^{1}1\cdot 1dx = 2, \qquad < x,x>=\int_{-1}^{1}x^{2}dx = \frac{x^{3}}{3}\Big|_{-1}^{1} = \frac{2}{3}$$
$$< x^{2}, x^{2}>=\int_{-1}^{1}x^{4}dx = \frac{x^{5}}{5}\Big|_{-1}^{1} = \frac{2}{5}$$

Notice that the standard basis of P_2 are not all <, >-orthogonal:

$$<1, x>=\int_{-1}^{1} x dx = 0 \qquad <1, x^{2}>==\int_{-1}^{1} x^{2} dx = \frac{2}{3} \qquad =\int_{-1}^{1} x^{3} dx = 0$$

We can use the Gram-Schmidt process on $\{1, x, x^2\}$ to find an orthonormal basis for P_2 on [-1, 1]. Let, $u_1(x) = 1$ and

$$u_{2}(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x$$

$$u_{3}(x) = x^{2} - \frac{\langle x^{2}, x \rangle}{\langle x, x \rangle} x - \frac{\langle x^{2}, 1 \rangle}{\langle 1, 1 \rangle} = x^{2} - \frac{1}{3}$$

We have an orthogonal set of functions $\{u_1, u_2, u_3\}$ we already calculated the length of u_1 and u_2 so we can immediately normalize those by dividing by their lengths; $v_1(x) = \frac{1}{\sqrt{2}}$ and $v_2(x) = \sqrt{\frac{3}{2}x}$. We need to calculate the length of u_3 so we can normalize it as well:

$$\langle u_3, u_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

Thus $v_3(x) = \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right)$ has length one. Therefore, $\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right)\right\}$ is an orthonormal basis for P_2 restricted to [-1, 1]. Other intervals would not have the same basis. This construction depends both on our choice of inner-product and the interval considered. Incidentally, these are the first three **Legendre Polynomials**. These arise naturally as solutions to certain differential equations. The theory of **orthogonal polynomials** is full of such calculations. Orthogonal polynomials are quite useful as approximating functions. If we offered a second course in differential equations we could see the full function of such objects.

Example 6.8.14. Clearly $f(x) = e^x \notin P_2$. What is the least-squares approximation of f? Use the projection onto P_2 : Proj $P_2(f) = \langle f, v_1 \rangle v_1 + \langle f, v_2 \rangle v_2 + \langle f, v_3 \rangle v_3$. We calculate,

$$< f, v_1 >= \int_{-1}^1 \frac{1}{\sqrt{2}} e^x dx = \frac{1}{\sqrt{2}} (e^1 - e^{-1}) \approx 1.661$$
$$< f, v_2 >= \int_{-1}^1 \sqrt{\frac{3}{2}} x e^x dx = \sqrt{\frac{3}{2}} (x e^x - e^x)|_{-1}^1 = \sqrt{\frac{3}{2}} [-(-e^{-1} - e^{-1})] = \sqrt{6}e^{-1} \approx 0.901$$

$$\langle f, v_3 \rangle = \int_{-1}^{1} \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right) e^x dx = \frac{2e}{3} - \frac{14e^{-1}}{3} \approx 0.0402$$

Thus,

$$Proj_{P_2}(f)(x) = 1.661v_1(x) + 0.901v_2(x) + 0.0402v_3(x)$$
$$= 1.03 + 1.103x + 0.017x^2$$

This is closest a quadratic can come to approximating the exponential function on the interval [-1, 1]. What's the giant theoretical leap we made in this example? We wouldn't face the same leap if we tried to approximate $f(x) = x^4$ with P_2 . What's the difference? Where does e^x live?

Example 6.8.15. Consider $C[-\pi,\pi]$ with inner product $\langle f,g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. The set of sine and cosine functions $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(kx), \sin(kx)\}$ is an orthogonal set of functions.

$$<\cos(mx),\cos(nx)>=\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx = \pi\delta_{mn}$$
$$<\sin(mx),\sin(nx)>=\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx = \pi\delta_{mn}$$
$$<\sin(mx),\cos(nx)>=\int_{-\pi}^{\pi}\sin(mx)\cos(nx)dx = 0$$

Thus we find the following is a set of <u>orthonormal</u> functions

$$\beta_{trig} = \{ \frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}}\cos(x), \ \frac{1}{\sqrt{\pi}}\sin(x), \ \frac{1}{\sqrt{\pi}}\cos(2x), \ \frac{1}{\sqrt{\pi}}\sin(2x), \ \dots, \ \frac{1}{\sqrt{\pi}}\cos(kx), \ \frac{1}{\sqrt{\pi}}\sin(kx) \}$$

6.8.2 Fourier analysis

The idea of Fourier analysis is based on the least-squares approximation and the last example of the preceding section. We wish to represent a function with a sum of sines and cosines, this is called a **Fourier sum**. Much like a power series, the more terms we use to approximate the function the closer the approximating sum of functions gets to the real function. In the limit the approximation can become exact, the Fourier sum goes to a Fourier series. I do not wish to confront the analytical issues pertaining to the convergence of Fourier series. As a practical matter, it's difficult to calculate infinitely many terms so in practice we just keep the first say 10 or 20 terms and it will come very close to the real function. The advantage of a Fourier sum over a polynomial is that sums of trigonometric functions have natural periodicities. If we approximate the function over the interval $[-\pi, \pi]$ we will also find our approximation repeats itself outside the interval. This is desireable if one wishes to model a wave-form of some sort. Enough talk. Time for an example. (there also an example in your text on pages 540-542 of Spence, Insel and Friedberg)

Example 6.8.16. Suppose
$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$
 and $f(t + 2n\pi) = f(t)$ for all $n \in \mathbb{Z}$.

This is called a square wave for the obvious reason (draw its graph). Find the first few terms in

a Fourier sum to represent the function. We'll want to use the projection: it's convenient to bring the normalizing constants out so we can focus on the integrals without too much clutter.¹¹

$$Proj_W(f)(t) = \frac{1}{2\pi} < f, 1 > +\frac{1}{\pi} < f, \cos t > \cos t + \frac{1}{\pi} < f, \sin t > \sin t + \frac{1}{\pi} < f, \cos 2t > \cos 2t + \frac{1}{\pi} < f, \sin 2t > \sin 2t + \cdots$$

Where $W = span(\beta_{trig})$. The square wave is constant on $(0, \pi]$ and $[-\pi, 0)$ and the value at zero is not defined (you can give it a particular value but that will not change the integrals that calculate the Fourier coefficients). Calculate,

$$\langle f, 1 \rangle = \int_{-\pi}^{\pi} f(t) dt = 0$$

$$< f, \cos t > = \int_{-\pi}^{\pi} \cos(t) f(t) dt = 0$$

Notice that f(t) and $\cos(t)f(t)$ are odd functions so we can conclude the integrals above are zero without further calculation. On the other hand, $\sin(-t)f(-t) = (-\sin t)(-f(t)) = \sin tf(t)$ thus $\sin(t)f(t)$ is an even function, thus:

$$\langle f, \sin t \rangle = \int_{-\pi}^{\pi} \sin(t) f(t) dt = 2 \int_{0}^{\pi} \sin(t) f(t) dt = 2 \int_{0}^{\pi} \sin(t) dt = 4$$

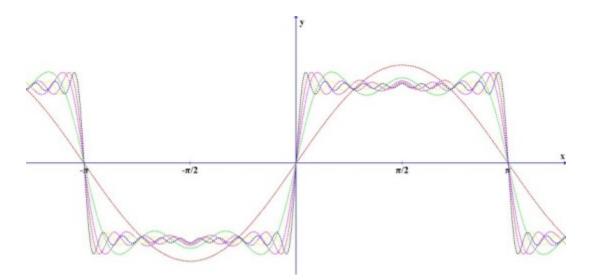
Notice that $f(t)\cos(kt)$ is odd for all $k \in \mathbb{N}$ thus $\langle f, \cos(kt) \rangle = 0$. Whereas, $f(t)\sin(kt)$ is even for all $k \in \mathbb{N}$ thus

$$< f, \sin kt > = \int_{-\pi}^{\pi} \sin(kt) f(t) dt = 2 \int_{0}^{\pi} \sin(kt) f(t) dt$$
$$= 2 \int_{0}^{\pi} \sin(kt) dt = \frac{2}{k} \left[1 - \cos(k\pi) \right] = \begin{cases} 0, & k \text{ even} \\ \frac{4}{k}, & k \text{ odd} \end{cases}$$

Putting it all together we find (the \sim indicates the functions are nearly the same except for a finite subset of points),

$$f(t) \sim \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)t)$$

 $^{^{11}}$ In fact, various texts put these little normalization factors in different places so when you look up results on Fourier series beware conventional discrepancies



I have graphed the Fourier sums up the sum with 11 terms.

Remark 6.8.17.

The treatment of Fourier sums and series is by no means complete in these notes. There is much more to say and do. Our goal here is simply to connect Fourier analysis with the more general story of orthogonality. In the math 334 course we use Fourier series to construct solutions to partial differential equations. Those calculations are foundational to describe interesting physical examples such as the electric and magnetic fields in a waveguide, the vibrations of a drum, the flow of heat through some solid, even the vibrations of a string instrument.

Chapter 7

eigenvalues and eigenvectors

The terms eigenvalue and vector originate from the German school of mathematics which was very influential in the early 20-th century. Heisenberg's formulation of quantum mechanics gave new importance to linear algebra and in particular the algebraic structure of matrices. In finite dimensional quantum systems the symmetries of the system were realized by linear operators. These operators acted on states of the system which formed a complex vector space called Hilbert Space. ¹ Operators representing momentum, energy, spin or angular momentum operate on a physical system represented by a sum of eigenfunctions. The eigenvalues then account for possible value which could be measured in an experiment. Generally, quantum mechanics involves complex vector spaces and infinite dimensional vector spaces however many of the mathematical difficulties are already present in our study of linear algebra. For example, one important question is how does one pick a set of states which diagonalize an operator? Moreover, if one operator is diagonalized by a particular basis then can a second operator be diagonalized simultaneously? Linear algebra, and in particular eigenvectors help give an answer for these questions. ²

Beyond, or perhaps I should say before, quantum mechanics eigenvectors have great application in classical mechanics, optics, population growth, systems of differential equations, chaos theory, difference equations and much much more. They are a fundmental tool which allow us to pick apart a matrix into its very core. Diagonalization of matrices almost always allow us to see the nature of a system more clearly.

However, not all matrices are diagonalizable. It turns out that any matrix is similar to a Jordan Block matrix. Moreover, the similarity transformation is accomplished via a matrix formed from concatenating generalized eigenvectors. When there are enough ordinary eigenvectors then the Jordan Form of the matrix is actually a diagonal matrix. The general theory for Jordan Forms, in particular the proof of the existence of a Jordan Basis, is rather involved. I will forego typical

¹Hilbert Spaces and infinite dimensional linear algebra are typically discussed in graduate linear algebra and/or the graduate course in functional analysis, we focus on the basics in this course.

 $^{^{2}}$ in addition to linear algebra one should also study group theory. In particular, matrix Lie groups and their representation theory explains most of what we call "chemistry". Magic numbers, electronic numbers, etc... all of these are eigenvalues which label the states of the so-called Casimir operators

worries about existence and just show you a few examples. I feel this is important because the Jordan Form actually does present itself in applications. The double root solution for constant coefficient 2nd order ODEs actually has the Jordan form hiding in the details. We will see how the matrix exponential allows for elegant solutions of any system of differential equations. I emphasize and motivate generalized eigenvectors from the viewpoint of differential equations. My approach is similar to that given in the text on DEqns by Nagel, Saff and Snider (the text for math 334). I should mention that if you wish to understand generalized eigenvectors and Jordan forms in the abstract then you should really engage in a serious study of <u>modules</u>. If you build a vector space over a ring instead of a field then you get a module. Many of the same theorems hold, if you are interested I would be happy to point you to some sources to begin reading. I would be a good topic for an independent study to follow this course.

Finally, there is the case of complex eigenvalues and complex eigenvectors. These have use in the real case. A general principle for linear systems is that if a complex system has a solution then the corresponding real system will inherit two solutions from the real and imaginary parts of the complex solution. Complex eigenvalues abound in applications. For example, rotation matrices have complex eigenvalues. We'll find that complex eigenvectors are useful and not much more trouble than the real case.

7.1 geometry of linear transformations

I'll focus on two dimensions to begin for the sake of illustration. Let's take a matrix A and a point x_o and study what happens as we multiply by the matrix. We'll denote $x_1 = Ax_o$ and generally $x_{k+1} = Ax_k$. It is customary to call x_k the "k-th state of the system". As we multiply the k-th state by A we generate the k + 1-th state.³

Example 7.1.1. Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and let $x_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Calculate,

 $\begin{aligned} x_1 &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ x_2 &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \end{bmatrix} \\ x_3 &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 27 \\ 54 \end{bmatrix} \\ x_4 &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 27 \\ 54 \end{bmatrix} = \begin{bmatrix} 81 \\ 162 \end{bmatrix} \end{aligned}$

Each time we multiply by A we scale the vector by a factor of three. If you want to look at x_o as a point in the plane the matrix A pushes the point x_k to the point $x_{k+1} = 3x_k$. Its not hard to see

³ask Dr. Mavinga and he will show you how a recursively defined linear difference equation can be converted into a matrix equation of the form $x_{k+1} = Ax_k$, this is much the same idea as saying that an n - th order ODE can be converted into a system of n- first order ODEs.

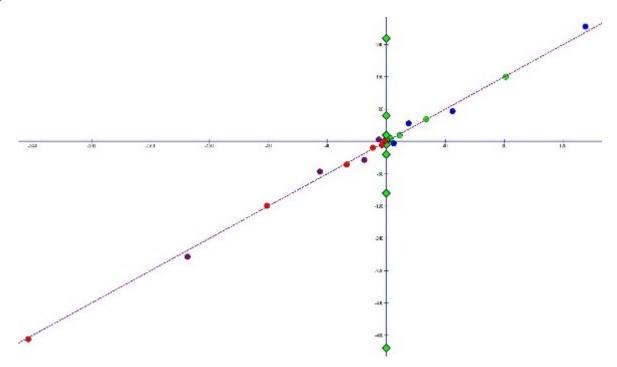
that $x_k = 3^k x_o$. What if we took some other point, say $y_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then what will A do?

$$y_{1} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$
$$y_{2} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$
$$y_{3} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 27 \\ 56 \end{bmatrix}$$
$$y_{4} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 27 \\ 48 \end{bmatrix} = \begin{bmatrix} 81 \\ 160 \end{bmatrix}$$

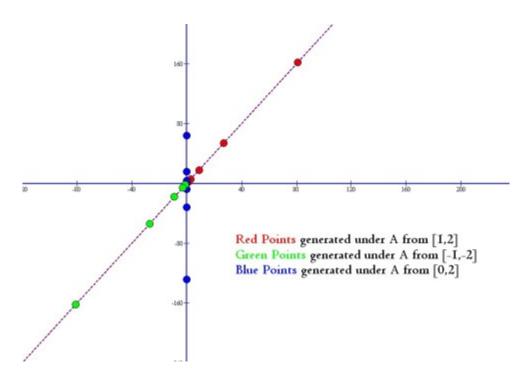
Now, what happens for arbitrary k? Can you find a formula for y_k ? This point is not as simple as x_o . The vector x_o is apparently a special vector for this matrix. Next study, $z_o = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

$$z_1 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$
$$z_2 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$z_3 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix}$$
$$z_4 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

Let me illustrate what is happening with a picture. I have used color to track the motion of a particular point. You can see that all points tend to get drawn into the line with direction vector x_o with the sole exception of the points along the y-axis which I have denoted via diamonds in the picture below:



The directions [1,2] and [0,1] are special, the following picture illustrates the motion of those points under A:

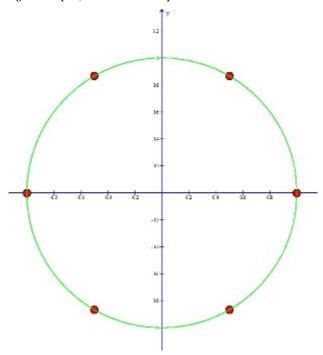


The line with direction vector [1,2] seems to attract almost all states to itself. On the other hand, if you could imagine yourself a solution walking along the y-axis then if you took the slightest mis-step to the right or left then before another dozen or so steps you'd find yourself stuck along the line in the [1,2]-direction. There is a connection of the system $x_{k+1} = Ax_k$ and the system of differential equations dx/dt = Bx if we have B = I + A. Perhaps we'll have time to explore the questions posed in this example from the viewpoint of the corresponding system of differential equations. In this case the motion is very discontinuous. I think you can connect the dots here to get a rough picture of what the corresponding system's solutions look like. Anywho, we will return to this discussion later in this chapter. For now we are simply trying to get a feeling for how one might discover that there are certain special vector(s) associated with a given matrix. We call these vectors the "eigenvectors" of A.

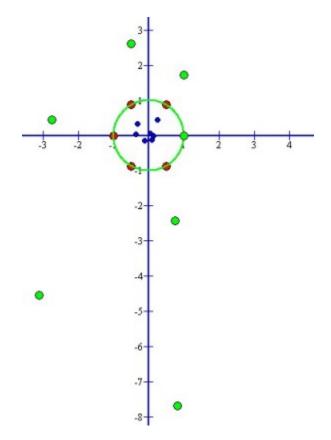
The next matrix will generate rather different motions on points in the plane.

Example 7.1.2. Let
$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
. Consider the trajectory of $x_o = [1,0]^T$,
 $x_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$
 $x_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$
 $x_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
 $x_4 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$
 $x_5 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$
 $x_6 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Past this point we just cycle back to the same points, clearly $x_k = x_{k+6}$ for all $k \ge 0$. If we started with a different initial point we would find this pattern again. The reason for this is that A is the matrix which rotates vectors by $\pi/3$ radians. The trajectories generated by this matrix are quite different then the preceding example, there is no special direction in this case.



