(warning, the sol^{n}'s are here, but the #'s might not match with your test) 9:06

MATH 321	TEST 1

Please work the problems in the white space provided and clearly box your solutions (where appropriate). If there is not enough space please indicate that you wrote additional work on the back of a sheet.

Problem 1 (10pts) Find the solution set and write it as span$\{v_1, v_2\}$ for appropriate $v_1, v_2 \in \mathbb{R}^4$.

$$x_1 + x_2 + x_3 - x_4 = 0$$
$$x_1 + 2x_3 + x_4 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 + r_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{-r_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

$x_1 = -2x_3 - x_4$
$x_2 = x_3 + 2x_4$

$$(x_1, x_2, x_3, x_4) = (-2x_3 - x_4,\ x_3 + 2x_4,\ x_3,\ x_4)$$
$$= x_3(-2, 1, 1, 0) + x_4(-1, 2, 0, 1)$$

$\therefore$ $\boxed{\text{span}\{(-2,1,1,0), (-1,2,0,1)\} = \text{Sol}^{n}\text{ Set}}$

Problem 2 (15pts) Find the formula for a cubic polynomial whose graphs contains the points $(1, 10)$, $(2, 26)$, $(-1, 2)$ and $(0, 4)$. 9:09

$$g(x) = Ax^3 + Bx^2 + Cx + D$$

$g(0) = D = 4$
$g(1) = A + B + C + 4 = 10$
$g(2) = 8A + 4B + 2C + 4 = 26$
$g(-1) = -A + B - C + 4 = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 8 & 4 & 2 & 22 \\ -1 & 1 & -1 & -2 \end{array}\right] \xrightarrow[r_3 + r_1]{r_2 - 8r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & -6 & -26 \\ 0 & 2 & 0 & 4 \end{array}\right] \xrightarrow[r_2 + 2r_3]{r_1 - \frac{1}{2}r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 0 & -6 & -18 \\ 0 & 2 & 0 & 4 \end{array}\right]$$

enough, I see $2B = 4$ and $-6C = -18$ $\therefore$ $B = 2$, $C = 3$

Also $A + C = 4 \Rightarrow A = 4 - C = 4 - 3 = 1 = A$.

$\therefore$ $\boxed{g(x) = x^3 + 2x^2 + 3x + 4}$

Problem 3 (10pts) Suppose the system of equations $Av = b$ where $v = (x_1, x_2, x_3, x_4)$ has a augmented coefficient matrix $[A|b]$ for which: $\text{rref}[A|b] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$. Find the solution set of $Av = b$ and determine if $(1, 2, 3, 4)$ is in the solution set. 9:14

$x_1 = x_2 + 7$
$x_3 = 8$
$x_4 = 9$

$\boxed{\{(t+7, t, 8, 9) \mid t \in \mathbb{R}\}}$

To have $(1, 2, 3, 4) = (t+7, t, 8, 9)$ → $t = 2$

But, $2 + 7 = 9 \neq 1$

$\therefore$ $\boxed{\text{No } (1,2,3,4) \text{ not a sol}^{n}}$

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Problem 4 (15pts) Given that $A^{-1} = \begin{bmatrix} 4 & 1 \\ 11 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ calculate $(A^{-1}BA)^2$.

$B^2 = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = 9I$

$$\begin{aligned}(A^{-1}BA)^2 &= (A^{-1}BA)(A^{-1}BA) \\ &= A^{-1}B^2A \\ &= A^{-1}(9I)A \\ &= 9A^{-1}A = 9I = \boxed{\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}}\end{aligned}$$

Problem 5 (10pts) Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Use induction to prove $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for all $n \in \mathbb{N}$. 9:18

Observe $n=1$ is true. Suppose inductively $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for some $n \in \mathbb{N}$, Consider, by ind. hypo. and def$^{\underline{n}}$ of A^{n+1},

$$A^{n+1} = A^nA = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^{n+1} & 0 \\ 0 & b^{n+1} \end{pmatrix}$$

Hence the claim true for $n \Rightarrow$ the claim true for $n+1$ thus by proof by mathematical induction $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ $\forall n \in \mathbb{N}$.

9:20 **Problem 6** (10pts) Let A, B, C be square matrices and define $[A, B] = AB - BA$. Show: $[AB, C] = A[B, C] + [A, C]B$.

$$\begin{aligned}[AB, C] &= (AB)C - C(AB) \\ &= A(BC - CB) + (AC - CA)B \\ &= \underline{A[B, C] + [A, C]B} \ //\end{aligned}$$

added zero in the form $-ACB + ACB = 0$.

