*Please work the problems in the white space provided and clearly box your solutions (where appropriate). If there is not enough space please indicate that you wrote additional work on the back of a sheet.*

**Problem 1** (10pts) Find the solution set and write it as span$\{v_1, v_2\}$ for appropriate $v_1, v_2 \in \mathbb{R}^4$.

$$x_1 + x_2 + x_3 - x_4 = 0$$
$$x_1 + 2x_3 + x_4 = 0$$

**Problem 2** (20pts) Find the formula for a cubic polynomial whose graphs contains the points $(1, 10)$, $(2, 26)$, $(-1, 2)$ and $(0, 4)$.

**Problem 3** (10pts) Suppose the system of equations $Av = b$ where $v = (x_1, x_2, x_3, x_4)$ has a augmented coefficient matrix $[A|b]$ for which: $\text{rref}[A|b] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$. Find the solution set of $Av = b$ and determine if $(1, 2, 3, 4)$ is in the solution set.

**Problem 4** (15pts) Given that $A^{-1} = \begin{bmatrix} 4 & 1 \\ 11 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ calculate $(A^{-1}BA)^2$.

**Problem 5** (10pts) Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Use induction to prove $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for all $n \in \mathbb{N}$.

**Problem 6** (10pts) Let $A, B, C$ be square matrices and define $[A, B] = AB - BA$. Show: $[AB, C] = A[B, C] + [A, C]B$.

**Problem 7** (10pts) Suppose $A$ is a symmetric matrix. Define $M = I - 2A$. Prove or disprove: $M$ is symmetric.

**Problem 8** (15pts) Let $A \in \mathbb{R}^{m\times n}$ and $B, C \in \mathbb{R}^{n\times p}$. Prove that $A(B+C) = AB + AC$.

**Problem 9** (10pts) Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 5 & 5 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} = R$. Find an elementary matrix $E$ such that $R = EA$.

**Problem 10** (10pts) Find the standard matrices of $S$ and $T$ and $S \circ T$ given that:

$$S(x, y) = (x + 2y, 3x + 4y) \qquad \& \qquad T(a, b, c) = (7a + 6b + 5c, 4a + 3b + 2c)$$

**Problem 11** (20pts) Let $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 6 & -4 & 1 \\ 3 & 9 & -6 & 0 \end{bmatrix}$. Also, let $v_1 = (1,2,3)$, $v_2 = (3,6,9)$, $v_3 = (-2,-4,-6)$, $v_4 = (0,1,0)$ and $w = (a,b,c)$. Calculate $\text{rref}[A|w]$ and explicitly indicate each row-operation in your calculation. Find the set of all $w$ which are **not** in $\text{span}\{v_1, v_2, v_3, v_4\}$

**Problem 12** (10pts) Provide a **set of vectors** which contains the maximum number of linearly independent column vectors of $A$. (the matrix $A$ is as was given in the previous problem)

**Problem 13** (10pts) Two distinct solutions $v_1$ and $v_2$ can be found for the linear system $Av = b$. Which of the following is necessarily true? (circle answer)

**(a.)** $b = 0$

**(b.)** $A$ is invertible

**(c.)** $A$ has more columns than rows

**(d.)** $v_1 = -v_2$

**(e.)** There exists a solution $v$ such that $v \neq v_1$ and $v \neq v_2$.

**Problem 14** (10pts) Let $P, Q$ be invertible $n \times n$ matrices. Prove $PQ$ is an invertible matrix.

**Problem 15** (10pts) Prove that if both $A^TBA$ and $B$ are invertible, then $A$ is invertible.

**Problem 16** (30pts) Let $J$ be an invertible square matrix. Define $g(v, w) = v^TJw$. We say $A \in \mathcal{R}_J$ iff $g(Av, Aw) = g(v, w)$ for all $v, w \in \mathbb{R}^n$. Your goal is to prove that $\mathcal{R}_J$ forms a subgroup of invertible matrices. To accomplish the goal complete the following logical tasks:

(a) show $\mathcal{R}_J$ is a set of invertible matrices,

(b) show $I \in \mathcal{R}_J$,

(c) show if $A, B \in \mathcal{R}_J$ then $AB \in \mathcal{R}_J$,

(d) show if $A \in \mathcal{R}_J$ then $A^{-1} \in \mathcal{R}_J$.