Although, generally this type of matrix generates elliptical orbits and then there are two special directions. Namely the major and minor axis of the elliptical orbits. Finally, this sort of matrix could have a scaling factor built in so that the trajectories spiral in or out of the origin. I provide a picture illustrating the various possibilities. The red dots in the picture below are generated from A as was given in the preceding example, the blue dots are generated from the matrix $[\frac{1}{2}col_1(A)|col_2(A)]$ whereas the green dots are obtained from the matrix $[2col_1(A)|col_2(A)]$. In each case I started with the point (1,0) and studied the motion of the point under repeated multiplications of matrix:



Let's summarize our findings so far: if we study the motion of a given point under successive multiplications of a matrix it may be pushed towards one of several directions or it may go in a circular/spiral-type motion.

Another thing to think back to is our study of stochastic matrices. For some systems we found that there was a special state we called the "steady-state" for the system. If the system was attracted to some particular final state as $t \to \infty$ then that state satisfied $PX^* = X^*$. We will soon be able to verify that X^* is an eigenvector of P with eigenvalue 1.

7.2 definition and properties of eigenvalues

The preceding section was motivational. We now begin the real⁴ material.

Definition 7.2.1.

Let $A \in \mathbb{R}^{n \times n}$. If $v \in \mathbb{R}^{n \times 1}$ is **nonzero** and $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is an **eigenvector** with **eigenvalue** λ of the matrix A.

We identify that the eigenvectors of A pointed in the direction where trajectories were asymptotically attracted in the examples of the preceding section. Although, the case of the circular trajectories broke from that general pattern. We'll discover those circular orbits correspond to the complex case.

Proposition 7.2.2.

Let $A \in \mathbb{R}^{n \times n}$ then λ is an eigenvalue of A iff $det(A - \lambda I) = 0$. We say $P(\lambda) = det(A - \lambda I)$ the **characteristic polynomial** and $det(A - \lambda I) = 0$ is the **characteristic equation**.

Proof: Suppose λ is an eigenvalue of A then there exists a nonzero vector v such that $Av = \lambda v$ which is equivalent to $Av - \lambda v = 0$ which is precisely $(A - \lambda I)v = 0$. Notice that $(A - \lambda I)0 = 0$ thus the matrix $(A - \lambda I)$ is singular as the equation $(A - \lambda I)x = 0$ has more than one solution. Consequently $det(A - \lambda I) = 0$.

Conversely, suppose $det(A - \lambda I) = 0$. It follows that $(A - \lambda I)$ is singular. Clearly the system $(A - \lambda I)x = 0$ is consistent as x = 0 is a solution hence we know there are infinitely many solutions. In particular there exists at least one vector $v \neq 0$ such that $(A - \lambda I)v = 0$ which means the vector v satisfies $Av = \lambda v$. Thus v is an eigenvector with eigenvalue λ for A^{5} . \Box

Let's collect the observations of the above proof for future reference.

Proposition 7.2.3.

The following are equivalent for $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{C}$,

- 1. λ is an eigenvalue of A
- 2. there exists $v \neq 0$ such that $Av = \lambda v$
- 3. there exists $v \neq 0$ such that $(A \lambda I)v = 0$
- 4. λ is a solution to $det(A \lambda I) = 0$
- 5. $(A \lambda I)v = 0$ has infinitely many solutions.

 $^{{}^{4}}$ I should mention that your text insists that e-vectors have real e-values. I make no such restriction. If we want to insist the e-values are real I will say that explicitly.

 $^{{}^{5}}$ It is worth mentioning that the theorems on uniqueness of solution and singular matrices and determinant hold for linear systems with complex coefficients and variables. We don't need a separate argument for the complex case

Example 7.2.4. Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. Find the eigenvalues of A from the characteristic equation:

$$det(A - \lambda I) = det \begin{bmatrix} 3 - \lambda & 0\\ 8 & -1 - \lambda \end{bmatrix} = (3 - \lambda)(-1 - \lambda) = (\lambda + 1)(\lambda - 3) = 0$$

Hence the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. Notice this is precisely the factor of 3 we saw scaling the vector in the first example of the preceding section.

Example 7.2.5. Let $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Find the eigenvalues of A from the characteristic equation:

$$det(A - \lambda I) = det \begin{bmatrix} \frac{1}{2} - \lambda & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} - \lambda \end{bmatrix} = (\frac{1}{2} - \lambda)^2 + \frac{3}{4} = (\lambda - \frac{1}{2})^2 + \frac{3}{4} = 0$$

Well how convenient is that? The determinant completed the square for us. We find: $\lambda = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$. It would seem that elliptical orbits somehow arise from complex eigenvalues

Proposition 3.5.3 proved that taking the determinant of a triagular matrix was easy. We just multiply the diagonal entries together. This has interesting application in our discussion of eigenvalues.

Example 7.2.6. Given A below, find the eigenvalues. Use Proposition 3.5.3 to calculate the determinant,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \implies det(A - \lambda I) = \begin{bmatrix} 2 - \lambda & 3 & 4 \\ 0 & 5 - \lambda & 6 \\ 0 & 0 & 7 - \lambda \end{bmatrix} = (2 - \lambda)(5 - \lambda)(7 - \lambda)$$

Therefore, $\lambda_1 = 2, \lambda_2 = 5$ and $\lambda_3 = 7$.

We conclude this section with an very useful proposition.

Proposition 7.2.7.

Let A be a upper or lower triangular matrix then the eigenvalues of A are the diagonal entries of the matrix.

Proof: follows immediately from Proposition 3.5.3 since the diagonal entries of $A - \lambda I$ are of the form $A_{ii} - \lambda$ hence the characteristic equation has the form $det(A - \lambda I) = (A_{11} - \lambda)(A_{22} - \lambda) \cdots (A_{nn} - \lambda)$ which has solutions $\lambda = A_{ii}$ for i = 1, 2, ..., n. \Box

We saw how this is useful in Example 3.6.5. The LU-factorization together with the proposition above gives a calculationally superior method for calculation the determinant. In addition, once you have the LU-factorization of A there are many other questions about A which are easier to answer. See your text for more on this if you are interested.

7.2.1 eigenwarning

Calculation of eigenvalues does not need to be difficult. However, I urge you to be careful in solving the characteristic equation. More often than not I see students make a mistake in calculating the eigenvalues. If you do that wrong then the eigenvector calculations will often turn into inconsistent equations. This should be a clue that the eigenvalues were wrong, but often I see what I like to call the "principle of minimal calculation" take over and students just adhoc set things to zero, hoping against all logic that I will somehow not notice this. Don't be this student. If the eigenvalues are correct, the eigenvector equations are consistent and you will be able to find nonzero eigenvectors. And don't forget, the eigenvectors must be nonzero.

7.3 eigenvalue properties

In this subsection we examine some foundational properties of e-values and their connection to determinants. In particular, we prepare to argue a seemingly obvious proposition, namely that an $n \times n$ matrix will have n eigenvalues. From the three examples in the earlier section that's pretty obvious, however we should avoid proof by example in as much is possible.

Theorem 7.3.1.

Fundamental Theorem of Algebra: if P(x) is an *n*-th order polynomial complex coefficients then the equation P(x) = 0 has *n*-solutions where some of the solutions may be repeated. Moreover, if P(x) is an *n*-th order polynomial with real coefficients then complex solutions to P(x) = 0 come in conjugate pairs. It follows that any polynomial with real coefficients can be factored into a unique product of repeated real and irreducible quadratic factors.

A proof of this theorem would take us far of topic here⁶. I state it to remind you what the possibilities are for the characteristic equation. Recall that the determinant is simply a product and sum of the entries in the matrix. Notice that $A - \lambda I$ has *n*-copies of λ and the determinant formula never repeats the same entry twice in the same summand. It follows that solving the characteristic equation for $A \in \mathbb{R}^{n \times n}$ boils down to factoring an *n*-th order polynomial in λ .

Proposition 7.3.2.

If $A \in \mathbb{R}^{n \times n}$ then A has n eigenvalues, however, some may be repeated and/or complex.

Proof: follows from definition of determinant and the Fundamental Theorem of Algebra 7

Notice that if $P(\lambda) = det(A - \lambda I)$ then λ_j is an e-value of the square matrix A iff $(\lambda - \lambda_j)$ divides⁸ the characteristic polynomial $P(\lambda)$.

⁶there is a nice proof which can be given in our complex variables course

⁷properties of eigenvalues and the characteristic equation can be understood from studying the *minimal* and *characteristic* polynomials. We take a less sophisticated approach in this course

⁸the term "divides" is a technical term. If f(x) divides g(x) then there exists h(x) such that g(x) = h(x)f(x). In other words, f(x) is a factor of g(x).

Proposition 7.3.3.

The constant term in the characteristic polynomial $P(\lambda) = det(A - \lambda I)$ is the determinant of A.

Proof: Suppose the characteristic polynomial P of A has coefficients c_i :

$$P(\lambda) = det(A - \lambda I) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_1 \lambda + c_0.$$

Notice that if $\lambda = 0$ then $A - \lambda I = A$ hence

$$P(0) = det(A) = c_n 0^n + \dots + c_1 0 + c_0$$

Thus $det(A) = c_0$. \Box

Proposition 7.3.4.

Zero is an eigenvalue of A iff A is a singular matrix.

Proof: Let $P(\lambda)$ be the characteristic polynomial of A. If zero is an eigenvalue then λ must factor the characteristic polynomial. Moreover, the factor theorem tells us that P(0) = 0 since $(\lambda - 0)$ factors $P(\lambda)$. Thus $c_0 = 0$ and we deduce using the previous proposition that $det(A) = c_0 = 0$. Which shows that A is singular.

Conversely, assume A is singular then det(A) = 0. Again, using the previous proposition, $det(A) = c_0$ hence $c_0 = 0$. But, this means we can factor out a λ in $P(\lambda)$ hence P(0) = 0 and we find zero is an e-value of A. \Box .

Proposition 7.3.5.

If $A \in \mathbb{R}^{n \times n}$ then A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Proof: If $A \in \mathbb{R}^{n \times n}$ then A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the characteristic polynomial factors over \mathbb{C} :

$$det(A - \lambda I) = k(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

Moreover, if you think about $A - \lambda I$ it is clear that the leading term obtains a coefficient of $(-1)^n$ hence $k = (-1)^n$. If c_0 is the constant term in the characteristic polynomial then algore reveals that $c_0 = (-1)^n (-\lambda_1) (-\lambda_2) \cdots (-\lambda_n) = \lambda_1 \lambda_2 \dots \lambda_n$. Therefore, using Proposition 7.3.3, $det(A) = \lambda_1 \lambda_2 \dots \lambda_n$. \Box .

7.4 real eigenvector examples

Example 7.4.1. Let $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ find the e-values and e-vectors of A. $det(A - \lambda I) = det \begin{bmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$ We find $\lambda_1 = 0$ and $\lambda_2 = 4$. Now find the e-vector with e-value $\lambda_1 = 0$, let $u_1 = [u, v]^T$ denote the e-vector we wish to find. Calculate,

$$(A - 0I)u_1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3u + v \\ 3u + v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously the equations above are redundant and we have infinitely many solutions of the form 3u + v = 0 which means v = -3u so we can write, $u_1 = \begin{bmatrix} u \\ -3u \end{bmatrix} = u \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. In applications we often make a choice to select a particular e-vector. Most modern graphing calculators can calculate e-vectors. It is customary for the e-vectors to be chosen to have length one. That is a useful choice for certain applications as we will later discuss. If you use a calculator it would likely give $u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ although the $\sqrt{10}$ would likely be approximated unless your calculator is smart.

Continuing we wish to find eigenvectors $u_2 = [u, v]^T$ such that $(A - 4I)u_2 = 0$. Notice that u, v are disposable variables in this context, I do not mean to connect the formulas from the $\lambda = 0$ case with the case considered now.

$$(A-4I)u_1 = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u+v \\ 3u-3v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again the equations are redundant and we have infinitely many solutions of the form v = u. Hence, $u_2 = \begin{bmatrix} u \\ u \end{bmatrix} = u \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for any $u \in \mathbb{R}$ such that $u \neq 0$.

Remark 7.4.2.

It was obvious the equations were redundant in the example above. However, we need not rely on pure intuition. The problem of calculating all the e-vectors is precisely the same as finding all the vectors in the null space of a matrix. We already have a method to do that without ambiguity. We find the rref of the matrix and the general solution falls naturally from that matrix. I don't bother with the full-blown theory for simple examples because there is no need. However, with 3×3 examples it may be advantageous to keep our earlier null space theorems in mind.

Example 7.4.3. Let $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ find the e-values and e-vectors of A. $0 = det(A - \lambda I) = det \begin{bmatrix} -\lambda & 0 & -4 \\ 2 & 4 - \lambda & 2 \\ 2 & 0 & 6 - \lambda \end{bmatrix}$ $= (4 - \lambda) [-\lambda(6 - \lambda) + 8]$ $= (4 - \lambda) [\lambda^2 - 6\lambda + 8]$ $= -(\lambda - 4)(\lambda - 4)(\lambda - 2)$ Thus we have a repeated e-value of $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$. Let's find the eigenvector $u_3 = [u, v, w]^T$ such that $(A - 2I)u_3 = 0$, we find the general solution by row reduction

$$rref \begin{bmatrix} -2 & 0 & -4 & | & 0 \\ 2 & 2 & 2 & | & 0 \\ 2 & 0 & 4 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \implies \begin{array}{c} u + 2w = 0 \\ v - w = 0 \end{array} \implies \begin{array}{c} u_3 = w \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Next find the e-vectors with e-value 4. Let $u_1 = [u, v, w]^T$ satisfy $(A - 4I)u_1 = 0$. Calculate,

$$rref \begin{bmatrix} -4 & 0 & -4 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies u + w = 0$$

Notice this case has two free variables, we can use v, w as parameters in the solution,

			$\begin{bmatrix} -w \end{bmatrix}$		0	[-1]			[0]		$\begin{bmatrix} -1 \end{bmatrix}$	
$u_1 =$	$v \\ w$	=	$v \\ w$	= v	$\begin{array}{c} 1 \\ 0 \end{array}$	0	\Rightarrow	$u_1 = v$	1	and $u_2 = w$	0	

I have boxed two linearly independent eigenvectors u_1, u_2 . These vectors will be linearly independent for any pair of nonzero constants v, w.

You might wonder if it is always the case that repeated e-values get multiple e-vectors. In the preceding example the e-value 4 had *multiplicity* two and there were likewise two linearly independent e-vectors. The next example shows that is not the case.

Example 7.4.4. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ find the e-values and e-vectors of A.

$$det(A - \lambda I) = det \begin{bmatrix} 1 - \lambda & 1\\ 0 & 1 - \lambda \end{bmatrix} = (1 - \lambda)(1 - \lambda) = 0$$

Hence we have a repeated e-value of $\lambda_1 = 1$. Find all e-vectors for $\lambda_1 = 1$, let $u_1 = [u, v]^T$,

$$(A-I)u_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v = 0 \implies u_1 = u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have only one e-vector for this system.

Incidentally, you might worry that we could have an e-value (in the sense of having a zero of the characteristic equation) and yet have no e-vector. Don't worry about that, we always get at least one e-vector for each distinct e-value. See Proposition 7.2.3

Example 7.4.5. Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
 find the e-values and e-vectors of A .

$$0 = det(A - \lambda I) = det \begin{bmatrix} 1 - \lambda & 2 & 3 \\ 4 & 5 - \lambda & 6 \\ 7 & 8 & 9 - \lambda \end{bmatrix}$$

$$= (1 - \lambda) [(5 - \lambda)(9 - \lambda) - 48] - 2[4(9 - \lambda) - 42] + 3[32 - 7(5 - \lambda)]$$

$$= -\lambda^3 + 15\lambda^2 + 18\lambda$$

$$= -\lambda(\lambda^2 - 15\lambda - 18)$$

Therefore, using the quadratic equation to factor the ugly part,

$$\lambda_1 = 0, \quad \lambda_2 = \frac{15 + 3\sqrt{33}}{2}, \quad \lambda_3 = \frac{15 - 3\sqrt{33}}{2}$$

The e-vector for e-value zero is not too hard to calculate. Find $u_1 = [u, v]^T$ such that $(A - 0I)u_1 = 0$. This amounts to row reducing A itself:

$$rref \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{c} u - w = 0 \\ v + 2w = 0 \end{array} \implies \begin{array}{c} u_1 = w \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The e-vectors corresponding e-values λ_2 and λ_3 are hard to calculate without numerical help. Let's discuss Texas Instrument calculator output. To my knowledge, TI-85 and higher will calculate both e-vectors and e-values. For example, my ancient TI-89, displays the following if I define our matrix A = mat2,

$$eigVl(mat2) = \{16.11684397, -1.11684397, 1.385788954e - 13\}$$

Calculators often need a little interpretation, the third entry is really zero in disguise. The e-vectors will be displayed in the same order, they are given from the "eigVc" command in my TI-89,

$$eigVc(mat2) = \begin{bmatrix} .2319706872 & .7858302387 & .4082482905 \\ .5253220933 & .0867513393 & -.8164965809 \\ .8186734994 & -.6123275602 & .4082482905 \end{bmatrix}$$

From this we deduce that eigenvectors for λ_1, λ_2 and λ_3 are

$$u_1 = \begin{bmatrix} .2319706872 \\ .5253220933 \\ .8186734994 \end{bmatrix} \qquad u_2 = \begin{bmatrix} .7858302387 \\ .0867513393 \\ -.6123275602 \end{bmatrix} \qquad u_3 = \begin{bmatrix} .4082482905 \\ -.8164965809 \\ .4082482905 \end{bmatrix}$$

Notice that $1/\sqrt{6} \approx 0.408248905$ so you can see that u_3 above is simply the $u = 1/\sqrt{6}$ case for the family of e-vectors we calculated by hand already. The calculator chooses e-vectors so that the vectors have length one.

While we're on the topic of calculators, perhaps it is worth revisiting the example where there was only one e-vector. How will the calculator respond in that case? Can we trust the calculator?

Example 7.4.6. Recall Example 7.4.4. We let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and found a repeated e-value of $\lambda_1 = 1$ and single e-vector $u_1 = u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hey now, it's time for technology, let A = a, $eigVl(a) = \{1,1\}$ and $eigVc(a) = \begin{bmatrix} 1. & -1. \\ 0. & 1.e - 15 \end{bmatrix}$

Behold, the calculator has given us two alleged e-vectors. The first column is the genuine e-vector we found previously. The second column is the result of machine error. The calculator was tricked by round-off error into claiming that [-1, 0.000000000000001] is a distinct e-vector for A. It is not. Moral of story? When using calculator you must first master the theory or else you'll stay mired in ignorance as presribed by your robot masters.