Problem 7 (10pts) Suppose A is a symmetric matrix. Define $M = I - 2A$. Prove or disprove: M is symmetric.

given $A^T = A$.

$$M^T = (I - 2A)^T = I^T - 2A^T = I - 2A = M$$

$\therefore$ $\boxed{M \text{ is symmetric}}$

Problem 8 (10pts) Let $A \in \mathbb{R}^{m \times n}$ and $B, C \in \mathbb{R}^{n \times p}$. Prove that $A(B + C) = AB + AC$.

$$\begin{aligned}(A(B+C))_{ij} &= \sum_{k=1}^{n} A_{ik}(B+C)_{kj} \quad &: \text{def}^{\underline{n}} \text{ of matrix mult.} \\ &= \sum_{k=1}^{n} (A_{ik}B_{kj} + A_{ik}C_{kj}) \quad &: \text{def}^{\underline{n}} \text{ of matrix add.} \\ &= \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj} = (AB)_{ij} + (AC)_{ij}\end{aligned}$$

prop. of finite sum

Hence the claim is true as the above holds $\forall i,j$.

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Problem 9 (10pts) Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 5 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} = R$. Find an elementary matrix E such that $R = EA$.

$$\boxed{E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$ (take I and swap rows 1 & 3)

Problem 10 (15pts) Let $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 6 & -4 & 1 \\ 3 & 9 & -6 & 0 \end{bmatrix}$. Also, let $v_1 = (1,2,3)$, $v_2 = (3,6,9)$, $v_3 = (-2,-4,-6)$, $v_4 = (0,1,0)$ and $w = (a,b,c)$. Calculate rref$[A|w]$ and explicitly indicate each row-operation in your calculation. Find the set of all w which are **not** in span$\{v_1, v_2, v_3, \cancel{v_4}\}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 2 & 6 & -4 & 1 & b \\ 3 & 9 & -6 & 0 & c \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 0 & 0 & -8 & 1 & b-2a \\ 0 & 0 & -12 & 0 & c-3a \end{array}\right] \xrightarrow{r_3/-12} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 0 & 0 & -8 & 1 & b-2a \\ 0 & 0 & 1 & 0 & \frac{3a-c}{12} \end{array}\right]$$

$$\xrightarrow[r_1 - 2r_3]{r_2 + 8r_3} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & a - \frac{2}{12}(3a-c) \\ 0 & 0 & 0 & 1 & b - 2a + \frac{8}{12}(3a-c) \\ 0 & 0 & 1 & 0 & \frac{1}{12}(3a-c) \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \boxed{\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2}a + \frac{1}{6}c \\ 0 & 0 & 1 & 0 & a/4 - c/12 \\ 0 & 0 & 0 & 1 & b - 2c/3 \end{array}\right]}$$

rref$[A|w]$

It follows $w \notin$ span$\{v_1, v_2, v_3\}$ if $w = (a,b,c)$ with $\frac{a}{4} - \frac{c}{12} \neq 0$

That is, $\boxed{\{(a,b,c) \mid a,b,c \in \mathbb{R} \text{ with } b - \frac{2c}{3} \neq 0\}}$

Problem 11 (10pts) Provide a **set of vectors** which contains the maximum number of linearly independent column vectors of A. (the matrix A is as was given in the previous problem)

pivot columns work here $\{V_1, V_3, V_4\}$.

also $\{V_2, V_3, V_4\}$ just cannot use both V_1 & V_2.

Problem 12 (15pts) Find the standard matrices of S and T and $S \circ T$ given that:

$$S(x,y) = (x + 2y, 3x + 4y) \quad \& \quad T(a,b,c) = (7a + 6b + 5c, 4a + 3b + 2c)$$

$$\boxed{[S] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \quad \boxed{[T] = \begin{bmatrix} 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix}}$$

$$[S \circ T] = [S][T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 15 & 12 & 9 \\ 37 & 30 & 23 \end{bmatrix}}$$

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Problem 11 (20pts) Let $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 6 & -4 & 1 \\ 3 & 9 & -6 & 0 \end{bmatrix}$. Also, let $v_1 = (1,2,3)$, $v_2 = (3,6,9)$, $v_3 = (-2,-4,-6)$, $v_4 = (0,1,0)$ and $w = (a,b,c)$. Calculate rref$[A|w]$ and explicitly indicate each row-operation in your calculation. Find the set of all w which are **not** in span$\{v_1, v_2, v_3$ [redacted]$\}$

$$[A|w] = \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 2 & 6 & -4 & 1 & b \\ 3 & 9 & -6 & 0 & c \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 0 & 0 & -8 & 1 & b-2a \\ 0 & 0 & -12 & 0 & c-3a \end{array}\right]$$

$$\xrightarrow{r_3/-12} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & a \\ 0 & 0 & -8 & 1 & b-2a \\ 0 & 0 & 1 & 0 & a/4 - c/12 \end{array}\right] \xrightarrow[r_2 + 8r_3]{r_1 - \frac{1}{2}r_3} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & a - \frac{1}{2}(a/4 - c/12) \\ 0 & 0 & 0 & 1 & b - 2a + 8(a/4 - c/12) \\ 0 & 0 & 1 & 0 & a/4 - c/12 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 7a/8 + c/24 \\ 0 & 0 & 1 & 0 & a/4 - c/12 \\ 0 & 0 & 0 & 1 & b - 2c/3 \end{array}\right]$$

correspond to V_1, V_2, V_3 see $b - 2c/3 \neq 0$ gives no sol^{n} to $C_1V_1 + C_2V_2 + C_3V_3 = (a,b,c)$

thus $\boxed{\{(a,b,c) \mid a,b,c \in \mathbb{R},\ b \neq \frac{2c}{3}\}}$ are not in span of $\{V_1, V_2, V_3\}$.

