Copying answers and steps is strictly forbidden. Working together is encouraged, share ideas not calculations. Show work on other paper. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Let $T(x, y, z, w) = (x+y+z+w, y+z+w, z+w)$. Find bases for Ker$(T)$ and Range(T).

$$T(x,y,z,w) = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$[T] \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \text{rref}([T])$$

1. $\text{Ker}(T) = \{x \in \mathbb{R}^4 \mid T(x) = 0\} = \text{Null}([T]) = \{(0,0,z,-z) \mid z \in \mathbb{R}\}$

Hence $\boxed{\beta = \{(0,0,1,-1)\} \text{ serves as basis for Ker}(T).}$

2. $\text{Range}(T) = \{T(x) \mid x \in \mathbb{R}^4\} = \text{Col}([T]) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

By CCP and fact columns 1, 2, 3 are pivot columns of $[T]$.

~~and~~ $\therefore \boxed{\gamma = \{(1,0,0), (1,1,0), (1,1,1)\} \text{ is basis for Range}(T)}$

Remark: $\#$ of columns $[T] = 4 = \underbrace{\text{rank}[T]}_{3} + \underbrace{\text{nullity}([T])}_{1}$

OR $\dim(\mathbb{R}^4) = \dim(\text{Range}(T)) + \dim(\text{Ker}(T))$

Problem 2 Let $T(x,y,z) = (x+2y+3z, y-z, x+z)$ define $T : \mathbb{R}^3 \to \mathbb{R}^3$. If $\beta = \{(1,0,0),(0,1,1),(0,1,-1)\}$ then find $\Phi_\beta(a,b,c)$ for arbitrary $a,b,c \in \mathbb{R}$ and $[T]_{\beta,\beta}$.

$$T(x,y,z) = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \hookrightarrow [\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Note $$[\beta]^{-1} = \begin{bmatrix} 1^{-1} & 0 \quad 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Thus, $$\underbrace{\Phi_\beta(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \boxed{\begin{bmatrix} a \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(b-c) \end{bmatrix} = \Phi_\beta(a,b,c)}$$

Coord. map $\Phi_\beta : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\Phi_\beta(v) = [\beta]^{-1} v$ (special case)

Then

$$[T]_{\beta\beta} = [\beta]^{-1}[T][\beta]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 5 & -1 \\ 1/2 & 1/2 & 1/2 \\ -1/2 & 0 & 1 \end{bmatrix}}$$

(I used results derived for $T : \mathbb{R}^n \to \mathbb{R}^n$ and basis β given to both domain & codomain this is special case of our general $[T]_{\beta,\gamma}$)

Problem 3 Find the general solution of the system of equations $x + y + z = 3$ and $y + z = 4$.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 4 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \longrightarrow x = -1 \\ \longrightarrow y = 4 - z \end{array}$$

Thus $(x, y, z) = (-1, 4-z, z)$ for $z \in \mathbb{R}$ is general sol$^{\underline{n}}$ for the given system. Notice

$$(x, y, z) = \underbrace{(-1, 4, 0)}_{\substack{x_p \\ \text{particular} \\ \text{sol}^{n}}} + c_1 \underbrace{(0, -1, 1)}_{\substack{x_n \\ \text{homogeneous} \\ \text{sol}^{n}}}$$

$Ax_p = b$
$Ax_n = 0$
$x_n \in \text{Null}(A)$.

Problem 4 Suppose $\beta = \{x^2, e^x, \cos(x), \sin(x)\}$ and $[v]_\beta = (1, 2, 4, 8)$. Find v.

$$[v]_\beta = (1, 2, 4, 8) \iff v = 1 \cdot x^2 + 2 \cdot e^x + 4 \cdot \cos(x) + 8 \cdot \sin(x)$$

Thus $\boxed{v = x^2 + 2e^x + 4\cos(x) + 8\sin(x)}$

Problem 5 Let $S = \{v, w, x\}$ be a linearly independent set in a vector space V. Prove or disprove: $T = \{v + w + x, w + x, w - x\}$ is also a linearly independent set in V.