Finally, I should mention that TI-calculators may or may not deal with complex e-vectors adequately. There are doubtless many web resources for calculating e-vectors/values. I would wager if you Googled it you'd find an online calculator that beats any calculator. Many of you have a laptop with wireless so there is almost certainly a way to check your answers if you just take a minute or two. I don't mind you checking your answers. If I assign it in homework then I do want you to work it out <u>without</u> technology. Otherwise, you could get a false confidence before the test. Technology is to supplement not replace calculation.

7.5 complex eigenvector examples

Let us begin with a proposition. Let $v \in \mathbb{C}^n$ we define the complex conjugate of the vector to be the vector of complex conjugates; $v^* = [v_i^*]$. For example, $[1 + i, 2, 3 - i]^* = [1 - i, 2, 3 + i]$. Likewise, let $w \in \mathbb{C}^{n \times 1}$ we define $w^* = [[w^T]^*]^T$. We can always steal definitions from columns to rows or vice-versa by doing a transpose sandwhich. If this confused you, don't worry about it.

Proposition 7.5.1.

If $A \in \mathbb{R}^{n \times n}$ has e-value λ and e-vector v then λ^* is likewise an e-value with e-vector v^* for A.

Proof: We assume $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^{n \times 1}$ with $v \neq 0$. We can write v = a + ib and $\lambda = \alpha + i\beta$ for some $a, b \in \mathbb{R}^{n \times 1}$ and $\alpha, \beta \in \mathbb{R}$. These real quantities are known as the real and imaginary components of v and λ respective:

If
$$\lambda = \alpha + i\beta \in \mathbb{C}$$
 for $\alpha, \beta \in \mathbb{R}$ then $Re(\lambda) = \alpha$, $Im(\lambda) = \beta$
If $v = a + ib \in \mathbb{C}^{n \times 1}$ for $a, b \in \mathbb{R}^{n \times 1}$ then $Re(v) = a$, $Im(v) = b$.

Take the complex conjugate of $Av = \lambda v$ to find $A^*v^* = \lambda^*v^*$. But, $A \in \mathbb{R}^{n \times n}$ thus $A^* = A$ and we find $Av^* = \lambda^*v^*$. Moreover, if v = a + ib and $v \neq 0$ then we cannot have a = 0 and b = 0. Thus $v = a - ib \neq 0$. Therefore, v^* is an e-vector with e-value λ^* . \Box

This is a useful proposition. It means that once we calculate one complex e-vectors we almost automatically get a second e-vector merely by taking the complex conjugate.

Example 7.5.2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and find the e-values and e-vectors of the matrix. Observe that $det(A - \lambda I) = \lambda^2 + 1$ hence the eigevalues are $\lambda = \pm i$. Find $u_1 = [u, v]^T$ such that $(A - iI)u_1 = 0$

$$0 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -iu+v \\ -u-iv \end{bmatrix} \Rightarrow \begin{array}{c} -iu+v=0 \\ -u-iv=0 \end{array} \Rightarrow \begin{array}{c} v = iu \end{array} \Rightarrow \begin{bmatrix} u_1 = u \begin{bmatrix} 1 \\ i \end{bmatrix}$$

We find infinitely many complex eigenvectors, one for each nonzero complex constant u. In applications, in may be convenient to set u = 1 so we can write, $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let's generalize the last example.

Example 7.5.3. Let $\theta \in \mathbb{R}$ and define $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ and find the e-values and e-vectors of the matrix. Observe

$$0 = det(A - \lambda I) = det \begin{bmatrix} \cos \theta - \lambda & \sin \theta \\ -\sin \theta & \cos \theta - \lambda \end{bmatrix}$$
$$= (\cos \theta - \lambda)^2 + \sin^2 \theta$$
$$= \cos^2 \theta - 2\lambda \cos \theta + \lambda^2 + \sin^2 \theta$$
$$= \lambda^2 - 2\lambda \cos \theta + 1$$
$$= (\lambda - \cos \theta)^2 - \cos^2 \theta + 1$$
$$= (\lambda - \cos \theta)^2 + \sin^2 \theta$$

Thus $\lambda = \cos \theta \pm i \sin \theta = e^{\pm i\theta}$. Find $u_1 = [u, v]^T$ such that $(A - e^{i\theta}I)u_1 = 0$

$$0 = \begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -iu\sin\theta + v\sin\theta = 0$$

If $\sin \theta \neq 0$ then we divide by $\sin \theta$ to obtain v = iu hence $u_1 = [u, iu]^T = u[1, i]^T$ which is precisely what we found in the preceding example. However, if $\sin \theta = 0$ we obtain no condition what-so-ever on the matrix. That special case is not complex. Moreover, if $\sin \theta = 0$ it follows $\cos \theta = 1$ and in fact A = I in this case. The identity matrix has the repeated eigenvalue of $\lambda = 1$ and every vector in $\mathbb{R}^{2 \times 1}$ is an e-vector.

Example 7.5.4. Let
$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
 find the e-values and e-vectors of A .
$$0 = det(A - \lambda I) = \begin{bmatrix} 1 - \lambda & 1 & 0 \\ -1 & 1 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix}$$
$$= (3 - \lambda)[(1 - \lambda)^2 + 1]$$

Hence $\lambda_1 = 3$ and $\lambda_2 = 1 \pm i$. We have a pair of complex e-values and one real e-value. Notice that for any $n \times n$ matrix we must have at least one real e-value since all odd polynomials possess at least one zero. Let's begin with the real e-value. Find $u_1 = [u, v, w]^T$ such that $(A - 3I)u_1 = 0$:

$$rref \begin{bmatrix} -2 & 1 & 0 & | & 0 \\ -1 & -2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \implies u_1 = w \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Next find e-vector with $\lambda_2 = 1 + i$. We wish to find $u_2 = [u, v, w]^T$ such that $(A - (1 + i)I)u_2 = 0$:

$$\begin{bmatrix} -i & 1 & 0 & | & 0 \\ -1 & -i & 0 & | & 0 \\ 0 & 0 & -1 - i & | & 0 \end{bmatrix} \xrightarrow[\frac{1}{-1-i}r_3 \to r_3]{} \begin{bmatrix} -i & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

One more row-swap and a rescaling of row 1 and it's clear that

$$rref\begin{bmatrix} -i & 1 & 0 & | & 0 \\ -1 & -i & 0 & | & 0 \\ 0 & 0 & -1 - i & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & i & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{array}{c} u + iv = 0 \\ w = 0 \end{array} \Rightarrow \begin{bmatrix} u \\ 1 \\ 0 \end{bmatrix}$$

I chose the free parameter to be v. Any choice of a nonzero complex constant v will yield an e-vector with e-value $\lambda_2 = 1 + i$. For future reference, it's worth noting that if we choose v = 1 then we find

$$u_2 = \begin{bmatrix} 0\\1\\0 \end{bmatrix} + i \begin{bmatrix} 1\\0\\0 \end{bmatrix}$$

We identify that $Re(u_2) = e_2$ and $Im(u_2) = e_1$

Example 7.5.5. Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and let $C = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Define A to be the block matrix

$$A = \begin{bmatrix} B & 0 \\ \hline 0 & C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

find the e-values and e-vectors of the matrix. Block matrices have nice properties: the blocks behave like numbers. Of course there is something to prove here, and I have yet to discuss block multiplication in these notes.

$$det(A - \lambda I) = det \begin{bmatrix} B - \lambda I & 0\\ 0 & C - \lambda I \end{bmatrix} = det(B - \lambda I)det(C - \lambda I)$$

Notice that both B and C are rotation matrices. B is the rotation matrix with $\theta = \pi/2$ whereas C is the rotation by $\theta = \pi/3$. We already know the e-values and e-vectors for each of the blocks if we ignore the other block. It would be nice if a block matrix allowed for analysis of each block one at a time. This turns out to be true, I can tell you without further calculation that we have e-values $\lambda_1 = \pm i$ and $\lambda_2 = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ which have complex e-vectors

$$u_1 = \begin{bmatrix} 1\\i\\0\\0 \end{bmatrix} = e_1 + ie_2 \qquad u_2 = \begin{bmatrix} 0\\0\\1\\i \end{bmatrix} = e_3 + ie_4$$

I invite the reader to check my results through explicit calculation. Technically, this is bad form as I have yet to prove anything about block matrices. Perhaps this example gives you a sense of why we should talk about the blocks at some point.

Finally, you might wonder are there matrices which have a repeated complex e-value. And if so are there always as many complex e-vectors as there are complex e-values? The answer: sometimes. Take for instance $A = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$ (where B is the same B as in the preceding example) this matrix will have a repeated e-value of $\lambda = \pm i$ and you'll be able to calculate $u_1 = e_1 \pm ie_2$ and $u_2 = e_3 \pm ie_4$ are linearly independent e-vectors for A. However, there are other matrices for which only one complex e-vector is available despite a repeat of the e-value. Bonus point if you can give me an example soon (it'll need to be at least a 4×4 matrix).

7.6 linear independendence of real eigenvectors

You might have noticed that e-vectors with distinct e-values are linearly independent. This is no accident.

Proposition 7.6.1.

If $A \in \mathbb{R}^{n \times n}$ has e-vector v_1 with e-value λ_1 and e-vector v_2 with e-value λ_2 such that $\lambda_1 \neq \lambda_2$ then $\{v_1, v_2\}$ is linearly independent.

Proof: Let v_1, v_2 have e-values λ_1, λ_2 respective and assume towards a contradiction that $v_2 = kv_2$ for some nonzero constant k. Multiply by the matrix A,

$$Av_1 = A(kv_2) \Rightarrow \lambda_1 v_1 = k\lambda_2 v_2$$

But we can replace v_1 on the l.h.s. with kv_2 hence,

$$\lambda_1 k v_2 = k \lambda_2 v_2 \quad \Rightarrow \quad k(\lambda_1 - \lambda_2) v_2 = 0$$

Note, $k \neq 0$ and $v_2 \neq 0$ by assumption thus the equation above indicates $\lambda_1 - \lambda_2 = 0$ therefore $\lambda_1 = \lambda_2$ which is a contradiction. Therefore there does not exist such a k and the vectors are linearly independent. \Box

Proposition 7.6.2.

If $A \in \mathbb{R}^{n \times n}$ has e-vectors v_1, v_2, \ldots, v_k with e-values $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_i \neq \lambda_j$ for all i, j then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof: Let e-vectors v_1, v_2, \ldots, v_k have e-values $\lambda_1, \lambda_2, \ldots, \lambda_k$ with respect to A and assume towards a contradiction that there is some vector v_j which is a nontrivial linear combination of the other vectors:

$$v_j = c_1 v_1 + c_2 v_2 + \dots + \widehat{c_j v_j} + \dots + c_k v_k$$

Multiply by A,

$$Av_j = c_1 Av_1 + c_2 Av_2 + \dots + c_j Av_j + \dots + c_k Av_k$$

Which yields,

$$\lambda_j v_j = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + \widehat{c_j \lambda_j v_j} + \dots + c_k \lambda_k v_k$$

But, we can replace v_i on the l.h.s with the linear combination of the other vectors. Hence

$$\lambda_j \left[c_1 v_1 + c_2 v_2 + \dots + \widehat{c_j v_j} + \dots + c_k v_k \right] = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + \widehat{c_j \lambda_j v_j} + \dots + c_k \lambda_k v_k$$

Consequently,

$$c_1(\lambda_j - \lambda_1)v_1 + c_2(\lambda_j - \lambda_2)v_2 + \dots + c_j(\widehat{\lambda_j - \lambda_j})v_j + \dots + c_k(\lambda_j - \lambda_k)v_k = 0$$

Since $v_i \neq 0$ and c_i are not all zero it follows at least one factor $\lambda_j - \lambda_i = 0$ for $i \neq j$ but this is a contradiction since we assumed the e-values were distinct. \Box

Notice the proof of the preceding two propositions was essentially identical. I provided the k = 2 proof to help make the second proof more accessible.

Definition 7.6.3.

Let $A \in \mathbb{R}^{n \times n}$ then a basis $\{v_1, v_2, \ldots, v_n\}$ for $\mathbb{R}^{n \times 1}$ is called an **eigenbasis** if each vector in the basis is an e-vector for A. Notice we assume these are real vectors since they form a basis for $\mathbb{R}^{n \times 1}$. **Example 7.6.4.** We calculated in Example 7.4.3 the e-values and e-vectors of $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$

were $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$ with e-vectors

$$u_1 = \begin{bmatrix} 0\\1\\0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1\\0\\1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$

Linear indpendence of u_3 from u_1, u_2 is given from the fact the e-values of u_3 and u_1, u_2 are distinct. Then is is clear that u_1 is not a multiple of u_2 thus they are linearly independent. It follows that $\{u_1, u_2, u_3\}$ form a linearly independent set of vectors in \mathbb{R}^3 , therefore $\{u_1, u_2, u_3\}$ is an eigenbasis.

Definition 7.6.5.

Let $A \in \mathbb{R}^{n \times n}$ then we call the set of all real e-vectors with real e-value λ unioned with the zero-vector the λ -eigenspace and we denote this set by W_{λ} .

Example 7.6.6. Again using Example 7.4.3 we have two eigenspaces,

$$W_4 = span\left\{ \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\1 \end{bmatrix} \right\} \qquad W_2 = span\left\{ \begin{bmatrix} -2\\1\\1 \end{bmatrix} \right\}$$

Proposition 7.6.7.

Eigenspaces are subspaces of $\mathbb{R}^{n \times 1}$. Moreoever, $dim(W_{\lambda}) \leq m$ where m is multiplicity of the λ solution in the characteristic equation.

Proof: By definition zero is in the eigenspace W_{λ} . Suppose $x, y \in W_{\lambda}$ note that $A(x + cy) = Ax + cAy = \lambda x + c\lambda y = \lambda(x + cy)$ hence $x + cy \in W_{\lambda}$ for all $x, y \in W_{\lambda}$ and $c \in \mathbb{R}$ therefore $W_{\lambda} \leq \mathbb{R}^{n \times 1}$. To prove $dim(W_{\lambda}) \leq m$ we simply need to show that $dim(W_{\lambda}) > m$ yields a contradiction. This can be seen from showing that if there were more than m e-vectors with e-value λ then the chacteristic equation would likewise more than m solutions of λ . The question then is why does each linearly independent e-vector give a factor in the characteristic equation? Answer this question for bonus points. \Box

Definition 7.6.8.

Let A be a real square matrix with real e-value λ . The dimension of W_{λ} is called the **geometric multiplicity** of λ . The number of times the λ solution is repeated in the characteristic equation is called the **algebraic multiplicity** of λ .

We already know from the examples we've considered thus far that there will not always be an eigenbasis for a given matrix A. Let me remind you:

- 1. we could have complex e-vectors (see Example 7.5.2)
- 2. we could have less e-vectors than needed for a basis (see Example 7.4.4)

We can say case 2 is caused from the geometric multiplicity being less than the algebraic multiplicity. What can we do about this? If we want to adjoin vectors to make-up for the lack of e-vectors then how should we find them in case 2?

Definition 7.6.9.

A generalized eigenvector of order k with eigenvalue λ with respect to a matrix $A \in \mathbb{R}^{n \times n}$ is a nonzero vector v such that

$$(A - \lambda I)^k v = 0$$

It's useful to construct generalized e-vectors from a *chain-condition* if possible.

Proposition 7.6.10.

Suppose $A \in \mathbb{R}^{n \times n}$ has e-value λ and e-vector v_1 then if $(A - \lambda I)v_2 = v_1$ has a solution then v_2 is a generalized e-vector of order 2 which is linearly independent from v_1 .

Proof: Suppose $(A - \lambda I)v_2 = v_1$ is consistent then multiply by $(A - \lambda I)$ to find $(A - \lambda I)^2 v_2 = (A - \lambda I)v_1$. However, we assumed v_1 was an e-vector hence $(A - \lambda I)v_1 = 0$ and we find v_2 is a generalized e-vector of order 2. Suppose $v_2 = kv_1$ for some nonzero k then $Av_2 = Akv_1 = k\lambda v_1 = \lambda v_2$ hence $(A - \lambda I)v_2 = 0$ but this contradicts the construction of v_2 as the solution to $(A - \lambda I)v_2 = v_1$. Consequently, v_2 is linearly independent from v_1 by virtue of its construction. \Box .

Example 7.6.11. Let's return to Example 7.4.4 and look for a generalized e-vector of order 2. Recall $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and we found a repeated e-value of $\lambda_1 = 1$ and single e-vector $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (fix u = 1 for convenience). Let's complete the chain: find $v_2 = [u, v]^T$ such that $(A - I)u_2 = u_1$,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies v = 1, \ u \ is \ free$$

Any choice of u will do, in this case we can even set u = 0 to find

$$u_2 = \left[\begin{array}{c} 0\\1 \end{array} \right]$$

Clearly, $\{u_1, u_2\}$ forms a basis of $\mathbb{R}^{2 \times 1}$. It is not an eigenbasis with respect to A, however it is what is known as a **Jordan basis** for A.

Theorem 7.6.12.

Any matrix with real eigenvalues has a Jordan basis. We can always find enough generalized e-vectors to form a basis for $\mathbb{R}^{n \times 1}$ with respect to A in the case that the e-values are all real.

Proof: not here, not now. This is a hard one. \Box

Proposition 7.6.13.

Let $A \in \mathbb{R}^{n \times n}$ and suppose λ is an e-value of A with e-vector v_1 then if $(A - \lambda I)v_2 = v_1$, $(A - \lambda I)v_3 = v_2, \ldots, (A - \lambda I)v_k = v_{k-1}$ are all consistent then $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set of vectors and v_j is a generalized vector of order j for each $j = 1, 2, \ldots, k$.

Proof: worth a bonus points if you can do it. \Box

Usually we can find a chain of generalized e-vectors for each e-value and that will product a Jordan basis. However, there is a trap that you will not likely get caught in for a while. It is not always possible to use a single chain for each e-value. Sometimes it takes a couple chains for a single e-value.

