MATH 321 QUIZ 3 (TAKEHOME, DUE START OF CLASS 4-27)

Copying answers and steps is strictly forbidden. Working together is encouraged, share ideas not calculations. Show work on other paper. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Suppose $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{4\times 4}$ and $\det(A) = 2$ and $\det(B) = 13$. Calculate:

$$\det \left[\begin{array}{c|c} -A & 0 \\ \hline 0 & 3B \end{array}\right].$$

Problem 2 Define $W = \text{span}\{(1,0,1,1),\ (0,2,2,3)\}$. Find an orthonormal basis γ_1 for W and an orthonormal basis γ_2 for $W^\perp$. Set $\beta = \gamma_1 \cup \gamma_2$ and calculate $[v]_\beta$ for $v = (2,2,2,2)$. Also, calculate $\text{Proj}_W(v)$ and $\text{Proj}_{W^\perp}(v)$

Problem 3 Let $T(f(x)) = f''(x)$ and define $V = \text{span}\{e^{2x},\ e^{3x},\ \sin(x)\}$. Show that T is diagonalizable by choosing an e-basis β for which $[T]_{\beta\beta}$ is diagonal. Calculate $\det(T)$ and $\text{trace}(T)$.

Problem 4 Find the real Jordan form of $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}$.

Problem 5 Find the formula for $Q(v)$ in terms of eigencoordinates y_1, y_2, y_3 given that

$$Q(v) = x^2 + y^2 + z^2 + 4xy - 4xz + 4yz$$

for the usual Cartesian coordinates $v = (x, y, z)$.

Problem 6 Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

Problem 7 Find an orthonormal basis for $W^\perp$ where $W = \text{span}\{(1,1,1,0,0),(2,2,2,2,2),(0,1,1,,0,2)\}$

Problem 8 If $A = \text{diag}\left(\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \right)$ where this notation indicates that A is block-diagonal with the diagonal blocks as given. Find the eigenvalues of A and state the algebraic and geometric multiplicity of each eigenvalue. In addition, find the characteristic and minimal polynomials for A. Exhibit the cycle Tableau for appropriate nilpotent maps associated to A.

Problem 9 Let $T : V \to V$ be a linear transformation and $\dim(V) = 6$ over $\mathbb{R}$. Find the characteristic and minimal polynomials of T given that: there exist linearly independent vectors v_1, v_2, v_3, v_4, v_5 in V such that:

$$T(v_1) = 3v_1, \quad T(v_2) - 3v_2 = v_1, \quad T(v_3 + iv_4) = (2+i)(v_3 + iv_4), \quad T^2(v_5) = 0.$$

Problem 10 Suppose A and B are nilpotent matrices for which $AB = BA$. Is $cA + B$ nilpotent for any $c \in \mathbb{R}$? Prove or disprove.

QUIZ 3 SOLUTION

P1 $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^{4\times 4}$, $\det A = 2$, $\det B = 13$ $\Big\}$ given

$$\det\left(\begin{array}{c|c} -A & 0 \\ \hline 0 & 3B \end{array}\right) = \det(-A)\,\det(3B)$$

$$= (-1)^3 \det A \cdot 3^4 \det B$$

$$= -(81)(2)(13)$$

$$= \boxed{-2106}$$

P2 $W = \text{span}\{\underbrace{(1,0,1,1)}_{v_1}, \underbrace{(0,2,2,3)}_{v_2}\}$

$$\underline{u_1 = \frac{1}{\sqrt{3}}(1,0,1,1)}.$$

$$\tilde{u}_2 = v_2 - (v_2 \cdot u_1)u_1 = (0,2,2,3) - \frac{1}{3}(5)(1,0,1,1)$$

$$\Rightarrow \tilde{u}_2 = (-5/3, 2, 2-5/3, 3-5/3)$$

$$\Rightarrow \tilde{u}_2 = \frac{1}{3}(-5, 6, 1, 4)$$

$$\underline{u_2 = \frac{1}{\sqrt{78}}(-5, 6, 1, 4)}.$$

Thus $\boxed{\gamma_1 = \left\{\frac{1}{\sqrt{3}}(1,0,1,1), \frac{1}{\sqrt{78}}(-5,6,1,4)\right\}}$ forms an orthonormal basis for W.