Suppose $\{v, w, x\} = S$ is a LI subset of V.
Let $T = \{v+w+x, w+x, w-x\}$. Consider,

$$c_1(v+w+x) + c_2(w+x) + c_3(w-x) = 0$$

$$\Rightarrow c_1 v + (c_1 + c_2 + c_3) w + (c_1 - c_3) x = 0$$

Hence by LI of S we deduce

$$c_1 = 0, \quad c_1 + c_2 + c_3 = 0 \quad \text{and} \quad c_1 - c_3 = 0$$

Thus $c_3 = c_1 = 0$ and $c_2 = -c_1 - c_3 = -0 - 0 = 0$

$\therefore c_1 = c_2 = c_3 = 0$ and we find T is LI. //

Problem 6 Let $T(ax^2 + bx + c) = 2ax + b$ for all $a, b, c \in \mathbb{C}$ (this means $a = a_1 + ia_2$ for $a_1, a_2 \in \mathbb{R}$ etc...). We have that $T : P_2(\mathbb{C}) \to P_1(\mathbb{C})$. Let $\beta = \{x^2, ix^2, x, ix, 1, i\}$ be a basis for $P_2(\mathbb{C})$ and $\gamma = \{x, ix, 1, i\}$ serve as the basis for $P_1(\mathbb{C})$. Find $[T]_{\beta,\gamma}$.

$$T\Big(\underbrace{(a_1+ia_2)x^2 + (b_1+ib_2)x + (c_1+ic_2)}_{V}\Big) = \underbrace{2(a_1+ia_2)x + b_1+ib_2}_{T(V)}$$

$$[V]_\beta = (a_1, a_2, b_1, b_2, c_1, c_2) \qquad = \underbrace{(2a_1, 2a_2, b_1, b_2)}_{[T(V)]_\gamma}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{[T]_{\beta,\gamma}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix}}_{[V]_\beta} = \underbrace{\begin{bmatrix} 2a_1 \\ 2a_2 \\ b_1 \\ b_2 \end{bmatrix}}_{[T(V)]_\gamma}$$

Alternatively,

$$[T]_{\beta\gamma} = \Big[[T(x^2)]_\gamma \,\Big|\, [T(ix^2)]_\gamma \,\Big|\, [T(x)]_\gamma \,\Big|\, [T(ix)]_\gamma \,\Big|\, [T(1)]_\gamma \,\Big|\, [T(i)]_\gamma \Big]$$
$$= \Big[[2x]_\gamma \,\Big|\, [2ix]_\gamma \,\Big|\, [1]_\gamma \,\Big|\, [i]_\gamma \,\Big|\, [0]_\gamma \,\Big|\, [0]_\gamma \Big]$$
$$= \left[\begin{array}{c|c|c|c|c|c} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

<u>Comment:</u> over $\mathbb{C}$ we have $\tilde{\beta} = \{x^2, x, 1\}$, $\tilde{\gamma} = \{x, 1\}$

$$[T]_{\tilde{\beta},\tilde{\gamma}} = \Big[[2x]_{\tilde{\gamma}} \,\Big|\, [1]_{\tilde{\gamma}} \,\Big|\, [0]_{\tilde{\gamma}} \Big] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ You might like to see $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [T]_{\tilde{\beta},\tilde{\gamma}} = [T]_{\beta,\gamma}$. Neat.

Problem 7 Let $v_1 = x^2 + x$ and $v_2 = x^2 + 3x$ and $v_3 = x + 1$. Show $\beta = \{v_1, v_2, v_3\}$ forms a basis for P_2 and if $v = ax^2 + bx + c$ find $[v]_\beta$.

Let $\underline{c_1 V_1 + c_2 V_2 + c_3 V_3 = ax^2 + bx + c}_{\,*}$ then if we can solve this $*$ we'll be able to argue $[v]_\beta = (c_1, c_2, c_3)$ provided we are also able to show β is LI.

$$c_1(x^2+x) + c_2(x^2+3x) + c_3(x+1) = ax^2+bx+c$$

$$x^2(c_1+c_2) + x(c_1+3c_2+c_3) + c_3 = ax^2+bx+c$$

Hence by LI of $\{x^2, x, 1\}$ we equate coeff. to see

$$\underline{c_3 = c}.$$

$$c_1 + c_2 = a \qquad \Longrightarrow \quad 2c_2 + c_3 = b - a$$

$$c_1 + 3c_2 + c_3 = b \qquad \therefore\ 2c_2 = b - a - c$$

$$\underline{c_2 = \tfrac{1}{2}(b-a-c)}.$$

And $c_1 = a - c_2 = a - \frac{1}{2}(b-a-c) = \underline{\frac{3}{2}a - \frac{1}{2}b + \frac{c}{2} = c_1}$.