Example 7.6.14. Suppose
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 it is not hard to show that $\det(A - \lambda I) = (\lambda - 1)^4 = (\lambda - 1)^4$

0. We have a quadruple e-value $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$.

$$0 = (A - I)\vec{u} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{u} = \begin{bmatrix} s_1 \\ 0 \\ s_3 \\ 0 \end{bmatrix}$$

Any nonzero choice of s_1 or s_3 gives us an e-vector. Let's define two e-vectors which are clearly linearly independent, $\vec{u}_1 = [1, 0, 0, 0]^T$ and $\vec{u}_2 = [0, 0, 1, 0]^T$. We'll find a generalized e-vector to go with each of these. There are two length two chains to find here. In particular,

$$(A-I)\vec{u}_3 = \vec{u}_1 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad s_2 = 1, s_4 = 0, \ s_1, s_3 \ free$$

I choose $s_1 = 0$ and $s_3 = 1$ since I want a new vector, define $\vec{u}_3 = [0, 0, 1, 0]^T$. Finally solving $(A - I)\vec{u}_4 = \vec{u}_2$ for $\vec{u}_4 = [s_1, s_2, s_3, s_4]^T$ yields conditions $s_4 = 1, s_2 = 0$ and s_1, s_3 free. I choose $\vec{u}_4 = [0, 0, 0, 1]^T$. To summarize we have four linearly independent vectors which form two chains:

$$(A-I)\vec{(u)}_3 = \vec{u}_1, \ (A-I)\vec{u}_1 = 0 \qquad (A-I)\vec{u}_4 = \vec{u}_2, \ (A-I)\vec{u}_2 = 0$$

7.7 linear independendence of complex eigenvectors

The complex case faces essentially the same difficulties. Complex e-vectors give us pair of linearly independent vectors with which we are welcome to form a basis. However, the complex case can also fail to provide a sufficient number of complex e-vectors to fill out a basis. In such a case we can still look for generalized complex e-vectors. Each generalized complex e-vector will give us a pair of linearly independent real vectors which are linearly independent from the pairs already constructed from the complex e-vectors. Although many of the arguments transfer directly from pervious sections there are a few features which are uniquely complex.

Proposition 7.7.1.

If $A \in \mathbb{R}^{m \times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and e-vector $v = a + ib \in \mathbb{C}^{n \times 1}$ such that $a, b \in \mathbb{R}^{n \times 1}$ then $\lambda^* = \alpha - i\beta$ is a complex e-value with e-vector $v^* = a - ib$ and $\{v, v^*\}$ is a linearly independent set of vectors over \mathbb{C} .

Proof: Proposition 7.5.1 showed that v^* is an e-vector with e-value $\lambda^* = \alpha - i\beta$. Notice that $\lambda \neq \lambda^*$ since $b \neq 0$. Therefore, v and v^* are e-vectors with distinct e-values. Note that Proposition 7.6.2 is equally valid for complex e-values and e-vectors. Hence, $\{v, v^*\}$ is linearly independent since these are e-vectors with distinct e-values. \Box

Proposition 7.7.2.

If $A \in \mathbb{R}^{m \times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and e-vector $v = a + ib \in \mathbb{C}^{n \times 1}$ such that $a, b \in \mathbb{R}^{n \times 1}$ then $a \neq 0$ and $b \neq 0$.

Proof: Expand $Av = \lambda v$ into the real components,

$$\lambda v = (\alpha + i\beta)(a + ib) = \alpha a - \beta b + i(\beta a + \alpha b)$$

and

$$Av = A(a+ib) = Aa + iAb$$

Equating real and imaginary components yields two real matrix equations,

$$Aa = \alpha a - \beta b$$
 and $Ab = \beta a + \alpha b$

Suppose a = 0 towards a contradiction, note that $0 = -\beta b$ but then b = 0 since $\beta \neq 0$ thus v = 0 + i0 = 0 but this contradicts v being an e-vector. Likewise if b = 0 we find $\beta a = 0$ which implies a = 0 and again v = 0 which contradicts v being an e-vector. Therefore, $a, b \neq 0$. \Box

Proposition 7.7.3.

If $A \in \mathbb{R}^{n \times n}$ and $\lambda = \alpha + i\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$ is an e-value with e-vector $v = a + ib \in \mathbb{C}^{n \times 1}$ and $a, b \in \mathbb{R}^{n \times 1}$ then $\{a, b\}$ is a linearly independent set of real vectors.

Proof: Add and subtract the equations v = a + ib and $v^* = a - ib$ to deduce

$$a = \frac{1}{2}(v + v^*)$$
 and $b = \frac{1}{2i}(v - v^*)$

Let $c_1, c_2 \in \mathbb{R}$ then consider,

$$c_1 a + c_2 b = 0 \quad \Rightarrow \quad c_1[\frac{1}{2}(v + v^*)] + c_2[\frac{1}{2i}(v - v^*)] = 0$$
$$\Rightarrow \quad [c_1 - ic_2]v + [c_1 + ic_2]v^* = 0$$

But, $\{v, v^*\}$ is linearly independent hence $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$. Adding these equations gives $2c_1 = 0$. Subtracting yields $2ic_2 = 0$. Thus $c_1 = c_2 = 0$ and we conclude $\{a, b\}$ is linearly independent. \Box

Proposition 7.7.4.

If $A \in \mathbb{R}^{m \times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and chain of generalized e-vectors $v_k = a_k + ib_k \in \mathbb{C}^{n \times 1}$ of orders $k = 1, 2, \ldots, m$ such that $a_k, b_k \in \mathbb{R}^{n \times 1}$ then $\{a_1, b_1, a_2, b_2, \ldots, a_m, b_m\}$ is linearly independent.

Proof: will earn bonus points. It's not that this is particularly hard, I'm just tired of typing at the moment. \Box

Remark 7.7.5.

I claim (but do not prove) that if $A \in \mathbb{R}^{m \times n}$ then we can find a basis assembled from real e-vectors and generalized e-vectors togther with the real and imaginary vector components of complex e-vectors and generalized e-vectors. If we use this basis to perform similarity transformation on A we'll find a matrix with Jordan blocks and rotation matrix blocks. This is not a Jordan basis because of the complex e-vectors, but it's close. We can always find a complex Jordan form for the real matrix A. The rational Jordan form takes care of the complex case all in real notation. However, the rational Jordan form is not of particular interest for us at this juncture, I leave it for your next course in linear algebra. While this dicussion is certainly interesting for algebra's sake alone, the full-story necessarily involves a good dose of modules and abstract polynomial theory. We'll instead focus on how the Jordan form comes into play in the context of systems of differential equations. This is simply my choice, there are others.

7.8 diagonalization

If a matrix has n-linearly independent e-vectors then we'll find that we can perform a similarity transformation to transform the matrix into a diagonal form.

Proposition 7.8.1.

Suppose that $A \in \mathbb{R}^{n \times n}$ has e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with linearly independent e-vectors v_1, v_2, \ldots, v_n . If we define $V = [v_1|v_2|\cdots|v_n]$ then $D = V^{-1}AV$ where D is a diagonal matrix with the eigenvalues down the diagonal: $D = [\lambda_1 e_1 | \lambda_2 e_2 | \cdots | \lambda_n e_n]$.

Proof: Notice that V is invertible since we assume the e-vectors are linearly independent. Moreover, $V^{-1}V = I$ in terms of columns translates to $V^{-1}[v_1|v_2|\cdots|v_n] = [e_1|e_2|\cdots|e_n]$. From which we deduce that $V^{-1}v_j = e_j$ for all j. Also, since v_j has e-value λ_j we have $Av_j = \lambda_j v_j$. Observe,

$$V^{-1}AV = V^{-1}A[v_1|v_2|\cdots|v_n]$$

= $V^{-1}[Av_1|Av_2|\cdots|Av_n]$
= $V^{-1}[\lambda_1v_1|\lambda_2v_2|\cdots|\lambda_nv_n]$
= $V^{-1}[\lambda_1v_1|\lambda_2v_2|\cdots|\lambda_nv_n]$
= $[\lambda_1V^{-1}v_1|\lambda_2V^{-1}v_2|\cdots|\lambda_nV^{-1}v_n]$
= $[\lambda_1e_1|\lambda_2e_2|\cdots|\lambda_ne_n]$

Example 7.8.2. Revisit Example 7.4.1 where we learned $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ had e-values $\lambda_1 = 0$ and $\lambda_2 = 4$ with e-vectors: $u_1 = [1, -3]^T$ and $u_2 = [1, 1]^T$. Let's follow the advice of the proposition above and diagonalize the matrix. We need to construct $U = [u_1|u_2]$ and calculate U^{-1} , which is easy since this is a 2 × 2 case:

$$U = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \qquad \Rightarrow \qquad U^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$$

Now multiply,

$$U^{-1}AU = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$$

Therefore, we find confirmation of the proposition, $U^{-1}AU = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

Notice there is one very unsettling aspect of diagonalization; we need to find the inverse of a matrix. Generally this is not pleasant. Orthogonality has another insight to help us here. Recall that we can apply the Gram-Schmidt process to orthogonalize the set of e-vectors. If the resulting set of orthogonal vectors is still an eigenbasis then we can prove the matrix formed from e-vectors is an orthogonal matrix.

Proposition 7.8.3.

If $A \in \mathbb{R}^{n \times n}$ has e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with orthonormal e-vectors v_1, v_2, \ldots, v_n and if we define $V = [v_1|v_2|\cdots|v_n]$ then $V^{-1} = V^T$ and $D = V^T A V$ where D is a diagonal matrix with the eigenvalues down the diagonal: $D = [\lambda_1 e_1 | \lambda_2 e_2 | \cdots | \lambda_n e_n]$.

Proof: Orthonormality implies $v_i^T v_j = \delta_{ij}$. Observe that

$$V^{T}V = \begin{bmatrix} \frac{v_{1}^{T}}{v_{2}^{T}} \\ \vdots \\ \hline v_{n}^{T} \end{bmatrix} \begin{bmatrix} v_{1}|v_{2}|\cdots|v_{n} \end{bmatrix} = \begin{bmatrix} v_{1}^{T}v_{1} & v_{1}^{T}v_{2} & \cdots & v_{1}^{T}v_{n} \\ v_{1}^{T}v_{1} & v_{1}^{T}v_{2} & \cdots & v_{1}^{T}v_{n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n}^{T}v_{1} & v_{n}^{T}v_{2} & \cdots & v_{n}^{T}v_{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus $V^{-1} = V^T$. The proposition follows from Proposition 7.8.1. \Box

This is great news. We now have hope of finding the diagonalization of a matrix without going to the trouble of inverting the e-vector matrix. Notice that there is no gaurantee that we can find *n*-orthonormal e-vectors. Even in the case we have *n*-linearly independent e-vectors it could happen that when we do the Gram-Schmidt process the resulting vectors are not e-vectors. That said, there is one important, and common, type of example where we are in fact gauranteed the existence of an orthonormal eigenbases for A.

Theorem 7.8.4.

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric iff there exists an orthonormal eigenbasis for A.

Proof: I'll prove the reverse implication in these notes. Your text has a complete proof of the forward implication in Appendix C, it's very neat, but we don't have that much time. Assume there exists and orthonormal eigenbasis $\{v_1, v_2, \ldots, v_n\}$ for A. Let $V = [v_1|v_2|\cdots|v_n]$ and use Proposition 7.8.3, $V^T A V = D$ where D is a diagonal matrix with the e-values down the diagonal. Clearly $D^T = D$. Transposing the equation yields $(V^T A V)^T = D$. Use the socks-shoes property for transpose to see $(V^T A V)^T = V^T A^T (V^T)^T = V^T A^T V$. We find that $V^T A^T V = V^T A V$. Multiply on the left by V and on the right by V^T and we find $A^T = A$ thus A is symmetric. \Box .

This theorem is a useful bit of trivia to know. But, be careful not to overstate the result. This theorem does not state that all diagonalizable matrices are symmetric.

Example 7.8.5. In Example 7.4.3 we found the e-values and e-vectors of
$$A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$$
 were

 $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$ with e-vectors

$$u_1 = \begin{bmatrix} 0\\1\\0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1\\0\\1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$

We argued in Example 7.6.4 that $\{u_1, u_2, u_3\}$ is an eigenbasis. In view of the Theorem above we know there is no way to perform the Gram-Schmidt process and get and orthonormal set of e-vectors for A. We could orthonormalize the basis, but it would not result in a set of e-vectors. We can be certain of this since A is not symmetric. I invite you to try Gram-Schmidt and see how the process spoils the e-values. The principle calculational observation is simply that when you add e-vectors with different e-values there is no reason to expect the sum is again an e-vector. There is an exception to my last observation, what is it?

Example 7.8.6. Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Observe that $det(A - \lambda I) = -\lambda(\lambda + 1)(\lambda - 3)$ thus $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3$. We can calculate orthonormal e-vectors of $v_1 = [1, 0, 0]^T$, $v_2 = \frac{1}{\sqrt{2}}[0, 1, -1]^T$

and $v_3 = \frac{1}{\sqrt{2}}[0,1,1]^T$. I invite the reader to check the validity of the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Its really neat that to find the inverse of a matrix of orthonormal e-vectors we need only take the $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

transpose; note	$\begin{bmatrix} 0\\ 0 \end{bmatrix}$	$\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$	$\frac{\frac{-1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$		$\frac{\frac{1}{\sqrt{2}}}{\frac{-1}{\sqrt{2}}}$	$\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$	=	0 0	$\begin{array}{c} 1 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 1 \end{array}$	
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We have all the tools we need to diagonalize a matrix. What then is the point? Why would we care if a matrix is diagonalized? One reason is the proposition that follows. Other reasons that come immediately to mind is taking the trace or determinant. Since we proved in Chapter 5 that the trace and determinant are invariant under a similarity transformation it follows that we can calculate them most easily if we have done a similarity transformation to the diagonal form of the matrix. If the matrix is diagonal then the determinant is just the product of the diagonals. In the final section of this chapter we return to the topic and examine a vareity of physical and geometric applications of diagonalization.

Proposition 7.8.7.

If $A \in \mathbb{R}^{n \times n}$ has e-vector v with eigenvalue λ then v is a e-vector of A^k with e-value λ^k .

Proof: let $A \in \mathbb{R}^{n \times n}$ have e-vector v with eigenvalue λ . Consider,

$$A^{k}v = A^{k-1}Av = A^{k-1}\lambda v = \lambda A^{k-2}Av = \lambda^{2}A^{k-2}v = \dots = \lambda^{k}v.$$

The \cdots is properly replaced by a formal induction argument. \Box .

7.9 calculus of matrices

A more apt title would be "calculus of matrix-valued functions of a real variable".

Definition 7.9.1.

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{m \times n}$. Suppose $A : I \subseteq \mathbb{R} \to \mathbb{R}^{m \times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for each i, j then we define

$$\frac{dA}{dt} = \left[\frac{dA_{ij}}{dt}\right]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int Adt = [\int A_{ij}dt]$ for A with integrable components. Definite integrals and higher derivatives are also defined component-wise.

Example 7.9.2. Suppose $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \qquad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \qquad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Integrate by integrating each component:

$$\int A(t)dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \qquad \int_0^2 A(t)dt = \begin{bmatrix} t^2 \Big|_0^2 & t^3 \Big|_0^2 \\ \\ t^4 \Big|_0^2 & t^5 \Big|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

Proposition 7.9.3.

Suppose A, B are matrix-valued functions of a real variable, f is a function of a real variable, c is a constant, and C is a constant matrix then

- 1. (AB)' = A'B + AB' (product rule for matrices)
- 2. (AC)' = A'C
- 3. (CA)' = CA'
- 4. (fA)' = f'A + fA'
- 5. (cA)' = cA'
- 6. (A+B)' = A' + B'

where each of the functions is evaluated at the same time t and I assume that the functions and matrices are differentiable at that value of t and of course the matrices A, B, C are such that the multiplications are well-defined. **Proof:** Suppose $A(t) \in \mathbb{R}^{m \times n}$ and $B(t) \in \mathbb{R}^{n \times p}$ consider,

$$\begin{aligned} (AB)'_{ij} &= \frac{d}{dt} ((AB)_{ij}) & \text{defn. derivative of matrix} \\ &= \frac{d}{dt} (\sum_k A_{ik} B_{kj}) & \text{defn. of matrix multiplication} \\ &= \sum_k \frac{d}{dt} (A_{ik} B_{kj}) & \text{linearity of derivative} \\ &= \sum_k \left[\frac{dA_{ik}}{dt} B_{kj} + A_{ik} \frac{dB_{kj}}{dt} \right] & \text{ordinary product rules} \\ &= \sum_k \frac{dA_{ik}}{dt} B_{kj} + \sum_k A_{ik} \frac{dB_{kj}}{dt} & \text{algebra} \\ &= (A'B)_{ij} + (AB')_{ij} & \text{defn. of matrix multiplication} \\ &= (A'B + AB')_{ij} & \text{defn. matrix addition} \end{aligned}$$

this proves (1.) as i, j were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since C constant means C' = 0. Proof of (4.) is similar to (1.):

 $(fA)'_{ij} = \frac{d}{dt}((fA)_{ij})$ defn. derivative of matrix $= \frac{d}{dt}(fA_{ij})$ defn. of scalar multiplication $= \frac{df}{dt}A_{ij} + f\frac{dA_{ij}}{dt}$ ordinary product rule $= (\frac{df}{dt}A + f\frac{dA}{dt})_{ij}$ defn. matrix addition $= (\frac{df}{dt}A + f\frac{dA}{dt})_{ij}$ defn. scalar multiplication.

The proof of (5.) follows from taking f(t) = c which has f' = 0. I leave the proof of (6.) as an exercise for the reader. \Box .

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

Since we're discussing this type of differentiation perhaps it would be worthwhile for me to insert a comment about complex functions here. Differentiation of functions from \mathbb{R} to \mathbb{C} is defined by splitting a given function into its real and imaginary parts then we just differentiate with respect to the real variable one component at a time. For example:

$$\frac{d}{dt}(e^{2t}\cos(t) + ie^{2t}\sin(t)) = \frac{d}{dt}(e^{2t}\cos(t)) + i\frac{d}{dt}(e^{2t}\sin(t))
= (2e^{2t}\cos(t) - e^{2t}\sin(t)) + i(2e^{2t}\sin(t) + e^{2t}\cos(t))$$
(7.1)

$$= e^{2t}(2+i)(\cos(t) + i\sin(t))
= (2+i)e^{(2+i)t}$$

where I have made use of the identity $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. We just saw that

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

 $^{^9 {\}rm or}$ definition, depending on how you choose to set-up the complex exponential, I take this as the definition in calculus II

which seems obvious enough until you appreciate that we just proved it for $\lambda = 2 + i$. We make use of this calculation in the next section in the case we have complex e-values.