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There are several ways to go from here. I'll find $W^\perp$ directly then run GSA on the basis for $W^\perp = \text{Null}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/2 \end{bmatrix} \text{ thus,}$$

$$x = (x_1, x_2, x_3, x_4) \in W^\perp \Rightarrow \begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= -x_3 - \tfrac{3}{2}x_4 \end{aligned}$$

$$x = x_3(-1,-1,1,0) + x_4(-1,-3/2,0,1)$$

$$\therefore \underline{W^\perp = \text{span}\{(1,1,-1,0), (2,3,0,-2)\}}.$$

P2 continued:

$$W^{\perp} = \text{span}\{\underbrace{(1,1,-1,0)}_{w_1}, \underbrace{(2,3,0,-2)}_{w_2}\}$$

$$\underline{u_3 = \frac{1}{\sqrt{3}}(1,1,-1,0)}.$$ (you can check) $u_1 \cdot u_3 = 0$, $u_1 \cdot u_2 = 0$

$$\tilde{u}_4 = w_2 - (w_2 \cdot u_3)u_3 = (2,3,0,-2) - \frac{1}{3}(5)(1,1,-1,0)$$

$$\tilde{u}_4 = (1/3, 4/3, 5/3, -6/3)$$

$$\underline{u_4 = \frac{1}{\sqrt{78}}(1,4,5,-6)}.$$

$$\boxed{\text{Hence } \gamma_2 = \left\{\frac{1}{\sqrt{3}}(1,1,-1,0), \frac{1}{\sqrt{78}}(1,4,5,-6)\right\} \text{ is orthonormal basis for } W^{\perp}}$$

Let $\beta = \gamma_1 \cup \gamma_2$ and let $v = (2,2,2,2)$

$$\left.\begin{aligned} v \cdot u_1 &= 6/\sqrt{3} \\ v \cdot u_2 &= 6/\sqrt{3} \\ v \cdot u_3 &= 2/\sqrt{3} \\ v \cdot u_4 &= 4/\sqrt{3} \end{aligned}\right\} \quad v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3 + (v \cdot u_4)u_4$$

Thus

$$\boxed{v = \overbrace{\frac{6}{3}(1,0,1,1) + \frac{6}{3\sqrt{26}}(-5,6,1,4)}^{\text{Proj}_W(v)} + \underbrace{\frac{2}{3}(1,1,-1,0) + \frac{4}{3\sqrt{26}}(1,4,5,-6)}_{\text{Proj}_{W^{\perp}}(v)}}$$

$\boxed{P3}$ $T(f(x)) = f''(x)$

$T(e^{2x}) = (e^{2x})'' = 4e^{2x}$

$T(e^{3x}) = (e^{3x})'' = 9e^{3x}$

$T(\sin x) = (\sin x)'' = -\sin x$

all e-vectors for T with

$\lambda_1 = 4$

$\lambda_2 = 9$

$\lambda_3 = -1$

Let $\beta = \{e^{2x}, e^{3x}, \sin(x)\}$

and observe

$$[T]_{\beta\beta} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus,

$$\det(T) = \det [T]_{\beta\beta} = \boxed{-36}$$

$$\text{Trace}(T) = \text{trace} [T]_{\beta\beta} = 4+9-1 = \boxed{12}$$

P4 continued

$$A-(2+i)I = \begin{bmatrix} -i & 3 & 1 \\ 0 & -1-i & 0 \\ -1 & 2 & -i \end{bmatrix} \sim \begin{bmatrix} -i & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -1 & 0 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x = -iz \\ y = 0 \end{matrix}$$

Thus $(-iz, 0, z) \in \text{Null}\,(A-(2+i)I)$

Obtain $\underbrace{\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}}_{V_2} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{a_2} + i \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{b_2}$ complex e-vector

$\text{Re}(V_2) = a_2$
$\text{Im}(V_2) = b_2$

If $\beta = \{(3,-6,15), (0,0,1), (-1,0,0)\}$ then we can calculate directly that $[\beta]^{-1} A [\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} = J_A$.

(But, technically I just asked for J_A, I didn't say show me the similarity trans. to produce it.)