Thus we find $*$ has sol^{n}

$$c_1 = \frac{3}{2}a - \frac{1}{2}b + \frac{c}{2}, \quad c_2 = \frac{1}{2}(b-a-c), \quad c_3 = c$$

If $c_1V_1 + c_2V_2 + c_3V_3 = 0 \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$ by the above formula $\therefore \beta$ is LI. Indeed,

$$ax^2+bx+c = \left(\frac{3}{2}a - \frac{1}{2}b + \frac{c}{2}\right)(x^2+x) + \left(\frac{1}{2}b - \frac{1}{2}a - \frac{1}{2}c\right)(x^2+3x) + c(x+1)$$

Thus β is a spanning set for P_2; $P_2 = \mathrm{span}(\beta)$. Thus β is LI spanning set for P_2. That is β is a basis for P_2.

Moreover,

$$\boxed{\Phi_\beta(ax^2+bx+c) = \left(\frac{3a-b+c}{2}, \frac{b-a-c}{2}, c\right)}$$

(coordinate mapping w.r.t. β for P_2)

Problem 8 Let $\gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Show γ is a linearly independent set. Moreover, show $W = \{A \in \mathbb{R}^{2\times 2} \mid \text{trace}(A) = 0\}$ takes γ as a basis.

Observe,

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} c_1 + c_2 & c_1 + c_3 \\ \hline c_2 + c_3 & -c_1 - c_2 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

implies $c_1 + c_2 = 0, \quad c_1 + c_3 = 0, \quad c_2 + c_3 = 0$

thus $(c_1 + c_2) - (c_1 - c_3) = c_2 - c_3 = 0 \quad \therefore c_2 = c_3$

and so $c_2 + c_3 = 2c_2 = 0 \quad \therefore c_2 = 0 \Rightarrow c_3 = 0$

and finally $c_1 = -c_2 = -0 = 0$. Hence γ is LI.

Let $A \in W$ then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 0$

hence $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

We must solve the following to show span $\gamma = W$.

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} c_1 + c_2 & c_1 + c_3 \\ \hline c_2 + c_3 & -c_1 - c_2 \end{array}\right]$$

Hence,

$$\begin{aligned} c_1 + c_2 &= a \\ c_1 + c_3 &= b \\ c_2 + c_3 &= c \end{aligned} \longrightarrow \begin{array}{c} (c_1 \;\; c_2 \;\; c_3) \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right] \end{array} \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b - a \\ 0 & 1 & 1 & c \end{array}\right]$$

$$\xrightarrow[r_2 + r_3]{r_1 - r_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a - c \\ 0 & 0 & 2 & b - a + c \\ 0 & 1 & 1 & c \end{array}\right]$$

enough, we find $\underline{c_2 = \frac{1}{2}(b - a + c)}$.
$c_3 = c - c_2$
$c_1 = c_3 + a - c$

That is $c_3 = c - \frac{1}{2}(b - a + c) = \underline{\frac{1}{2}(a - b + c) = c_3}$.

and $c_1 = c_3 + a - c = \underline{\frac{1}{2}(3a - b - c) = c_1}$.

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \frac{3a - b - c}{2} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \frac{b - a + c}{2} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + \frac{a - b + c}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence span $\gamma = W \;\therefore\;$ $\underline{\gamma \text{ basis for } W}$.