7.10 introduction to systems of linear differential equations

A differential equation (DEqn) is simply an equation that is stated in terms of derivatives. The highest order derivative that appears in the DEqn is called the *order* of the DEqn. In calculus we learned to integrate. Recall that $\int f(x)dx = y$ iff $\frac{dy}{dx} = f(x)$. Everytime you do an integral you are solving a first order DEqn. In fact, it's an *ordinary* DEnq (ODE) since there is only one indpendent variable (it was x). A system of ODEs is a set of differential equations with a common independent variable. It turns out that any linear differential equation can be written as a system of ODEs in *normal form*. I'll define *normal form* then illustrate with a few examples.

Definition 7.10.1.

Let t be a real variable and suppose x_1, x_2, \ldots, x_n are functions of t. If A_{ij}, f_i are functions of t for all $1 \le i \le m$ and $1 \le j \le n$ then the following set of differential equations is defined to be a system of linear differential equations in **normal form**:

$$\frac{dx_1}{dt} = A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n + f_1$$
$$\frac{dx_2}{dt} = A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n + f_2$$
$$\vdots = \vdots \quad \vdots \quad \dots \quad \vdots$$
$$\frac{dx_m}{dt} = A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n + f_n$$

In matrix notation, $\frac{dx}{dt} = Ax + f$. The system is called homogeneous if f = 0 whereas the system is called nonhomogeneous if $f \neq 0$. The system is called **constant coefficient** if $\frac{d}{dt}(A_{ij}) = 0$ for all i, j. If m = n and a set of initial conditions $x_1(t_0) = y_1, x_2(t_0) = y_2, \ldots, x_n(t_0) = y_n$ are given then this is called an **initial value problem** (IVP).

Example 7.10.2. If x is the number of tigers and y is the number of rabbits then

$$\frac{dx}{dt} = x + y \qquad \qquad \frac{dy}{dt} = -100x + 20y$$

is a model for the population growth of tigers and bunnies in some closed environment. My logic for my made-up example is as follows: the coefficient 1 is the growth rate for tigers which don't breed to quickly. Whereas the growth rate for bunnies is 20 since bunnies reproduce like, well bunnies. Then the y in the $\frac{dx}{dt}$ equation goes to account for the fact that more bunnies means more tiger food and hence the tiger reproduction should speed up (this is probably a bogus term, but this is my made up example so deal). Then the -100x term accounts for the fact that more tigers means more tigers eating bunnies so naturally this should be negative. In matrix form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -100 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

How do we solve such a system? This is the question we seek to answer.

The preceding example is a *predator-prey* model. There are many other terms that can be added to make the model more realistic. Ultimately all population growth models are only useful if they can account for all significant effects. History has shown population growth models are of only limited use for humans.

Example 7.10.3. Reduction of Order in calculus II you may have studied how to solve y'' + by' + cy = 0 for any choice of constants b, c. This is a second order ODE. We can reduce it to a system of first order ODEs by introducing new variables: $x_1 = y$ and $x_2 = y'$ then we have

$$x_1' = y' = x_2$$

and,

$$x_2' = y'' = -by' - cy = -bx_2 - cx_1$$

As a matrix DEqn,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly if y''' + 2y'' + 3y'' + 4y' + 5y = 0 we can introduce variables to reduce the order: $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ then you can show:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is equivalent to y'''' + 2y''' + 3y'' + 4y' + 5y = 0. We call the matrix above the **companion matrix** of the n-th order constant coefficient ODE. There is a beautiful interplay between solutions to n-th order ODEs and the linear algebra of the compansion matrix.

Example 7.10.4. Suppose y'' + 4y' + 5y = 0 and x'' + x = 0. The is a system of linear second order ODEs. It can be recast as a system of 4 first order ODEs by introducing new variables: $x_1 = y, x_2 = y'$ and $x_3 = x, x_4 = x'$. In matrix form the given system in normal form is:

$\begin{bmatrix} x_1 \end{bmatrix}' \parallel$	0	1	0	0]	$\begin{bmatrix} x_1 \end{bmatrix}$
x_2	-5	$-4 \\ 0$	$\begin{array}{c} 0 \\ 0 \end{array}$	0	x_2
$\left \begin{array}{c} x_2 \\ x_3 \end{array}\right =$	0	0	-	1	x_3
$\begin{bmatrix} x_4 \end{bmatrix}$	0	0	-1	0	$\begin{bmatrix} x_4 \end{bmatrix}$

The companion matrix above will be found to have eigenvalues $\lambda = -2 \pm i$ and $\lambda = \pm i$. I know this without further calculation purely on the basis of what I know from DEqns and the interplay I alluded to in the last example.

Example 7.10.5. If y''' + 2y'' + y = 0 we can introduce variables to reduce the order: $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ then you can show:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

7.11. THE MATRIX EXPONENTIAL

is equivalent to y''' + 2y'' + y = 0. If we solve the matrix system then we solve the equation in y and vice-versa. I happen to know the solution to the y equation is $y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$. From this I can deduce that the companion matrix has a repeated e-value of $\lambda = \pm i$ and just one complex e-vector and its conjugate. This matrix would answer the bonus point question I posed a few sections back. I invite the reader to verify my claims.

Remark 7.10.6.

For those of you who will or have taken math 334 my guesswork above is predicated on two observations:

- 1. the "auxillarly" or "characteristic" equation in the study of the constant coefficient ODEs is identical to the characteristic equation of the companion matrix.
- 2. ultimately eigenvectors will give us exponentials and sines and cosines in the solution to the matrix ODE whereas solutions which have multiplications by t stem from generalized e-vectors. Conversely, if the DEqn has a t or t^2 multiplying cosine, sine or exponential functions then the companion matrix must in turn have generalized e-vectors to account for the t or t^2 etc...

I will not explain (1.) in this course, however we will hopefully make sense of (2.) by the end of this section.

7.11 the matrix exponential

Perhaps the most important first order ODE is $\frac{dy}{dt} = ay$. This DEqn says that the rate of change in y is simply proportional to the amount of y at time t. Geometrically, this DEqn states the solutions value is proportional to its slope at every point in its domain. The solution¹⁰ is the exponential function $y(t) = e^{at}$.

We face a new differential equation; $\frac{dx}{dt} = Ax$ where x is a vector-valued function of t and $A \in \mathbb{R}^{n \times n}$. Given our success with the exponential function for the scalar case is it not natural to suppose that $x = e^{tA}$ is the solution to the matrix DEqn? The answer is yes. However, we need to define a few items before we can understand the true structure of the claim.

Definition 7.11.1.

Let $A\mathbb{R}^{n \times n}$ define $e^A \in \mathbb{R}^{n \times n}$ by the following formula

$$e^{A} = \sum_{n=0}^{\infty} \frac{1}{n!} A^{n} = I + A + \frac{1}{2}A^{2} + \frac{1}{3!}A^{3} + \cdots$$

We also denote $e^A = exp(A)$ when convenient.

¹⁰ok, technically separation of variables yields the general solution $y = ce^{at}$ but I'm trying to focus on the exponential function for the moment.

This definition is the natural extension of the Taylor series formula for the exponential function we derived in calculus II. Of course, you should be skeptical of this definition. How do I even know the series converges for an arbitrary matrix A? And, what do I even mean by "converge" for a series of matrices? (skip the next subsection if you don't care)

7.11.1 analysis for matrices

Remark 7.11.2.

The purpose of this section is to alert the reader to the gap in the development here. We will use the matrix exponential despite our inability to fully grasp the underlying analysis. Much in the same way we calculate series in calculus without proving every last theorem. I will attempt to at least sketch the analytical underpinnings of the matrix exponential. The reader will be happy to learn this is not part of the required material.

We use the Frobenius norm for $A \in \mathbb{R}^{n \times n}$, $||A|| = \sqrt{\sum_{i,j} (A_{ij})^2}$. We already proved this was a norm in a previous chapter. A sequence of square matrices is a function from \mathbb{N} to $\mathbb{R}^{n \times n}$. We say the sequence $\{A_n\}_{n=1}^{\infty}$ converges to $L \in \mathbb{R}^{n \times n}$ iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $||A_n - L|| < \epsilon$ for all n > M. This is the same definition we used in calculus, just now the norm is the Frobenius norm and the functions are replaced by matrices. The definition of a series is also analogue to the definition you learned in calculus II.

Definition 7.11.3.

Let $A_k \in \mathbb{R}^{m \times m}$ for all k, the sequence of partial sums of $\sum_{k=0}^{\infty} A_k$ is given by $S_n = \sum_{k=1}^{n} A_k$. We say the series $\sum_{k=0}^{\infty} A_k$ converges to $L \in \mathbb{R}^{m \times m}$ iff the sequence of partial sums converges to L. In other words,

$$\sum_{k=1}^{\infty} A_k = \lim_{n \to \infty} \sum_{k=1}^n A_k$$

Many of the same theorems hold for matrices:

Proposition 7.11.4.

Let $t \to S_A(t) = \sum A_k(t)$ and $t \to S_B(t) = \sum_k B_k(t)$ be matrix valued functions of a real variable t where the series are uniformly convergent and $c \in \mathbb{R}$ then

1.
$$\sum_{k} cA_{k} = c \sum_{k} A_{k}$$

2.
$$\sum_{k} (A_k + B_k) = \sum_{k} A_k + \sum_{k} B_k$$

3.
$$\frac{d}{dt} \left[\sum_k A_k \right] = \sum_k \frac{d}{dt} \left[A_k \right]$$

4. $\int \left[\sum_{k} A_{k}\right] dt = C + \sum_{k} \int A_{k} dt$ where C is a constant matrix.

The summations can go to infinity and the starting index can be any integer.

Uniform convergence means the series converge without regard to the value of t. Let me just refer you to the analysis course, we should discuss uniform convergence in that course, the concept equally well applies here. It is the crucial fact which one needs to interchange the limits which are implicit within \sum_k and $\frac{d}{dt}$. There are counterexamples in the case the series is not uniformly convergent. Fortunately,

Proposition 7.11.5.

Let A be a square matrix then $exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ is a uniformly convergent series of matrices.

Basically, the argument is as follows: The set of square matrices with the Frobenius norm is isometric to \mathbb{R}^{n^2} which is a complete space. A complete space is one in which every Cauchy sequence converges. We can show that the sequence of partial sums for exp(A) is a Cauchy sequence in $\mathbb{R}^{n \times n}$ hence it converges. Obviously I'm leaving some details out here. You can look at the excellent *Calculus* text by Apostle to see more gory details. Also, if you don't like my approach to the matrix exponential then he has several other ways to look it.

(Past this point I expect you to start following along again.)

7.11.2 formulas for the matrix exponential

Now for the fun part.

Proposition 7.11.6.

Let A be a square matrix then $\frac{d}{dt} [exp(tA)] = Aexp(tA)$

Proof: I'll give the proof in two notations. First,

$$\frac{d}{dt} \left[exp(tA) \right] = \frac{d}{dt} \left[\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k \right]$$
$$= \sum_{k=0}^{\infty} \frac{d}{dt} \left[\frac{1}{k!} t^k A^k \right]$$
$$= \sum_{k=0}^{\infty} \frac{k}{k!} t^{k-1} A^k$$
$$= A \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} A^{k-1}$$
$$= A exp(tA)$$

defn. of matrix exponential

since matrix exp. uniformly conv.

 A^k constant and $\frac{d}{dt}(t^k) = kt^{k-1}$

since
$$k! = k(k-1)!$$
 and $A^k = AA^{k-1}$.

defn. of matrix exponential.

I suspect the following argument is easier to follow:

$$\frac{d}{dt}(exp(tA)) = \frac{d}{dt}(I + tA + \frac{1}{2}t^{2}A^{2} + \frac{1}{3!}t^{3}A^{3} + \cdots)
= \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \frac{1}{2}\frac{d}{dt}(t^{2}A^{2}) + \frac{1}{3\cdot 2}\frac{d}{dt}(t^{3}A^{3}) + \cdots
= A + tA^{2} + \frac{1}{2}t^{2}A^{3} + \cdots
= A(I + tA + \frac{1}{2}t^{2}A^{2} + \cdots)
= Aexp(tA).$$

Notice that we have all we need to see that exp(tA) is a matrix of solutions to the differential equation x' = Ax. The following prop. follows from the preceding prop. and Prop. 2.3.11.

Proposition 7.11.7.

If X = exp(tA) then X' = Aexp(tA) = AX. This means that each column in X is a solution to x' = Ax.

Let us illustrate this proposition with a particularly simple example.

Example 7.11.8. Suppose x' = x, y' = 2y, z' = 3z then in matrix form we have:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The coefficient matrix is diagonal which makes the k-th power particularly easy to calculate,

$$\begin{aligned} A^{k} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{k} & 0 \\ 0 & 0 & 3^{k} \end{bmatrix} \\ \Rightarrow & exp(tA) = \sum_{k=0}^{\infty} \frac{t^{k}}{k!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{k} & 0 \\ 0 & 0 & 3^{k} \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^{k}}{k!} 1^{k} & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^{k}}{k!} 2^{k} & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^{k}}{k!} 3^{k} \end{bmatrix} \\ \Rightarrow & exp(tA) = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \end{aligned}$$

Thus we find three solutions to x' = Ax,

$$x_1(t) = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} \qquad x_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

In turn these vector solutions amount to the solutions $x = e^t$, y = 0, z = 0 or x = 0, $y = e^{2t}$, z = 0 or x = 0, y = 0, $z = e^{3t}$. It is easy to check these solutions.

Usually we cannot calculate the matrix exponential explicitly by such a straightforward calculation. We need e-vectors and sometimes generalized e-vectors to reliably calculate the solutions of interest.

Proposition 7.11.9.

If A, B are square matrices such that
$$AB = BA$$
 then $e^{A+B} = e^A e^B$

Proof: I'll show how this works for terms up to quadratic order,

$$e^{A}e^{B} = (1 + A + \frac{1}{2}A^{2} + \dots)(1 + B + \frac{1}{2}B^{2} + \dots) = 1 + (A + B) + \frac{1}{2}A^{2} + AB + \frac{1}{2}B^{2} + \dots$$

However, since AB = BA and

$$(A+B)^{2} = (A+B)(A+B) = A^{2} + AB + BA + B^{2} = A^{2} + 2AB + B^{2}.$$

Thus,

$$e^{A}e^{B} = 1 + (A+B) + \frac{1}{2}(A+B)^{2} + \dots = e^{A+B}$$

You might wonder what happens if $AB \neq BA$. In this case we can account for the departure from commutativity by the **commutator** of A and B.

Definition 7.11.10.

Let $A, B \in \mathbb{R}^{n \times n}$ then the **commutator** of A and B is [A, B] = AB - BA.

Proposition 7.11.11.

Let $A, B, C \in \mathbb{R}^{n \times n}$ then 1. [A, B] = -[B, A]2. [A + B, C] = [A, C] + [B, C]3. [AB, C] = A[B, C] + [A, C]B4. [A, BC] = B[A, C] + [A, B]C5. [[A, B], C] + [[B, C], A] + [[C, A], B] = 0

The proofs of the properties above are not difficult. In contrast, the following formula known as the Baker-Campbell-Hausdorff (BCH) relation takes considerably more calculation:

 $e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]++\frac{1}{12}[[B,A],A]+\cdots}$ BCH-formula

The higher order terms can also be written in terms of nested commutators. What this means is that if we know the values of the commutators of two matrices then we can calculate the product of their exponentials with a little patience. This connection between multiplication of exponentials and commutators of matrices is at the heart of Lie theory. Actually, mathematicians have greatly abstracted the idea of Lie algebras and Lie groups way past matrices but the concrete example of matrix Lie groups and algebras is perhaps the most satisfying. If you'd like to know more just ask. It would make an excellent topic for an independent study that extended this course.

Remark 7.11.12.

In fact the *BCH* holds in the abstract as well. For example, it holds for the Lie algebra of derivations on smooth functions. A *derivation* is a linear differential operator which satisfies the product rule. The derivative operator is a derivation since D[fg] = D[f]g + fD[g]. The commutator of derivations is defined by [X, Y][f] = X(Y(f)) - Y(X(f)). It can be shown that [D, D] = 0 thus the BCH formula yields

$$e^{aD}e^{bD} = e^{(a+b)D}.$$

If the coefficient of D is thought of as position then multiplication by e^{bD} generates a translation in the position. By the way, we can state Taylor's Theorem rather compactly in this operator notation: $f(x+h) = exp(hD)f(x) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

Proposition 7.11.13.

Let $A, P \in \mathbb{R}^{n \times n}$ and assume P is invertible then

$$exp(P^{-1}AP) = P^{-1}exp(A)P$$

Proof: this identity follows from the following observation:

$$(P^{-1}AP)^{k} = P^{-1}APP^{-1}APP^{-1}AP \cdots P^{-1}AP = P^{-1}A^{k}P.$$

Thus $exp(P^{-1}AP) = \sum_{k=0}^{\infty} \frac{1}{k!} (P^{-1}AP)^k = P^{-1} (\sum_{k=0}^{\infty} \frac{1}{k!} A^k) P = P^{-1} exp(A) P.$

Proposition 7.11.14.

Let A be a square matrix, det(exp(A)) = exp(trace(A)).