P4 Find "real" Jordan form of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 2 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)(1-\lambda)(2-\lambda) - 1(-1)(1-\lambda)$$

$$= (1-\lambda)\left[(2-\lambda)^2 + 1\right]$$

$$= (1-\lambda)\left[(\lambda-2)^2 + 1\right] \begin{array}{l} \to \lambda_1 = 1 \\ \to \lambda_2 = 2 \pm i \end{array}$$

Thus,

$$J_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & +1 & 2 \end{bmatrix}$$

How to obtain J_A from A?

$$A - I = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix} \begin{array}{l} \to x = -3y - z \\ 5y = -2z \end{array}$$

$$\sim \begin{bmatrix} 5 & 15 & 5 \\ 0 & 15 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & -1 \\ 0 & 15 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to z = 5x \\ \to y = \frac{-6z}{15} \\ x = \frac{z}{5} \end{array}$$

Thus $u_1 = (1/5, -6/15, 1)\, z$

or $\boxed{v_1 = (3, -6, 15) \text{ has } \lambda_1 = 1}$

$\boxed{P5}$ $$Q(x,y,z) = [x,y,z]\underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 2 & -2 \\ 2 & 1-\lambda & 2 \\ -2 & 2 & 1-\lambda \end{bmatrix}$$

$$= \det\begin{bmatrix} 1-\lambda & 2 & -2 \\ 2 & 1-\lambda & 2 \\ 0 & 3-\lambda & 3-\lambda \end{bmatrix}$$

$$= \det\begin{bmatrix} 3-\lambda & 3-\lambda & 0 \\ 2 & 1-\lambda & 2 \\ 0 & 3-\lambda & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)\big[(1-\lambda)(3-\lambda) - 2(3-\lambda)\big] +$$
$$- (3-\lambda)\big[2(3-\lambda)\big]$$

$$= (3-\lambda)^2\big[1-\lambda-2-2\big]$$

$$= (3-\lambda)^2(-3-\lambda) \longrightarrow \underline{\lambda_1 = 3,\ \lambda_2 = -3}.$$

Hence, in e-coordinates where $AV_1 = 3V_1$ and $AV_2 = 3V_2$ and $AV_3 = -3V_3$ we have

$$Q(y_1V_1 + y_2V_2 + y_3V_3) = \boxed{3y_1^2 + 3y_2^2 - 3y_3^2}.$$

Remark: I could find V_1, V_2, V_3 if need be. But, for the formula alone, I don't need to...

$\boxed{\text{P6}}$ Find e-values and basis for each e-space for $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

<u>Observation</u>: A has lin. dep. rows $\therefore$ A^{-1} d.n.e. and, I expect $\lambda = 0$ as one e-value.

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & 8-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 8) - 8$$
$$= \lambda^2 - 9\lambda + 8 - 8$$
$$= \underbrace{\lambda(\lambda - 9)}$$
$$\underline{\lambda_1 = 0} \ \& \ \underline{\lambda_2 = 9}.$$

$\underline{\lambda_1 = 0}$

$A \sim \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

$A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

$u + 4v = 0$

$u = -4v$

$\Rightarrow$ $\text{Null}(A) = \text{span}\{(-4, 1)\} = E_1$

$\boxed{\{(-4,1)\} \text{ basis for } E_1}$

$\underline{\lambda_2 = 9}$

$A - 9I = \begin{bmatrix} -8 & 4 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\therefore$ $(A - 9I)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has $2u - v = 0$

$v = 2u$.

Thus $\boxed{\{(1,2)\} \text{ is basis for } E_2}$

That is, $\underline{E_2 = \text{Null}(A - 9I) = \text{span}\{(1,2)\}}$.

<u>Remark</u>: you can check, $\beta = \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ has

$[\beta]^{-1} A [\beta] = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$. But, it was not req$^{\text{red}}$ here.

$\boxed{P7}$ Find orthonormal basis for $W^{\perp}$ where $W = \text{span}\{(1,1,1,0,0), (2,2,2,2,2), (0,1,1,0,2)\}$

$$W^{\perp} = \text{Null}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \quad \left(\begin{array}{c}\text{this gives } \perp \\ \text{vectors to } W\end{array}\right)$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $x = (x_1, x_2, x_3, x_4, x_5) \in W^{\perp}$ has $\begin{cases} x_1 = 2x_5 \\ x_2 = -x_3 - 2x_5 \\ x_4 = -x_5 \end{cases}$

we have free variables x_3, x_5

hence $x = x_3(0,-1,1,0,0) + x_5(2,-2,0,-1,1)$

$$\therefore \; \underline{W^{\perp} = \text{span}\{\underbrace{(0,-1,1,0,0)}_{v_1}, \underbrace{(2,-2,0,-1,1)}_{v_2}\}}.$$