Problem 12 (10pts) Provide a **set of vectors** which contains the maximum number of linearly independent column vectors of A. (the matrix A is as was given in the previous problem)

Problem 13 (10pts) Two distinct solutions v_1 and v_2 can be found for the linear system $Av = b$. Which of the following is necessarily true? (circle answer)

→ ∞ly many sol^{ns}.

(a.) $b = 0$

(b.) A is invertible

(c.) A has more columns than rows

(d.) $v_1 = -v_2$

(e.) There exists a solution v such that $v \neq v_1$ and $v \neq v_2$.

$$Av_1 = b \longrightarrow A\left(\frac{V_1}{2} + \frac{V_2}{2}\right) = \frac{b}{2} + \frac{b}{2} = b$$
$$AV_2 = b$$

not V_1 or V_2 if it is then we adjust to account for that case also...

Problem 16 (10pts) Let P, Q be invertible $n \times n$ matrices. Prove PQ is an invertible matrix.

Observe $Q^{-1}P^{-1}$ is a well-defined matrix as P^{-1}, Q^{-1} are given to exist. Moreover $(PQ)(Q^{-1}P^{-1}) = PQQ^{-1}P^{-1} = PIP^{-1} = PP^{-1} = I$ thus $(PQ)^{-1} = Q^{-1}P^{-1}$ and this shows PQ is invertible.

Problem 17 (10pts) Prove that if both A^TBA and B are invertible, then A is invertible.

Suppose A^TBA and B are invertible. $\S$ towards $\rightarrow\leftarrow$ that A^{-1} d.n.e. $\Rightarrow \exists x_0 \neq 0$ such that $Ax_0 = 0$.

Observe $(A^TB)Ax_0 = A^TBAx_0 = A^TB(0) = 0$ and $x_0 \neq 0$

$\therefore (A^TBA)^{-1}$ d.n.e. (by the Prop. that M^{-1} exists $\Leftrightarrow$ $Mx = 0$ iff $x = 0$)

which $\rightarrow\leftarrow$ A^TBA^{-1} existing. Thus A^{-1} exists.//

(the fact that B^{-1} was invertible was not needed)

Problem 18 (20pts) Let J be an invertible square matrix. Define $g(v, w) = v^T Jw$. We say $A \in \mathcal{R}_J$ iff $g(Av, Aw) = g(v, w)$ for all $v, w \in \mathbb{R}^n$. Your goal is to prove that $\mathcal{R}_J$ forms a subgroup of invertible matrices. To accomplish the goal complete the following logical tasks:

(a) show $\mathcal{R}_J$ is a set of invertible matrices,

(b) show $I \in \mathcal{R}_J$,

(c) show if $A, B \in \mathcal{R}_J$ then $AB \in \mathcal{R}_J$,

(d) show if $A \in \mathcal{R}_J$ then $A^{-1} \in \mathcal{R}_J$.

$$R_J = \{ A \in \mathbb{R}^{n\times n} \mid g(Av, Aw) = g(v,w) \ \forall v, w \in \mathbb{R}^n \}$$

But, $g(Av, Aw) = g(v,w) \Leftrightarrow (Av)^T JAw = v^T Jw$

$\Leftrightarrow \underline{v^TA^TJAw = v^TJw}$ $*$

However, $*$ holds $\forall v, w \in \mathbb{R}^n$ hence we may select $v = e_i$ and $w = e_j$ to see $(A^TJA)_{ij} = J_{ij}$ $\therefore A^TJA = J$.

Thus $R_J = \{ A \mid A^TJA = J \} \subseteq \mathbb{R}^{n\times n}$.

(a.) Let $A \in R_J$ then $A^TJA = J$ $\therefore (A^TJA)^{-1} = J^{-1}$ which is given to exist. Thus, by PROBLEM 17, we find A^{-1} exists. But, A was arbitrary hence all matrices in R_J are invertible.//

(b.) Note $I^TJI = IJI = IJ = J$ $\therefore I \in R_J$.//

(c.) $A, B \in R_J \Rightarrow A^TJA = J$ and $B^TJB = J$. Consider, $(AB)^TJAB = B^TA^TJAB = B^TJB = J$ $\therefore (AB) \in R_J$.

(d.) If $A \in R_J \Rightarrow A^TJA = J$. Let $M = (A^{-1})^TJA^{-1}$. We must show $M = J$.

Observe: $A^TJ = \underline{JA^{-1}}$. Hence $M = (A^{-1})^TA^TJ = (A^T)^{-1}A^TJ = IJ = J \Rightarrow \underline{A^{-1} \in R_J}$ //