Proof: If the matrix A is diagonalizable then the proof is simple. Diagonalizability means there exists invertible $P = [v_1|v_2|\cdots|v_n]$ such that $P^{-1}AP = D = [\lambda_1v_1|\lambda_2v_2|\cdots|\lambda_nv_n]$ where v_i is an e-vector with e-value λ_i for all *i*. Use the preceding proposition to calculate

$$\det(exp(D)) = \det(exp(P^{-1}AP) = \det(P^{-1}exp(A)P) = \det(P^{-1}P)\det(exp(A)) = \det(exp(A))$$

On the other hand, the trace is cyclic trace(ABC) = trace(BCA)

$$trace(D) = trace(P^{-1}AP) = trace(PP^{-1}A) = trace(A)$$

But, we also know D is diagonal with eigenvalues on the diagonal hence exp(D) is diagonal with e^{λ_i} on the corresponding diagonals

$$\det(exp(D)) = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} \text{ and } trace(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Finally, use the laws of exponents to complete the proof,

$$e^{trace(A)} = e^{trace(D)} = e^{\lambda_1 + \lambda_2 + \dots + \lambda_n} = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = det(exp(D)) = det(exp(A)).$$

I've seen this proof in texts presented as if it were the general proof. But, not all matrices are diagonalizable so this is a curious proof. I stated the proposition for an arbitrary matrix and I meant it. The proof, the real proof, is less obvious. Let me sketch it for you:

better proof: The preceding proof shows it may be hopeful to suppose that det(exp(tA)) = exp(t trace(A)) for $t \in \mathbb{R}$. Notice that $y = e^{kt}$ satisfies the differential equation $\frac{dy}{dt} = ky$. Conversely, if $\frac{dy}{dt} = ky$ for some constant k then the general solution is given by $y = c_o e^{kt}$ for some $c_o \in \mathbb{R}$. Let f(t) = det(exp(tA)). If we can show that f'(t) = trace(A)f(t) then we can conclude $f(t) = c_0 e^{ttrace(A)}$. Consider:

$$\begin{aligned} f'(t) &= \frac{d}{dh} \left(f(t+h) \Big|_{h=0} \\ &= \frac{d}{dh} \left(det(exp[(t+h)A]) \Big|_{h=0} \\ &= \frac{d}{dh} \left(det(exp[tA+hA]) \right|_{h=0} \\ &= \frac{d}{dh} \left(det(exp[tA]exp[hA]) \Big|_{h=0} \\ &= det(exp[tA]) \frac{d}{dh} \left(det(exp[hA]) \Big|_{h=0} \\ &= f(t) \frac{d}{dh} \left(det(I+hA+\frac{1}{2}h^2A^2+\frac{1}{3!}h^3A^3+\cdots) \right|_{h=0} \\ &= f(t) \frac{d}{dh} \left(det(I+hA) \right) \Big|_{h=0} \end{aligned}$$

Let us discuss the $\frac{d}{dh}(det(I + hA))$ term separately for a moment:¹¹

$$\begin{aligned} \frac{d}{dh}(det(I+hA)) &= \frac{d}{dh} [\sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2\dots i_n} (I+hA)_{i_1 1} (I+hA)_{i_2 2} \cdots (I+hA)_{i_n n}]_{h=0} \\ &= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2\dots i_n} \frac{d}{dh} [(I+hA)_{1 i_1} (I+hA)_{1 i_2} \cdots (I+hA)_{n i_n}]_{h=0} \\ &= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2\dots i_n} (A_{1 i_1} I_{1 i_2} \cdots I_{n i_n} + I_{1 i_1} A_{2 i_2} \cdots I_{n i_n} + \dots + I_{1 i_1} I_{2 i_2} \cdots A_{n i_n}) \\ &= \sum_{i_1} \epsilon_{i_1 2\dots n} A_{1 i_1} + \sum_{i_2} \epsilon_{1 i_2 \dots n} A_{2 i_2} + \dots + \sum_{i_n} \epsilon_{1 2\dots I_n} A_{n i_n} \\ &= A_{11} + A_{22} + \dots + A_{nn} \\ &= trace(A) \end{aligned}$$

Therefore, f'(t) = trace(A)f(t) consequently, $f(t) = c_o e^{t trace(A)} = det(exp(tA))$. However, we can resolve c_o by calculating $f(0) = det(exp(0)) = det(I) = 1 = c_o$ hence

$$e^{t\,trace(A)} = det(exp(tA))$$

Take t = 1 to obtain the desired result. \Box

Remark 7.11.15.

The formula det(exp(A)) = exp(trace(A)) is very important to the theory of matrix Lie groups and Lie algebras. Generically, if G is the Lie group and \mathfrak{g} is the Lie algebra then they are connected via the matrix exponential: $exp : \mathfrak{g} \to G_o$ where I mean G_o to denoted the connected component of the identity. For example, the set of all nonsingular matrices GL(n) forms a Lie group which is disconnected. Half of GL(n) has positive determinant whereas the other half has negative determinant. The set of all $n \times n$ matrices is denoted gl(n) and it can be shown that exp(gl(n)) maps onto the part of GL(n) which has positive determinant. One can even define a matrix logarithm map which serves as a local inverse for the matrix exponential near the identity. Generally the matrix exponential is not injective thus some technical considerations must be discussed before we could put the matrix log on a solid footing. This would take us outside the scope of this course. However, this would be a nice topic to do a follow-up independent study. The theory of matrix Lie groups and their representations is ubiqitious in modern quantum mechanical physics.

¹¹I use the definition of the identity matrix $I_{ij} = \delta_{ij}$ in eliminating all but the last summation in the fourth line. Then the levi-civita symbols serve the same purpose in going to the fifth line as $\epsilon_{i_12...n} = \delta_{1i_1}, \epsilon_{1i_2...n} = \delta_{2i_2}$ etc...

Finally, we come to the formula that is most important to our study of systems of DEqns. Let's call this the magic formula.

Proposition 7.11.16.

Let
$$\lambda \in \mathbb{C}$$
 and suppose $A \in \mathbb{R}^{n \times n}$ then

$$exp(tA) = e^{\lambda t} (I + t(A - \lambda I) + \frac{t^2}{2} (A - \lambda I)^2 + \frac{t^3}{3!} (A - \lambda I)^3 + \cdots).$$

Proof: Notice that $tA = t(A - \lambda I) + t\lambda I$ and $t\lambda I$ commutes with all matrices thus,

$$exp(tA) = exp(t(A - \lambda I) + t\lambda I)$$

= $exp(t(A - \lambda I))exp(t\lambda I)$
= $e^{\lambda t}exp(t(A - \lambda I))$
= $e^{\lambda t}(I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots)$

In the third line I used the identity proved below,

$$exp(t\lambda I) = I + t\lambda I + \frac{1}{2}(t\lambda)^2 I^2 + \dots = I(1 + t\lambda + \frac{(t\lambda)^2}{2} + \dots) = Ie^{t\lambda}. \qquad \Box$$

While the proofs leading up to the magic formula only dealt with real matrices it is not hard to see the proofs are easily modified to allow for complex matrices.

7.12 solutions for systems of DEqns with real eigenvalues

Let us return to the problem of solving $\vec{x}' = A\vec{x}$ for a constant square matrix A where $\vec{x} = [x_1, x_2, \ldots, x_n]$ is a vector of functions of t. I'm adding the vector notation to help distinguish the scalar function x_1 from the vector function \vec{x}_1 in this section. Let me state one theorem from the theory of differential equations. The existence of solutions theorem which is the heart of of this theorem is fairly involved to prove, you'll find it in one of the later chapters of the differential equations text by Nagel Saff and Snider.

Theorem 7.12.1.

If $\vec{x}' = A\vec{x}$ and A is a constant matrix then any solution to the system has the form

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \dots + c_n \vec{x}_n(t)$$

where $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a linearly independent set of solutions defined on \mathbb{R} (this is called the **fundamental solution set**). Moreover, these fundamental solutions can be concatenated into a single invertible solution matrix called the **fundamental matrix** $X = [\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n]$ and the general solution can be expressed as $\vec{x}(t) = X(t)\vec{c}$ where \vec{c} is an arbitrary vector of real constants. If an initial condition $\vec{x}(t_o) = \vec{x}_o$ is given then the solution to the IVP is $\vec{x}(t) = X^{-1}(t_o)X(t)\vec{x}_o$.

We proved in the previous section that the matrix exponential exp(tA) is a solution matrix and the inverse is easy enought to guess: $exp(tA)^{-1} = exp(-tA)$. This proves the columns of exp(tA) are solutions to $\vec{x}' = A\vec{x}$ which are linearly independent and as such form a fundamental solution set.

<u>Problem</u>: we cannot directly calculate exp(tA) for most matrices A. We have a solution we can't calculate. What good is that?

When can we explicitly calculate exp(tA) without much thought? Two cases come to mind: (1.) if A is diagonal then it's easy, saw this in Example 7.11.8, (2.) if A is a **nilpotent** matrix then there is some finite power of the matrix which is zero; $A^k = 0$. In the nilpotent case the infinite series defining the matrix exponential truncates at order k:

$$exp(tA) = I + tA + \frac{t^2}{2}A^2 + \dots + \frac{t^{k-1}}{(k-1)!}A^{k-1}$$

Example 7.12.2. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we calculate $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ thus $exp(tA) = I + tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

Incidentally, the solution to $\vec{x}' = A\vec{x}$ is generally $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}$. In other words, $x_1(t) = c_2 + c_2 t$ whereas $x_2(t) = c_2$. These solutions are easily seen to solve the system $x'_1 = x_2$ and $x'_2 = 0$.

Unfortunately, the calculation we just did in the last example almost never works. For example, try to calculate an arbitrary power of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, let me know how it works out. We would like for all examples to truncate. The **magic formula** gives us a way around this dilemma:

Proposition 7.12.3.

Let $A \in \mathbb{R}^{n \times n}$. Suppose v is an e-vector with e-value λ then $exp(tA)v = e^{\lambda t}v$.

Proof: we are given that $(A - \lambda I)v = 0$ and it follows that $(A - \lambda I)^k v = 0$ for all $k \ge 1$. Use the magic formula,

$$exp(tA)v = e^{\lambda t}(I + t(A - \lambda I) + \dots)v = e^{\lambda t}(Iv + t(A - \lambda I)v + \dots = e^{\lambda t}v$$

noting all the higher order terms vanish since $(A - \lambda I)^k v = 0$. \Box

We can't hope for the matrix exponential itself to truncate, but when we multiply exp(tA) on an e-vector something special happens. Since $e^{\lambda t} \neq 0$ the set of vector functions $\{e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2, \ldots, e^{\lambda_k t}v_k\}$ will be linearly independent if the e-vectors v_i are linearly independent. If

the matrix A is diagonalizable then we'll be able to find enough e-vectors to construct a fundamental solution set using e-vectors alone. However, if A is not diagonalizable, and has only real e-values, then we can still find a Jordan basis $\{v_1, v_2, \ldots, v_n\}$ which consists of generalized e-vectors and it follows that $\{e^{tA}v_1, e^{tA}v_2, \ldots, e^{tA}v_n\}$ forms a fundamental solution set. Moreover, this is not just of theoretical use. We can actually calculate this solution set.

Proposition 7.12.4.

Let $A \in \mathbb{R}^{n \times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized e-vectors with e-value λ , meaning $(A - \lambda)v_1 = 0$ and $(A - \lambda)v_{k-1} = v_k$ for $k \ge 2$, then 1. $e^{tA}v_1 = e^{\lambda t}v_1$, 2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$, 3. $e^{tA}v_3 = e^{\lambda t}(v_3 + tv_2 + \frac{t^2}{2}v_1)$, 4. $e^{tA}v_k = e^{\lambda t}(v_k + tv_{k-1} + \dots + \frac{t^{k-1}}{(k-1)!}v_1)$.

Proof: Study the chain condition,

$$(A - \lambda I)v_2 = v_1 \quad \Rightarrow \quad (A - \lambda)^2 v_2 = (A - \lambda I)v_1 = 0$$
$$(A - \lambda I)v_3 = v_2 \quad \Rightarrow \quad (A - \lambda I)^2 v_3 = (A - \lambda I)v_2 = v_1$$

Continuing with such calculations¹² we find $(A - \lambda I)^j v_i = v_{i-j}$ for all i > j and $(A - \lambda I)^i v_i = 0$. The magic formula completes the proof:

$$e^{tA}v_2 = e^{\lambda t} (v_2 + t(A - \lambda I)v_2 + \frac{t^2}{2}(A - \lambda I)^2 v_2 \cdots) = e^{\lambda t} (v_2 + tv_1)$$

likewise,

$$e^{tA}v_3 = e^{\lambda t} \left(v_3 + t(A - \lambda I)v_3 + \frac{t^2}{2}(A - \lambda I)^2 v_3 + \frac{t^3}{3!}(A - \lambda I)^3 v_3 + \cdots \right)$$

= $e^{\lambda t} \left(v_3 + tv_2 + \frac{t^2}{2}(A - \lambda I)v_2 \right)$
= $e^{\lambda t} \left(v_3 + tv_2 + \frac{t^2}{2}v_1 \right).$

We already proved the e-vector case in the preceding proposition and the general case follows from essentially the same calculation. \Box

We have all the theory we need to solve systems of homogeneous constant coefficient ODEs.

 $^{^{12}}$ keep in mind these conditions hold because of our current labling scheme, if we used a different indexing system then you'd have to <u>think</u> about how the chain conditions work out, to test your skill perhaps try to find the general solution for the system with the matrix from Example 7.6.14

Example 7.12.5. Recall Example 7.4.1 we found $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ had e-values $\lambda_1 = 0$ and $\lambda_2 = 4$ and corresponding e-vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
 and $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

thus we find the general solution to $\vec{x}' = A\vec{x}$ is simply,

$$\vec{x}(t) = c_1 \begin{bmatrix} 1\\ -3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1\\ 1 \end{bmatrix}$$

just to illustrate the terms: we have fundmamental solution set and matrix:

$$\left\{ \left[\begin{array}{c} 1\\ -3 \end{array} \right], \left[\begin{array}{c} e^{4t}\\ e^{4t} \end{array} \right] \right\} \qquad X = \left[\begin{array}{c} 1 & e^{4t}\\ -3 & e^{4t} \end{array} \right]$$

Notice that a different choice of e-vector scaling would just end up adjusting the values of c_1, c_2 in the event an initial condition was given. This is why different choices of e-vectors still gives us the same general solution. It is the flexibility to change c_1, c_2 that allows us to fit any initial condition.

Example 7.12.6. We can modify Example 7.10.2 and propose a different model for a tiger/bunny system. Suppose x is the number of tigers and y is the number of rabbits then

$$\frac{dx}{dt} = x - 4y \qquad \qquad \frac{dy}{dt} = -10x + 19y$$

is a model for the population growth of tigers and bunnies in some closed environment. Suppose that there is initially 2 tigers and 100 bunnies. Find the populations of tigers and bunnies at time t > 0:

<u>Solution</u>: notice that we must solve $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & -4 \\ -10 & 19 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 2, 100 \end{bmatrix}^T$. We can calculate the eigenvalues and corresponding eigenvectors:

$$det(A - \lambda I) = 0 \implies \lambda_1 = -1, \ \lambda_2 = 21 \implies u_1 = \begin{bmatrix} 2\\1 \end{bmatrix}, \ u_2 = \begin{bmatrix} -1\\5 \end{bmatrix}$$

Therefore, using Proposition 7.12.4, the general solution has the form:

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2\\1 \end{bmatrix} + c_2 e^{21t} \begin{bmatrix} -1\\5 \end{bmatrix}.$$

However, we also know that $\vec{x}(0) = [2, 100]^T$ hence

$$\begin{bmatrix} 2\\100 \end{bmatrix} = c_1 \begin{bmatrix} 2\\1 \end{bmatrix} + c_2 \begin{bmatrix} -1\\5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2\\100 \end{bmatrix} = \begin{bmatrix} 2&-1\\1&5 \end{bmatrix} \begin{bmatrix} c_1\\c_2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} c_1\\c_2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5&1\\-1&2 \end{bmatrix} \begin{bmatrix} 2\\100 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 110\\198 \end{bmatrix}$$

Finally, we find the vector-form of the solution to the given initial value problem:

$$\vec{x}(t) = 10e^{-t} \begin{bmatrix} 2\\1 \end{bmatrix} + \frac{198}{11}e^{21t} \begin{bmatrix} -1\\5 \end{bmatrix}$$

Which means that $x(t) = 20e^{-t} - \frac{198}{11}e^{21t}$ and $y(t) = 1020e^{-t} + 90e^{21t}$ are the number of tigers and bunnies respective at time t.

Notice that a different choice of e-vectors would have just made for a different choice of c_1, c_2 in the preceding example. Also, notice that when an initial condition is given there ought not be any undetermined coefficients in the final answer¹³.

Example 7.12.7. We found that in Example 7.4.3 the matrix $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ has e-values

 $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$ with corresponding e-vectors

$$\vec{u}_1 = \begin{bmatrix} 0\\1\\0 \end{bmatrix} \vec{u}_2 = \begin{bmatrix} -1\\0\\1 \end{bmatrix} \vec{u}_3 = \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$

Hence, using Proposition 7.12.4 and Theorem 7.12.1 the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is simply:

$$\vec{x}(t) = c_1 e^{4t} \vec{u}_1 + c_2 e^{4t} \vec{u}_2 + c_3 e^{2t} \vec{u}_3 = c_1 e^{4t} \begin{bmatrix} 0\\1\\0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -1\\0\\1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2\\1\\1 \end{bmatrix}$$

Example 7.12.8. Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that:

$$A = \left[\begin{array}{rrrrr} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

We analyzed this matrix in Example 7.6.14. We found a pair of chains of generalized e-vectors all with eigenvalue $\lambda = 1$ which satisfied the following conditions:

$$(A-I)\vec{u}_3 = \vec{u}_1, \ (A-I)\vec{u}_1 = 0 \qquad (A-I)\vec{u}_4 = \vec{u}_2, \ (A-I)\vec{u}_2 = 0$$

In particular, $\vec{u}_j = e_j$ for j = 1, 2, 3, 4. We can use the magic formula to extract 4 solutions from the matrix exponential, by Proposition 7.12.4 we find:

$$\vec{x}_{1} = e^{At}\vec{u}_{1} = e^{t}\vec{u}_{1} = e^{t}e_{1}$$

$$\vec{x}_{2} = e^{At}\vec{u}_{2} = e^{t}(e_{2} + te_{1})$$

$$\vec{x}_{3} = e^{At}\vec{u}_{3} = e^{t}e_{3}$$

$$\vec{x}_{4} = e^{At}\vec{u}_{4} = e^{t}(e_{4} + te_{3})$$
(7.2)

¹³Assuming of course that there are enough initial conditions given to pick a unique solution from the family of solutions which we call the "general solution".

