Copying answers and steps is strictly forbidden. Working together is encouraged, share ideas not calculations. Show work on other paper. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Let $T(x, y, z, w) = (x+y+z+w, y+z+w, z+w)$. Find bases for $\text{Ker}(T)$ and $\text{Range}(T)$.

Problem 2 Let $T(x, y, z) = (x+2y+3z, y-z, x+z)$ define $T : \mathbb{R}^3 \to \mathbb{R}^3$. If $\beta = \{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}$ then find $\Phi_\beta(a, b, c)$ for arbitrary $a, b, c \in \mathbb{R}$ and $[T]_{\beta,\beta}$.

Problem 3 Find the general solution of the system of equations $x + y + z = 3$ and $y + z = 4$.

Problem 4 Suppose $\beta = \{x^2, e^x, \cos(x), \sin(x)\}$ and $[v]_\beta = (1, 2, 4, 8)$. Find v.

Problem 5 Let $S = \{v, w, x\}$ be a linearly independent set in a vector space V. Prove or disprove: $T = \{v + w + x, w + x, w - x\}$ is also a linearly independent set in V.

Problem 6 Let $T(ax^2 + bx + c) = 2ax + b$ for all $a, b, c \in \mathbb{C}$ (this means $a = a_1 + ia_2$ for $a_1, a_2 \in \mathbb{R}$ etc...). We have that $T : P_2(\mathbb{C}) \to P_1(\mathbb{C})$. Let $\beta = \{x^2, ix^2, x, ix, 1, i\}$ be a basis for $P_2(\mathbb{C})$ and $\gamma = \{x, ix, 1, i\}$ serve as the basis for $P_1(\mathbb{C})$. Find $[T]_{\beta,\gamma}$.

Problem 7 Let $v_1 = x^2 + x$ and $v_2 = x^2 + 3x$ and $v_3 = x + 1$. Show $\beta = \{v_1, v_2, v_3\}$ forms a basis for P_2 and if $v = ax^2 + bx + c$ find $[v]_\beta$.

Problem 8 Let $\gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Show γ is a linearly independent set. Moreover, show $W = \{A \in \mathbb{R}^{2\times 2} \mid \text{trace}(A) = 0\}$ takes γ as a basis.

Problem 9 Let $T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$ for $f(x) \in P_2$. Observe that $T : P_2 \to T(P_2)$ is a linear transformation with $T(P_2) = W$. Considering T, β, γ (as discussed in the previous two problems) calculate $[T]_{\beta,\gamma}$ and determine the rank and nullity for T.

Problem 10 Suppose $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is defined by $T(A) = BA^TB$ for some $B \in \mathbb{R}^{2\times 2}$. Show T is a linear transformation. Also, show that if B is invertible then T is invertible.

Problem 11 Let D be the set of diagonal 4×4 real matrices and let S be the set of symmetric 4×4 real matrices. Find a basis for S/D and an isomorphism to P_n for appropriate n.

Problem 12 Let $V_1 = \{f(x) \in P_2 \mid f(0) = 0\}$ and $V_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ find a basis for $V_1 \times V_2$ and an isomorphism to P_n for appropriate n.

Problem 13 A **derivation** on the set of smooth functions $\mathcal{F}$ is a linear transformation $T : \mathcal{F} \to \mathcal{F}$ such that $T(fg) = T(f)g + fT(g)$. Show that the set of derivations (denoted $\text{Der}(\mathcal{F})$) forms a subspace of the set of linear transformations $\mathcal{L}(\mathcal{F}, \mathcal{F})$. That is; show $\text{Der}(\mathcal{F}) \leq \mathcal{L}(\mathcal{F}, \mathcal{F})$. Before your proof, first two things:(1.) a linear transformation which is not a derivation and (2.) a linear transformation which is a derivation (hint: they're aptly named)

Problem 14 Suppose $A \oplus V_1 = A \oplus V_2$. Does it follow that $V_1 = V_2$? If not, what can you say about the relation of V_1 and V_2 ?

Problem 15 Let $V_1 \leq V$ and $V_2 \leq V$. Show that $V_1 \cap V_2 \leq V$.