Remark: $[A]_\gamma = \left(\frac{3a-b-c}{2}, \frac{b-a+c}{2}, \frac{a-b+c}{2} \right)$

Problem 9 Let $T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$ for $f(x) \in P_2$. Observe that $T : P_2 \to T(P_2)$ is a linear transformation with $T(P_2) = W$. Considering T, β, γ (as discussed in the previous two problems) calculate $[T]_{\beta,\gamma}$ and determine the rank and nullity for T.

$$T\Big(\underbrace{a(x^2+x)+b(x^2+3x)+c(x+1)}_{f(x)}\Big) = \left[\begin{array}{c|c} c & a+3b+c \\ \hline 2a+2b & -c \end{array}\right]$$

$$f'(x) = a(2x+1)+b(2x+3)+c$$
$$f''(x) = 2a+2b$$

$[f(x)]_\beta = (a,b,c)$ is why I wrote $f(x)$ as I did.

$$[T(f(x))]_\gamma = \left[\begin{array}{c|c} c & a+3b+c \\ 2a+2b & -c \end{array}\right]_\gamma$$

$\bar{a}, \bar{b}, \bar{c}$ def$^{\underline{n}}$ implicitly here.

$$= \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{bmatrix}_\gamma$$

$$= \left(\frac{3\bar{a}-\bar{b}-\bar{c}}{2}, \frac{\bar{b}-\bar{a}+\bar{c}}{2}, \frac{\bar{a}-\bar{b}+\bar{c}}{2} \right)$$

$$= \frac{1}{2}\big(3c-(a+3b+c)-(2a+2b),\ (a+3b+c)-c+(2a+2b),\ c-(a+3b+c)+(2a+2b)\big)$$

$$= \left(\frac{2c-3a-5b}{2}, \frac{3a+5b}{2}, \frac{a-b}{2} \right)$$

Consider then to make $[T]_{\beta,\gamma}[f(x)]_\beta = [T(f(x))]_\gamma$ we need that

$$[T]_{\beta,\gamma} = \begin{bmatrix} -3/2 & -5/2 & 1 \\ 3/2 & 5/2 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

We can check, $[T(x^2+x)]_\gamma = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}_\gamma = (-3/2, 3/2, 1/2)$

PROBLEM 9 continued

def$^{\underline{n}}$ from notes, or derived result, either way, it's true ☺

$$\text{rank}(T) = \text{rank}([T]_{\beta,\gamma}) = 3$$

$$\text{nullity}(T) = \text{nullity}([T]_{\beta,\gamma}) = 0$$

We can <u>prove</u> these many ways. One way,

Calculate

$$\text{rref}\begin{pmatrix} -3/2 & -5/2 & 1 \\ 3/2 & 5/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Col}([T]_{\beta,\gamma}) = \mathbb{R}^3 \Rightarrow \text{rank}([T]_{\beta,\gamma}) = 3.$$

And, by inspection $\text{Null}([T]_{\beta,\gamma}) = \{0\} \therefore \text{nullity}([T]_{\beta,\gamma}) = 0.$

Problem 10 Suppose $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is defined by $T(A) = BA^TB$ for some $B \in \mathbb{R}^{2\times 2}$. Show T is a linear transformation. Also, show that if B is invertible then T is invertible.

Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Consider,

$$\begin{aligned} T(A_1 + A_2) &= B(A_1 + A_2)^T B && : \text{def}^{\underline{n}} \text{ of } T \\ &= B(A_1^T + A_2^T)B && : \text{prop. of transpose} \\ &= BA_1^TB + BA_2^TB && : \text{matrix algebra.} \\ &= T(A_1) + T(A_2) && : \text{def}^{\underline{n}} \text{ of } T \end{aligned}$$

Hence T is additive. Moreover, for $c \in \mathbb{R}$ and $A \in \mathbb{R}^{2\times 2}$ note

$$T(cA) = B(cA)^TB = B(cA^T)B = cBA^TB = cT(A)$$

Hence T is homogeneous. Therefore, T is linear. //

Consider $T(A) = BA^TB = y \Rightarrow A^T = B^{-1}yB^{-1}$ if B^{-1} exists.
So, supposing B^{-1} exists, we find $A = (B^{-1}yB^{-1})^T$ and I propose $T^{-1}(y) = (B^{-1}yB^{-1})^T$. Consider,

$$T(T^{-1}(y)) = B((B^{-1}yB^{-1})^T)^TB = BB^{-1}yB^{-1}B = IyI = y.$$

Also,

$$T^{-1}(T(A)) = T^{-1}(BA^TB) = (B^{-1}(BA^TB)B^{-1})^T = (A^T)^T = A.$$

Thus my proposed T^{-1} is indeed the inverse fnct for T and this shows T is invertible.