Let,

$$u_1 = \frac{1}{\sqrt{2}}(0,-1,1,0,0)$$

Remark: easy to check v_1, v_2 $\perp$ to W

and,

$$\tilde{u}_2 = v_2 - (v_2 \cdot u_1)u_1 = (2,-2,0,-1,1) - \tfrac{1}{2}(2)(0,-1,1,0,0)$$

$$\tilde{u}_2 = (2,-1,-1,-1,1)$$

$$\Rightarrow \underline{u_2 = \frac{1}{\sqrt{8}}(2,-1,-1,-1,1)}.$$

Thus $\boxed{\gamma = \left\{\frac{1}{\sqrt{2}}(0,-1,1,0,0), \frac{1}{\sqrt{8}}(2,-1,-1,-1,1)\right\}}$

forms orthonormal basis for $W^{\perp}$

$\boxed{P8}$ $A = \operatorname{diag}\left(\begin{bmatrix} \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} & & \\ & \boxed{3} & \\ & & \boxed{3} \end{bmatrix}, \begin{bmatrix} \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & \\ & \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} \end{bmatrix} \right)$

We read e-values

$$\left.\begin{matrix} \lambda_1 = 3 \text{ with } m_1 = 4 \\ \lambda_2 = 4 \text{ with } m_2 = 4 \end{matrix}\right\} \text{algebraic multiplicities.}$$

$\dim(E_1) = 3 =$ geom. mult. of $\lambda_1 = 3$

$\dim(E_2) = 2 =$ geom. mult. of $\lambda_2 = 4$

The characteristic poly is,

$$P(t) = \det(A - tI) = (3-t)^4(4-t)^4$$

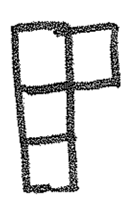
The minimal polynomial is <u>monic</u>

$$m(t) = g(t) = \underbrace{(t-3)^2(t-4)^2}$$

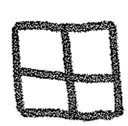
Let $T(v) = Av$ and
$K_1 = \operatorname{Ker}(T-3I)^4$
$K_2 = \operatorname{Ker}(T-4I)^4$
Then the Tableaus:

based on fact largest Jordan Block for both $\lambda_1 = 3$, $\lambda_2 = 4$ was 2×2.

$T|_{K_1} - 3I|_{K_1} = N_1$ has $N_1^4 = 0$ and



$(T-4I)|_{K_2} = N_2$ nilpotent $N_2^4 = 0$



$\boxed{P9}$ $T: V \longrightarrow V$ and $\dim V = 6$ over $\mathbb{R}$

Find char. & min. poly. of T given $\{v_1, v_2, v_3, v_4, v_5\}$ is LI and

$$T(v_1) = 3v_1 \longrightarrow (T - 3I)v_1 = 0$$

$$T(v_2) - 3v_2 = v_1 \longrightarrow (T - 3I)v_2 = v_1$$

tells me $\lambda_1 = 3$ with $m_1 \geq 2$.

$$T(\underbrace{v_3 + iv_4}_{v}) = (2+i)(\underbrace{v_3 + iv_4}_{v})$$

$$T(v) = (2+i)v$$

complex e-value $\lambda_2 = 2+i$

$$T^2(v_5) = 0 \Rightarrow \lambda_3 = 0 \text{ with } m_3 \geq 2$$

Thus $\dim(E_1) \geq 1$, $\dim(K_1) \geq 2$

and $\dim(\operatorname{span}\{v_3, v_4\}) = 2$

and $\dim(K_3) \geq 2$

But, as $\dim V = 6$ we must have equality in the above. Thus

$$\boxed{P(t) = (3-t)^2 \left((t-2)^2 + 1\right) t^2 \quad \text{char. poly.}}$$

(this is also the min. poly. as there is just one Jordan (real Jordan Block) for each e-value)

$\boxed{P10}$ If $A^n = 0$ and $B^k = 0$ where $A^{n-1}, B^{k-1} \neq 0$ and also $AB = BA$ we can show $(cA+B)^{nk} = 0$ as follows, since $AB = BA$ binomial Thm applies:

$$(cA+B)^{nk} = c^n A^m + m c^{m-1} A^{m-1} B^1 + \dots + m c A B^{m-1} + B^m$$

Now, argue in each term above

$$c^l A^i B^j \quad \text{has } i \geq n,\ j \geq k$$

Hence $(cA+B)^{nk} = 0 \therefore cA+B$ nilpotent.

— (this proof incomplete) —