Let's write the general solution in vector and scalar form, by Theorem 7.12.1,

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 + c_4 \vec{x}_4 = c_1 e^t e_1 + c_2 e^t (e_2 + te_1) + c_3 e^t e_3 + c_4 e^t (e_4 + te_3) = \begin{bmatrix} c_1 e^t + tc_2 e^t \\ c_2 e^t \\ c_3 e^t + tc_4 e^t \\ c_4 e^t \end{bmatrix}$$

In other words, $x_1(t) = c_1e^t + tc_2e^t$, $x_2(t) = c_2e^t$, $x_3(t) = c_3e^t + tc_4e^t$ and $x_4(t) = c_4e^t$ form the general solution to the given system of differential equations.

Example 7.12.9. Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given (generalized)eigenvectors \vec{u}_i , i = 1, 2, 3, 4, 5, 6, 7, 8, 9 such that:

$$(A-I)\vec{u}_1 = 0, \quad A\vec{u}_2 = \vec{u}_2, \quad A\vec{u}_3 = 7\vec{u}_3, \quad (A-I)\vec{u}_4 = \vec{u}_1$$

 $(A+5I)\vec{u}_5 = 0, \quad (A-3I)\vec{u}_6 = \vec{u}_7 \quad A\vec{u}_7 = 3\vec{u}_7, \quad A\vec{u}_8 = 0, \quad (A-3I)\vec{u}_9 = \vec{u}_6$

We can use the magic formula to extract 9 solutions from the matrix exponential, by Proposition 7.12.4 we find:

$$\vec{x}_{1} = e^{At}\vec{u}_{1} = e^{t}\vec{u}_{1} = e^{t}\vec{u}_{1}$$

$$\vec{x}_{2} = e^{At}\vec{u}_{2} = e^{t}\vec{u}_{2}$$

$$\vec{x}_{3} = e^{At}\vec{u}_{3} = e^{7t}\vec{u}_{3}$$

$$\vec{x}_{4} = e^{At}\vec{u}_{4} = e^{t}(\vec{u}_{4} + t\vec{u}_{1})$$

$$can you see why?$$

$$\vec{x}_{5} = e^{At}\vec{u}_{5} = e^{-5t}\vec{u}_{5}$$

$$\vec{x}_{6} = e^{At}\vec{u}_{6} = e^{3t}(\vec{u}_{6} + t\vec{u}_{7})$$

$$can you see why?$$

$$\vec{x}_{7} = e^{At}\vec{u}_{7} = e^{3t}\vec{u}_{7}$$

$$\vec{x}_{8} = e^{At}\vec{u}_{8} = \vec{u}_{8}$$

$$\vec{x}_{9} = e^{At}\vec{u}_{9} = e^{3t}(\vec{u}_{9} + t\vec{u}_{6} + \frac{1}{2}t^{2}\vec{u}_{7})$$

$$can you see why?$$

Let's write the general solution in vector and scalar form, by Theorem 7.12.1,

$$\vec{x}(t) = \sum_{i=1}^{9} c_i \vec{x}_i$$

where the formulas for each solution $\vec{x_i}$ was given above. If I was to give an explicit matrix A with the eigenvectors given above it would be a 9×9 matrix. Challenge: find the matrix exponential e^{At} in terms of the given (generalized)eigenvectors.

Hopefully the examples have helped the theory settle in by now. We have one last question to settle for systems of DEqns.

Theorem 7.12.10.

The nonhomogeneous case $\vec{x}' = A\vec{x} + \vec{f}$ the general solution is $\vec{x}(t) = X(t)c + \vec{x}_p(t)$ where X is a fundamental matrix for the corresponding homogeneous system and \vec{x}_p is a particular solution to the nonhomogeneous system. We can calculate $\vec{x}_p(t) = X(t) \int X^{-1} \vec{f} dt$.

Proof: suppose that $\vec{x}_p = X\vec{v}$ for X a fundamental matrix of $\vec{x}' = A\vec{x}$ and some vector of unknown functions \vec{v} . We seek conditions on \vec{v} which make \vec{x}_p satisfy $\vec{x}_p' = A\vec{x}_p + \vec{f}$. Consider,

$$(\vec{x}_p)' = (X\vec{v})' = X'\vec{v} + X\vec{v}' = AX\vec{v} + X\vec{v}$$

But, $\vec{x}_p' = A\vec{X}_p + \vec{f} = AX\vec{v} + \vec{f}$ hence

$$X\frac{d\vec{v}}{dt} = \vec{f} \quad \Rightarrow \quad \frac{d\vec{v}}{dt} = X^{-1}\vec{f}$$

Integrate to find $\vec{v} = \int X^{-1} \vec{f} dt$ therefore $x_p(t) = X(t) \int X^{-1} \vec{f} dt$. \Box

If you ever work through variation of parameters for higher order ODEqns then you should appreciate the calculation above. In fact, we can derive *n*-th order variation of parameters from converting the *n*-th order ODE by reduction of order to a system of *n* first order linear ODEs. You can show that the so-called Wronskian of the fundamental solution set is precisely the determinant of the fundamental matrix for the system $\vec{x}' = A\vec{x}$ where *A* is the companion matrix. I have this worked out in an old test from a DEqns course I taught at NCSU¹⁴

Example 7.12.11. Suppose that $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, find the general solution of the nonhomogenous DEqn $\vec{x}' = A\vec{x} + \vec{f}$. Recall that in Example 7.12.5 we found $\vec{x}' = A\vec{x}$ has fundamental matrix $X = \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}$. Use variation of parameters for systems of ODEs to constuct \vec{x}_p . First calculate the inverse of the fundamental matrix, for a 2 × 2 we know a formula:

$$X^{-1}(t) = \frac{1}{e^{4t} - (-3)e^{4t}} \begin{bmatrix} e^{4t} & -e^{4t} \\ 3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t} \end{bmatrix}$$

¹⁴see solution of Problem 6 in *www.supermath.info/ma*341f07test2_sol.pdf for the n = 2 case of this comment, also §6.4 of Nagel Saff and Snider covers *n*-th order variation of parameters if you want to see details

Thus,

$$\begin{split} x_p(t) &= X(t) \int \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3e^{-4t} & e^{-4t} \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt = \frac{1}{4} X(t) \int \begin{bmatrix} e^t - e^{-t} \\ 3e^{-3t} + e^{-5t} \end{bmatrix} dt \\ &= \frac{1}{4} \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix} \begin{bmatrix} e^t + e^{-t} \\ -e^{-3t} - \frac{1}{5}e^{-5t} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \\ -3(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \\ -3(e^t + e^{-t}) + e^{4t}(-e^{-3t} - \frac{1}{5}e^{-5t}) \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} e^t + e^{-t} - e^t - \frac{1}{5}e^{-t} \\ -3e^t - 3e^{-t} - e^t - \frac{1}{5}e^{-t} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} \frac{4}{5}e^{-t} \\ -4e^t - \frac{16}{5}e^{-t} \end{bmatrix} \end{split}$$

Therefore, the general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1\\ -3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1\\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} e^{-t}\\ -e^t - 4e^{-t} \end{bmatrix}.$$

The general scalar solutions implicit within the general vector solution $\vec{x}(t) = [x(t), y(t)]^T$ are

$$x(t) = c_1 + c_2 e^{4t} + \frac{1}{5}e^{-t} \qquad y(t) = -3c_1 + c_2 e^{4t} - \frac{1}{5}e^t - \frac{4}{5}e^{-t}.$$

I'll probably ask you to solve a 3×3 system in the homework. The calculation is nearly the same as the preceding example with the small inconvenience that finding the inverse of a 3×3 requires some calculation.

Remark 7.12.12.

You might wonder how would you solve a system of ODEs x' = Ax such that the coefficients A_{ij} are not constant. We will not cover such problems in this course. We do cover how to solve an n - th order ODE with nonconstant coefficients via series techniques in Math 334. It's probably possible to extend some of those techniques to systems. Laplace Transforms also extend to systems of ODEs. It's just a matter of algebra. Nontrivial algebra.

7.13 solutions for systems of DEqns with complex eigenvalues

The calculations in the preceding section still make sense for a complex e-value and complex e-vector. However, we usually need to find real solutions. How to fix this? The same way as always. We extract real solutions from the complex solutions. Fortunately, our previous work on linear independence of complex e-vectors insures that the resulting solution set will be linearly independent.

Proposition 7.13.1.

Let $A \in \mathbb{R}^{n \times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized complex e-vectors with e-value $\lambda = \alpha + i\beta$, meaning $(A - \lambda)v_1 = 0$ and $(A - \lambda)v_{k-1} = v_k$ for $k \ge 2$ and $v_j = a_j + ib_j$ for $a_j, b_j \in \mathbb{R}^{n \times 1}$ for each j, then 1. $e^{tA}v_1 = e^{\lambda t}v_1$, 2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$, 3. $e^{tA}v_3 = e^{\lambda t}(v_3 + tv_2 + \frac{t^2}{2}v_1)$, 4. $e^{tA}v_k = e^{\lambda t}(v_k + tv_{k-1} + \dots + \frac{t^{k-1}}{(k-1)!}v_1)$. Furthermore, the following are the 2k linearly independent real solutions that are implicit within the complex solutions above, 1. $x_1 = Re(e^{tA}v_1) = e^{\alpha t}[(\cos\beta t)a_1 - (\sin\beta t)b_1]$, 2. $x_2 = Im(e^{tA}v_1) = e^{\alpha t}[(\cos\beta t)a_1 + (\cos\beta t)b_1])$, 3. $x_3 = Re(e^{tA}v_2) = e^{\alpha t}[(\cos\beta t)(a_2 + ta_1) - (\sin\beta t)(b_2 + tb_1)]$, 4. $x_4 = Im(e^{tA}v_2) = e^{\alpha t}[(\cos\beta t)(a_2 + ta_1) + (\cos\beta t)(b_2 + tb_1)]$,

5.
$$x_5 = Re(e^{-t}v_3) = e^{\alpha t} [(\cos\beta t)(a_3 + ta_2 + \frac{t}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t}{2}b_1)],$$

6. $x_6 = Im(e^{tA}v_3) = e^{\alpha t} [(\cos\beta t)(a_3 + ta_2 + \frac{t^2}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t^2}{2}b_1)].$

Proof: the magic formula calculations of the last section just as well apply to the complex case. Furthermore, we proved that

$$Re\left[e^{\alpha t + i\beta t}(v + iw)\right] = e^{\alpha t}\left[(\cos\beta t)v - (\sin\beta t)w\right]$$

and

$$Im[e^{\alpha t+i\beta t}(v+iw)] = e^{\alpha t}[(\sin\beta tv + (\cos\beta t)w],$$

the proposition follows. \Box

Example 7.13.2. This example uses the results derived in Example 7.5.2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and find the e-values and e-vectors of the matrix. Observe that $det(A - \lambda I) = \lambda^2 + 1$ hence the eigevalues are $\lambda = \pm i$. We find $u_1 = [1, i]^T$. Notice that

$$u_1 = \begin{bmatrix} 1\\i \end{bmatrix} = \begin{bmatrix} 1\\0 \end{bmatrix} + i \begin{bmatrix} 0\\1 \end{bmatrix}.$$

This means that $\vec{x}' = A\vec{x}$ has general solution:

$$\vec{x}(t) = c_1 \left(\cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 \left(\sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

The solution above is the "vector-form of the solution". We can add the terms together to find the scalar solutions: denoting $\vec{x}(t) = [x(t), y(t)]^T$,

$$x(t) = c_1 \cos(t) + c_2 \sin(t)$$
 $y(t) = -c_1 \sin(t) + c_2 \cos(t)$

These are the parametric equations of a circle with radius $R = \sqrt{c_1^2 + c_2^2}$.

Example 7.13.3. We solved the e-vector problem for $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ in Example 7.5.4. We found one real e-value $\lambda_1 = 3$ and a pair of complex e-values $\lambda_2 = 1 \pm i$. The corresponding e-vectors were:

$$\vec{u}_1 = \begin{bmatrix} 0\\0\\1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0\\1\\0 \end{bmatrix} + i \begin{bmatrix} 1\\0\\0 \end{bmatrix}$$

We identify that $Re(\vec{u}_2) = e_2$ and $Im(\vec{u}_2) = e_1$. The general solution of $\vec{x}' = A\vec{x}$ should have the form:

$$\vec{x}(t) = c_1 e^{At} \vec{u}_1 + c_2 Re(e^{At} \vec{u}_2) + c_3 Im(e^{At} \vec{u}_2)$$

The vectors above are e-vectors so these solution simplify nicely:

$$\vec{x}(t) = c_1 e^{3t} e_3 + c_2 e^t (\cos(t)e_2 - \sin(t)e_1) + c_3 e^t (\sin(t)e_2 + \cos(t)e_1)$$

For fun let's look at the scalar form of the solution. Denoting $\vec{x}(t) = [x(t), y(t), z(t)]^T$,

$$x(t) = -c_2 e^t \sin(t) + c_3 e^t \cos(t), \qquad y(t) = c_2 e^t \cos(t) + c_3 e^t \sin(t), \qquad z(t) = c_1 e^{3t}$$

Believe it or not this is a spiral helix which has an exponentially growing height and radius.

Example 7.13.4. Let's suppose we have a chain of 2 complex eigenvectors \vec{u}_1, \vec{u}_2 with eigenvalue $\lambda = 2 + i3$. I'm assuming that

$$(A - (2+i)I)\vec{u}_2 = \vec{u}_1, \qquad (A - (2+i)I)\vec{u}_1 = 0.$$

We get a pair of complex-vector solutions (using the magic formula which truncates since these are *e*-vectors):

$$\vec{z}_1(t) = e^{At}\vec{u}_1 = e^{(2+i)t}\vec{u}_1, \qquad \vec{z}_2(t) = e^{At}\vec{u}_2 = e^{(2+i)t}(\vec{u}_2 + t\vec{u}_1),$$

The real and imaginary parts of these solutions give us 4 real solutions which form the general solution:

$$\begin{aligned} \vec{x}(t) &= c_1 e^{2t} \left[\cos(3t) Re(\vec{u}_1) - \sin(3t) Im(\vec{u}_1) \right] \\ &+ c_2 e^{2t} \left[\sin(3t) Re(\vec{u}_1) + \cos(3t) Im(\vec{u}_1) \right] \\ &+ c_3 e^{2t} \left[\cos(3t) [Re(\vec{u}_2) + tRe(\vec{u}_1)] - \sin(3t) [Im(\vec{u}_2) + tIm(\vec{u}_1)] \right] \\ &+ c_4 e^{2t} \left[\sin(3t) [Re(\vec{u}_2) + tRe(\vec{u}_1)] + \cos(3t) [Im(\vec{u}_2) + tIm(\vec{u}_1)] \right]. \end{aligned}$$

7.14 geometry and difference equations revisited

In Example 7.1.1 we studied $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and how it pushed the point $x_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ around the plane. We found x_i for i = 1, 2, 3, 4 by multiplication by A directly. That method is fine for small i but what is we wished to know the formula for the 1000-th state? We should hope there is some way to find that state without direct multiplication repeated 1000 times. One method is to make use of the diagonalization of the matrix. We know that e-vectors (if they exist) can be glued together to make the diagonalizing similarity transforming matrix; there exists $P \in \mathbb{R}^{n \times n}$ such that $P^{-1}AP = D$ where D is a diagonal matrix. Notice that D^k is easy to calculate. We can solve for $A = PDP^{-1}$ and find that $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$. The you can prove inductively that $A^k = PD^kP^{-1}$. It is much easier to calculate PD^kP^{-1} when k >> 1.

7.14.1 difference equations vs. differential equations

I mentioned that the equation $x_{k+1} = Ax_k$ is a *difference equation*. We can think of this as a differential equation where the time-step is always one-unit. To see this I should remind you how $\vec{x}' = B\vec{x}$ is defined in terms of a limiting process:

$$\vec{x}'(t) = \lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h} = B\vec{x}(t)$$

A gross approximation to the continuous limiting process would be to just take h = 1 and drop the limit. That approximation yields:

$$B\vec{x}(t) = \vec{x}(t+1) - \vec{x}(t).$$

We then suppose $t \in \mathbb{N}$ and denote $\vec{x}(t) = \vec{x}_t$ to obtain:

$$\vec{x}_{t+1} = (B+I)\vec{x}_t.$$

We see that the differential equation $\vec{x}' = B\vec{x}$ is crudely approximated by the difference equation $\vec{x}_{t+1} = A\vec{x}_t$. where A = B + I. Since we now have tools to solve differential equations directly it should be interesting to contrast the motion generated by the difference equation to the exact parametric equations which follow from the e-vector solution of the corresponding differential equation.

7.15 conic sections and quadric surfaces

Some of you have taken calculus III others have not but most of you still have much to learn about level curves and surfaces. Let me give two examples to get us started:

 $x^2 + y^2 = 4$ level curve; generally has form f(x, y) = k $x^2 + 4y^2 + z^2 = 1$ level surface; generally has form F(x, y, z) = k

Alternatively, some special surfaces can be written as a graph. The top half of the ellipsoid $F(x, y, z) = x^2 + 4y^2 + z^2 = 1$ is the graph(f) where $f(x, y) = \sqrt{1 - x^2 - 4y^2}$ and $graph(f) = \{x, y, f(x, y) | (x, y) \in dom(f)\}$. Of course there is a great variety of examples to offer here and I only intend to touch on a few standard examples in this section. Our goal is to see what linear algebra has to say about conic sections and quadric surfaces. Your text only treats the case of conic sections.

7.15.1 quadratic forms and their matrix

Definition 7.15.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^{n \times 1} \to \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A \vec{x}$ for all $\vec{x} \in \mathbb{R}^{n \times 1}$ where $A \in \mathbb{R}^{n \times n}$ such that $A^T = A$. In particular, if $\vec{x} = [x, y]^T$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then $\vec{x}^T A \vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2$.