<u>Remark</u>: Alternatively, and much more painfully, pick a basis β for $\mathbb{R}^{2\times 2}$ and calculate $[T]_{\beta\beta}$. If you can show $[T]_{\beta\beta}^{-1}$ exists (given B^{-1} exists of course) then you could also use that to argue T^{-1} exists.

Problem 11 Let D be the set of diagonal 4×4 real matrices and let S be the set of symmetric 4×4 real matrices. Find a basis for S/D and an isomorphism to P_n for appropriate n.

$$D = \left\{ \begin{pmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\} = \operatorname{span}\{E_{11}, E_{22}, E_{33}, E_{44}\}$$

$$S = \operatorname{span}\{E_{11}, E_{22}, E_{33}, E_{44}, E_{12}+E_{21}, E_{13}+E_{31}, E_{14}+E_{41}, E_{23}+E_{32}, E_{24}+E_{42}, E_{34}+E_{43}\}$$

$\dim(S) = 10$. From Mission 6 we had problem $\dim(V/W) = \dim V - \dim W$ and there I showed $\beta_V - \beta_W \Rightarrow \beta_{V/W} = \{x + W \mid x \in \beta_V - \beta_W\}$

Here,

$$S/D = \operatorname{span}\left\{ \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} + D \;\middle|\; a, b, c, d, e, f \in \mathbb{R} \right\}$$

And the basis to naturally pick is just

$$\beta_{S/D} = \{E_{12}+E_{21}+D, E_{13}+E_{31}+D, E_{14}+E_{41}+D, E_{23}+E_{32}+D, E_{24}+E_{42}+D, E_{34}+E_{43}+D\}$$

Moreover as $\dim(S/D) = 6 \hookrightarrow P_5 = \operatorname{span}\{x^5, x^4, x^3, x^2, x, 1\}$ is the right target,

$$\Psi\left(\begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix}\right) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

Problem 12 Let $V_1 = \{f(x) \in P_2 \mid f(0) = 0\}$ and $V_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ find a basis for $V_1 \times V_2$ and an isomorphism to P_n for appropriate n.

$$V_1 = \{f(x) \in P_2 \mid f(0)=0\} = \{x(Ax+B) \mid A, B \in \mathbb{R}\} = \mathrm{span}\{x^2, x\}.$$

$$V_2 = \left\{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A^T = A\right\} = \left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{R}\right\}$$

$$V_2 = \mathrm{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

For $\beta_1 = \{x^2, x\}$ and $\beta_2 = \{E_{11}, E_{12}+E_{21}, E_{22}\}$ we form basis $\beta_1 \times \{0\} \cup \{0\} \times \beta_2$ for $V_1 \times V_2$.

$$\beta = \left\{ \left(x^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right), \left(x, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \right\}$$

This β serves as basis for $V_1 \times V_2$ and $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2) = 2+3 = 5.$

$$\Psi\left(\left(Ax^2+Bx, \begin{bmatrix} \alpha & \gamma \\ \gamma & \delta \end{bmatrix}\right)\right) = Ax^4 + Bx^3 + \alpha x^2 + \gamma x + \delta$$

Gives $\Psi : V_1 \times V_2 \longrightarrow P_4$ an isomorphism.

Problem 13 A **derivation** on the set of smooth functions $\mathcal{F}$ is a linear transformation $T : \mathcal{F} \to \mathcal{F}$ such that $T(fg) = T(f)g + fT(g)$. Show that the set of derivations (denoted $\mathrm{Der}(\mathcal{F})$) forms a subspace of the set of linear transformations $\mathcal{L}(\mathcal{F}, \mathcal{F})$. That is; show $\mathrm{Der}(\mathcal{F}) \leq \mathcal{L}(\mathcal{F}, \mathcal{F})$. Before your proof, first two things:(1.) a linear transformation which is not a derivation and (2.) a linear transformation which is a derivation (hint: they're aptly named)

(1.) $T(f) = 2f$ is certainly linear. However, for $f, g \in \mathcal{F}$

~~note $T(f \circ g) = 2(f \circ g)$~~ sorry.

$T(fg) = 2fg$ But, $T(f)g + fT(g) = 2fg + f(2g) = 4fg$

Thus $T(fg) \neq T(f)g + fT(g)$ $\therefore$ $T \notin \mathrm{Der}(\mathcal{F})$.

(2.) $T(f) = f'$ ← differentiation is linear trans. on $\mathcal{F}$

$T(fg) = (fg)' = f'g + fg' = T(f)g + fT(g)$

Hence $T = d/dx \in \mathrm{Der}(\mathcal{F})$.

Notice $T(f) = 0$ $\forall f \in \mathcal{F}$ gives a linear transformation on $\mathcal{F}$ which trivially satisfies $T(fg) = T(f)g + fT(g)$ as $0 = 0$. Hence $\mathrm{Der}(\mathcal{F}) \neq \{0\}$. Let $T_1, T_2 \in \mathrm{Der}(\mathcal{F})$ and $c_1 \in \mathbb{R}$. clearly $c_1T_1 + T_2$ is once more in $\mathcal{L}(\mathcal{F}, \mathcal{F})$ so it remains to show $cT_1 + T_2$ satisfies the product rule,

$$
\begin{aligned}
(c_1T_1 + T_2)(fg) &= (c_1T_1)(fg) + T_2(fg) && : \text{def}^{\underline{n}} \text{ of op. } + \\
&= c_1T_1(fg) + T_2(fg) && : \text{def}^{\underline{n}} \text{ of op. scal. mult.} \\
&= c_1(T_1(f)g + fT_1(g)) + T_2(f)g + fT_2(g) && : T_1, T_2 \in \mathrm{Der}(\mathcal{F}). \\
&= c_1(T_1(f) + T_2(f))g + f(c_1T_1(g) + T_2(g)) && : \text{algebra} \\
&= (c_1T_1 + T_2)(f)g + f(c_1T_1 + T_2)(g)
\end{aligned}
$$

Thus $c_1T_1 + T_2$ satisfies the product rule $\therefore$ $c_1T_1 + T_2 \in \mathrm{Der}(\mathcal{F})$

~~But~~ Then it follows by subspace th$^{\underline{m}}$ that $\mathrm{Der}(\mathcal{F}) \leq \mathcal{L}(\mathcal{F}, \mathcal{F})$.

Problem 14 Suppose $A \oplus V_1 = A \oplus V_2$. Does it follow that $V_1 = V_2$? If not, what can you say about the relation of V_1 and V_2 ?

$\mathbb{R}^2$ has lots of counter-examples.

$$\mathbb{R}^2 = e_1\mathbb{R} \oplus e_2\mathbb{R} = e_1\mathbb{R} \oplus \langle 1,1\rangle\mathbb{R}$$

However, clearly $e_2\mathbb{R} \neq \langle 1,1\rangle\mathbb{R}$.

We do have $\dim(A \oplus V_1) = \dim A + \dim V_1$ and $\dim(A \oplus V_2) = \dim A + \dim V_2$ thus we deduce $\dim V_1 = \dim V_2$. It follows that $V_1 \approx V_2$.

Problem 15 Let $V_1 \leq V$ and $V_2 \leq V$. Show that $V_1 \cap V_2 \leq V$.

Suppose $V_1 \leq V$ and $V_2 \leq V$. Consider $V_1 \cap V_2$. Notice $0 \in V_2$ and $0 \in V_2$ $\therefore$ $0 \in V_1 \cap V_2 \neq \emptyset$. Next, suppose $c_1 \in \mathbb{R}$ and $x, y \in V_1 \cap V_2$. Then $x, y \in V_1$ and $x, y \in V_2$. But, $V_1 \leq V$ thus $c_1x + y \in V_1$ and $V_2 \leq V$ thus $c_1x + y \in V_2$. Therefore, $c_1x + y \in V_1 \cap V_2$. Thus $V_1 \cap V_2$ is closed under addition & scalar multiplication and the subspace theorem provides the nonempty $V_1 \cap V_2 \leq V$.//