The n = 3 case is similar, denote $A = [A_{ij}]$ and $\vec{x} = [x, y, z]^T$ so that

$$\vec{x}^T A \vec{x} = A_{11} x^2 + 2A_{12} xy + 2A_{13} xz + A_{22} y^2 + 2A_{23} yz + A_{33} z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{n \times n}$ and $\vec{x} = [x_i]^T$ then the quadratic form

$$\vec{x}^T A \vec{x} = \sum_{i,j} A_{ij} x_i x_j = \sum_{i=1}^n A_{ii} x_i^2 + \sum_{i < j} 2A_{ij} x_i x_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

Also, you may recall I said a **bilinear form** was a mapping from $V \times V \to \mathbb{R}$ which is linear in each slot. For example, an inner-product as defined in Definition 6.8.1 is a symmetric, positive definite bilinear form. When we discussed $\langle x, y \rangle$ we allowed $x \neq y$, in contrast a quadratic form is more like $\langle x, x \rangle$. Of course the dot-product is also an inner product and we can write a given quadratic form in terms of a dot-product:

$$\vec{x}^T A \vec{x} = \vec{x} \cdot (A \vec{x}) = (A \vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 7.15.2.

$$2x^{2} + 2xy + 2y^{2} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 7.15.3.

$$2x^{2} + 2xy + 3xz - 2y^{2} - z^{2} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 7.15.4.

The values of a quadratic form on $\mathbb{R}^{n \times 1} - \{0\}$ is completely determined by it's values on the (n-1)-sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^{n \times 1} \mid ||\vec{x}|| = 1\}$. In particular, $Q(\vec{x}) = ||\vec{x}||^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A \vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude ||x|| and its direction $\hat{x} = \frac{1}{||\vec{x}||} \vec{x}$,

$$Q(\vec{x}) = Q(||\vec{x}||\hat{x}) = (||\vec{x}||\hat{x})^T A||\vec{x}||\hat{x} = ||\vec{x}||^2 \hat{x}^T A \hat{x} = ||x||^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $||\vec{x}||^2$. \Box

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of $\mathbb{R}^{n \times 1}$. If $f : S \to \mathbb{R}$ then we could say f(S) > 0 iff f(s) > 0 for all $s \in S$. Likewise, f(S) < 0 iff f(s) < 0 for all $s \in S$. The proposition below follows from the proposition above since $||\vec{x}||^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 7.15.5.

If Q is a quadratic form on $\mathbb{R}^{n \times 1}$ and we denote $\mathbb{R}_{*}^{n \times 1} = \mathbb{R}^{n \times 1} - \{0\}$ 1.(negative definite) $Q(\mathbb{R}_{*}^{n \times 1}) < 0$ iff $Q(S_{n-1}) < 0$ 2.(positive definite) $Q(\mathbb{R}_{*}^{n \times 1}) > 0$ iff $Q(S_{n-1}) > 0$ 3.(non-definite) $Q(\mathbb{R}_{*}^{n \times 1}) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

Before I get too carried away with the theory let's look at a couple examples.

Example 7.15.6. Consider the quadric form $Q(x, y) = x^2 + y^2$. You can check for yourself that z = Q(x, y) is a cone and Q has positive outputs for all inputs except (0, 0). Notice that $Q(v) = ||v||^2$ so it is clear that $Q(S_1) = 1$. We find agreement with the preceding proposition.

Next, think about the application of Q(x, y) to level curves; $x^2 + y^2 = k$ is simply a circle of radius \sqrt{k} or just the origin.

Finally, let's take a moment to write $Q(x,y) = [x,y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = \lambda_2 = 1$.

Example 7.15.7. Consider the quadric form $Q(x,y) = x^2 - 2y^2$. You can check for yourself that z = Q(x,y) is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominent term. Notice that Q(1,0) = 1 whereas Q(0,1) = -2 hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition.

Next, think about the application of Q(x, y) to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically (k > 0) or horizontally (k < 0) or a pair of lines $y = \pm \frac{x}{2}$ in the k = 0 case.

Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.

Example 7.15.8. Consider the quadric form $Q(x, y) = 3x^2$. You can check for yourself that z = Q(x, y) is parabola-shaped trough along the y-axis. In this case Q has positive outputs for all inputs except (0, y), we would call this form **positive semi-definite**. A short calculation reveals that $Q(S_1) = [0, 3]$ thus we again find agreement with the preceding proposition (case 3).

Next, think about the application of Q(x, y) to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm \sqrt{k/3}$ or just the y-axis.

Finally, let's take a moment to write $Q(x,y) = [x,y] \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 3$ and $\lambda_2 = 0$.

Example 7.15.9. Consider the quadric form $Q(x, y, z) = x^2 + 2y^2 + 3z^2$. Think about the application of Q(x, y, z) to level surfaces; $x^2 + 2y^2 + 3z^2 = k$ is an ellipsoid.

Finally, let's take a moment to write $Q(x, y, z) = [x, y, z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$.

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples.

Proposition 7.15.10.

If Q is a quadratic form on $\mathbb{R}^{n \times 1}$ with matrix A and e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with orthonormal e-vectors v_1, v_2, \ldots, v_n then $Q(v_i) = \lambda_i^2$ for $i = 1, 2, \ldots, n$. Moreover, if $P = [v_1|v_2|\cdots|v_n]$ then $Q(\vec{x}) = (P^T \vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$ where we defined $\vec{y} = P^T \vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 7.15.11. Consider the quadric form $Q(x, y) = 2x^2 + 2xy + 2y^2$. It's not immediately obvious (to me) what the level curves Q(x, y) = k look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:

$$det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 1\\ 1 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

Therefore, the e-values are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$(A-I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved u + v = 0 to give v = -u choose u = 1 then normalize to get the vector above. Next,

$$(A-3I)\vec{u}_2 = \begin{bmatrix} -1 & 1\\ 1 & -1 \end{bmatrix} \begin{bmatrix} u\\ v \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\ 1 \end{bmatrix}$$

I just solved u - v = 0 to give v = u choose u = 1 then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\vec{x}, \vec{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be

inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c} x &= \frac{1}{2}(\bar{x} + \bar{y}) \\ y &= \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} or \begin{array}{c} \bar{x} &= \frac{1}{2}(x - y) \\ \bar{y} &= \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield¹⁵:

$$\tilde{Q}(\bar{x},\bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

It is clear that in the barred coordinate system the level curve Q(x, y) = k is an ellipse. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x, y) = 2x^2 + 2xy + 2y^2 = k$ is an ellipse rotated by 45 degrees.

Example 7.15.12. Consider the quadric form $Q(x, y) = x^2 + 2xy + y^2$. It's not immediately obvious (to me) what the level curves Q(x, y) = k look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:

$$det(A - \lambda I) = det \begin{bmatrix} 1 - \lambda & 1\\ 1 & 1 - \lambda \end{bmatrix} = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

Therefore, the e-values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A-0)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved u + v = 0 to give v = -u choose u = 1 then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 1\\ 1 & -1 \end{bmatrix} \begin{bmatrix} u\\ v \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\ 1 \end{bmatrix}$$

I just solved u - v = 0 to give v = u choose u = 1 then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\vec{x}, \vec{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c} x &= \frac{1}{2}(\bar{x} + \bar{y}) \\ y &= \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} or \begin{array}{c} \bar{x} &= \frac{1}{2}(x - y) \\ \bar{y} &= \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve Q(x,y) = k is a pair of paralell lines. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = x^2 + 2xy + y^2 = k$ is a line with slope -1. Indeed, with a little algebraic insight we could have anticipated this result since $Q(x,y) = (x+y)^2$ so Q(x,y) = k implies $x + y = \sqrt{k}$ thus $y = \sqrt{k} - x$.

¹⁵technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

Example 7.15.13. Consider the quadric form Q(x, y) = 4xy. It's not immediately obvious (to me) what the level curves Q(x, y) = k look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:

$$det(A - \lambda I) = det \begin{bmatrix} -\lambda & 2\\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0$$

Therefore, the e-values are $\lambda_1 = -2$ and $\lambda_2 = 2$.

$$(A+2I)\vec{u}_1 = \begin{bmatrix} 2 & 2\\ 2 & 2 \end{bmatrix} \begin{bmatrix} u\\ v \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\ -1 \end{bmatrix}$$

I just solved u + v = 0 to give v = -u choose u = 1 then normalize to get the vector above. Next,

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -2 & 2\\ 2 & -2 \end{bmatrix} \begin{bmatrix} u\\ v \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\ 1 \end{bmatrix}$$

I just solved u - v = 0 to give v = u choose u = 1 then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\vec{x}, \vec{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c} x &= \frac{1}{2}(\bar{x} + \bar{y}) \\ y &= \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} or \begin{array}{c} \bar{x} &= \frac{1}{2}(x - y) \\ \bar{y} &= \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve Q(x, y) = k is a hyperbola. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of Q(x, y) = 4xy = k is a hyperbola rotated by 45 degrees.

Remark 7.15.14.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. 16

We ought to do at least one 3-dimensional example.

Example 7.15.15. Consider the quadric form defined below:

$$Q(x, y, z) = [x, y, z] \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A and calculate the e-values/vectors:

$$det(A - \lambda I) = det \begin{bmatrix} 6 - \lambda & -2 & 0 \\ -2 & 6 - \lambda & 0 \\ 0 & 0 & 5 - \lambda \end{bmatrix}$$
$$= [(\lambda - 6)^2 - 4](5 - \lambda)$$
$$= (5 - \lambda)[\lambda^2 - 12\lambda + 32](5 - \lambda)$$
$$= (\lambda - 4)(\lambda - 8)(5 - \lambda)$$

Therefore, the e-values are $\lambda_1 = 4$, $\lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\1\\0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1\\-1\\0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0\\0\\1 \end{bmatrix}$$

Let $P = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}, \bar{z}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0\\ -1 & 1 & 0\\ 0 & 0 & \sqrt{2} \end{bmatrix} \Rightarrow \begin{array}{ccc} x & = \frac{1}{2}(\bar{x} + \bar{y}) & \bar{x} & = \frac{1}{2}(x - y)\\ y & = \frac{1}{2}(-\bar{x} + \bar{y}) & or & \bar{y} & = \frac{1}{2}(x + y)\\ z & = \bar{z} & \bar{z} & \bar{z} & = z \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}, \bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface Q(x, y, z) = k is an ellipsoid. If we draw the barred coordinate system superposed over the xyz-coordinate system then you'll see that the graph of Q(x, y, z) = k is an ellipsoid rotated by 45 degrees around the z-axis.

Remark 7.15.16.

If you would like to read more about conic sections or quadric surfaces and their connection to e-values/vectors I reccommend sections 9.6 and 9.7 of Anton's text. I have yet to add examples on how to include translations in the analysis. It's not much more trouble but I decided it would just be an uncessary complication this semester. Also, section 7.1,7.2 and 7.3 in Lay's text show a bit more about how to use this math to solve concrete applied problems. You might also take a look in Strang's text, his discussion of tests for positive-definite matrices is much more complete than I will give here.

It turns out that linear algebra and e-vectors can give us great insight into locating local extrema for a function of several variables. To summarize, we can calculate the multivariate Taylor series and we'll find that the quadratic terms correspond to an algebraic object known as a quadratic form. In fact, each quadratic form has a symmetric matrix representative. We know that symmetric matrices are diagonalizable hence the e-values of a symmetric matrix will be real. Moreover, the eigenvalues tell you what the min/max value of the function is at a critical point (usually). This is the n-dimensional generalization of the 2nd-derivative test from calculus. I'll only study the n = 2 and n = 3 case in this course. If you'd like to see these claims explained in more depth feel free to join us in math 331 next semester.

7.16 intertia tensor, an application of quadratic forms

Part I

sections for future courses or bonus work

Vector Spaces over $\mathbb C$

- 8.1 complex matrix calculations
- 8.2 inner products
- 8.3 Hermitian matrices

Additional Topics

- 9.1 minimal polynomial
- 9.2 quotient spaces
- 9.3 tensor products and blocks
- 9.4 Cayley-Hamiliton Theorem
- 9.5 singular value decomposition
- 9.6 spectral decomposition
- 9.7 QR-factorization

Multilinear Algebra

- 10.1 dual space
- 10.2 double dual space
- 10.3 multilinearity
- 10.4 tensor product
- 10.5 forms
- 10.6 determinants done right

Infinite Dimensional Linear Algebra

- 11.1 norms in infinite dimensions
- 11.2 examples
- 11.3 differentiation in finite dimensions
- 11.4 differentiation in infinite dimensions

appendix on finite sums

In this appendix we prove a number of seemingly obvious propositions about finite sums of arbitrary size. Most of these statements are "for all $n \in \mathbb{N}$ " thus proof by mathematical induction is the appropriate proof tool. That said, I will abstain from offering a proof for every claim. I offer a few sample arguments and leave the rest to the reader. Let's begin with defining the finite sum:

Definition 12.0.1.

A finite sum of n summands A_1, A_2, \ldots, A_n is $A_1 + A_2 + \cdots + A_n$. We use "summation notation" or "sigma" notation to write this succinctly:

$$A_1 + A_2 + \dots + A_n = \sum_{i=1}^n A_i$$

The index i is the "dummy index of summation". Technically, we define the sum above recursively. In particular,

$$\sum_{i=1}^{n+1} A_i = A_{n+1} + \sum_{i=1}^{n} A_i$$

for each $n \ge 1$ and $\sum_{i=1}^{1} A_i = A_1$ begins the recursion.

Proposition 12.0.2.

Let $A_i, B_i \in \mathbb{R}$ for each $i \in \mathbb{N}$ and suppose $c \in \mathbb{R}$ then for each $n \in \mathbb{N}$,

(1.)
$$\sum_{i=1}^{n} (A_i + B_i) = \sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i$$

(2.) $\sum_{i=1}^{n} cA_i = c \sum_{i=1}^{n} A_i.$

Proof: Let's begin with (1.). Notice the claim is trivially true for n = 1. Moreover, n = 2 follows from the calculation below:

$$\sum_{i=1}^{2} (A_i + B_i) = (A_1 + B_1) + (A_2 + B_2) = (A_1 + A_2) + (B_1 + B_2) = \sum_{i=1}^{2} A_i + \sum_{i=1}^{2} B_i.$$

Inductively assume that (1.) is true for $n \in \mathbb{N}$. Consider, the following calculations are justified either from the recursive definition of the finite sum or the induction hypothesis:

$$\sum_{i=1}^{n+1} (A_i + B_i) = \sum_{i=1}^n (A_i + B_i) + A_{n+1} + B_{n+1}$$
$$= \left(\sum_{i=1}^n A_i + \sum_{i=1}^n B_i\right) + A_{n+1} + B_{n+1}$$
$$= \left(\sum_{i=1}^n A_i\right) + A_{n+1} + \left(\sum_{i=1}^n B_i\right) + B_{n+1}$$
$$= \sum_{i=1}^{n+1} A_i + \sum_{i=1}^{n+1} B_i.$$

Thus (1.) is true for n + 1 and hence by PMI we find (1.) is true for all $n \in \mathbb{N}$. The proof of (2.) is similar. \Box

Proposition 12.0.3.

Let
$$A_i, B_{ij} \in \mathbb{R}$$
 for $i, j \in \mathbb{N}$ and suppose $c \in \mathbb{R}$ then for each $n \in \mathbb{N}$,

$$(1.) \sum_{i=1}^{n} c = cn$$

$$(2.) \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ij}.$$

$$(3.) \sum_{i=1}^{n} \sum_{j=1}^{n} A_i B_{ij} = \sum_{i=1}^{n} A_i \sum_{j=1}^{n} B_{ij}$$

$$(4.) \left(\sum_{i=1}^{n} A_i\right)c = \sum_{i=1}^{n} (A_ic).$$

Proof: Let's begin with (1.). Notice the claim is trivially true for n = 1. Assume that (1.) is true for a fixed, but arbitrary, $n \in \mathbb{N}$ and consider that

$$\sum_{i=1}^{n+1} c = c + \sum_{i=1}^{n} c = c + nc = (n+1)c$$

by the recursive definition of the finite sum and the induction hypothesis. We find that (1.) is true for n + 1 hence by PMI (1.) is true for all $n \in \mathbb{N}$.

The proof of (2.) proceeds by induction on n. If n = 1 then there is only one possible term, namely B_{11} and the sums trivially agree. Consider the n = 2 case as we prepare for the induction step,

$$\sum_{i=1}^{2} \sum_{j=1}^{2} B_{ij} = \sum_{i=1}^{2} [B_{i1} + B_{i2}] = [B_{11} + B_{12}] + [B_{21} + B_{22}]$$

On the other hand,

$$\sum_{j=1}^{2} \sum_{i=1}^{2} B_{ij} = \sum_{j=1}^{2} [B_{1j} + B_{2j}] = [B_{11} + B_{21}] + [B_{11} + B_{21}].$$

The sums in opposite order produce the same terms overall, however the ordering of the terms may differ¹. Fortunately, real number-addition commutes. Assume inductively that (2.) is true for $n \in \mathbb{N}$. Using the definition of sum and the induction hypothesis in the 4-th line:

$$\begin{split} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} B_{ij} &= \sum_{i=1}^{n+1} \left[B_{i,n+1} + \sum_{j=1}^{n} B_{ij} \right] \\ &= B_{n+1,n+1} + \sum_{j=1}^{n} B_{n+1,j} + \sum_{i=1}^{n} \left[B_{i,n+1} + \sum_{j=1}^{n} B_{ij} \right] \\ &= B_{n+1,n+1} + \sum_{j=1}^{n} B_{n+1,j} + \sum_{i=1}^{n} B_{i,n+1} + \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \\ &= B_{n+1,n+1} + \sum_{j=1}^{n} B_{n+1,j} + \sum_{i=1}^{n} B_{i,n+1} + \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ij} \\ &= \sum_{j=1}^{n+1} B_{n+1,j} + \sum_{j=1}^{n+1} \sum_{i=1}^{n} B_{ij} \\ &= \sum_{j=1}^{n+1} \left[B_{n+1,j} + \sum_{i=1}^{n} B_{ij} \right] \\ &= \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} B_{ij}. \end{split}$$

Thus *n* implies n + 1 for (2.) therefore by PMI we find (2.) is true for all $n \in \mathbb{N}$. The proofs of (3.) and (4.) involve similar induction arguments. \Box

¹reordering terms in the infinite series case can get you into trouble if you don't have absolute convergence. Riemann showed a conditionally convergent series can be reordered to force it to converge to any value you might choose.

Remark 12.0.4.

I included this appendix to collect a few common tricks we need to use for proofs involving arbitrary sums. In particular proofs for parts of Theorem 2.3.13 require identities from this appendix. You are free to use the facts in this appendix by referring to them as "a property of finite sums".