Please work the problems in the white space provided and clearly box your solutions (where appropriate). If there is not enough space please indicate that you wrote additional work on the back of a sheet.

Reminder: LI is read "linear independence". Also, $W \leq V$ reads "W is a subspace of V". If in doubt about whether I mean for something to be a vector space or linear transformation please ask.

Problem 1 Consider $W = \{(x+y, y-z, x+z) \mid x, y, z \in \mathbb{R}\}$. Prove or disprove: $W \leq \mathbb{R}^3$.

Problem 2 Prove: if $S = \{x, y\}$ is LI then $T = \{x+2y, 3x+4y\}$ is LI.

Problem 3 Let V and W be vector spaces and suppose T and S are linear transformations from V to W. Define $M = \{v \in V \mid T(v) + S(3v) = 0\}$. Prove that $M \leq V$.

Problem 4 Let $M \in \mathbb{R}^{2\times 2}$ and define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(X) = XM + 3X$. Prove that T is a linear transformation.

Problem 5 Let T be defined as in the previous problem and suppose $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the matrix of T with respect to the basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ for $\mathbb{R}^{2\times 2}$. That is: calculate $[T]_{\beta,\beta}$.

Problem 6 Let $T : V \to W$ be a linear transformation and $V = \text{span}\{v_1, v_2, v_3, v_4\}$ and $W = \text{span}\{w_1, w_2, w_3\}$ for some nonzero vectors $v_1, v_2, v_3, v_4, w_1, w_2, w_3$. List the possible dimensions for $\text{Ker}(T)$ and $\text{Range}(T)$. Write your answer in tabular form as indicated below (explain your reseasoning briefly before table)

Problem 7 Let $W = \{A \in \mathbb{R}^{2\times 2} \mid \text{trace}(A) = 0\}$. Prove that $\beta = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$ is a basis for W. Also, find $\Phi_\beta(A)$ for a generic element A in W.

Problem 8 Let $\beta = \{1, x, f(x)\}$ form a basis for P_2. Furthermore, $v = x^2 + x - 3$ has $[v]_\beta = (2, 3, 4)$. Calculate $f(x)$.

Problem 9 Given that $V = V_1 \oplus V_2$. **Prove:** $V/V_2 \approx V_1$.

Problem 10 Let V be a n-dimensional vector space over $\mathbb{R}$. Suppose there exists a basis $\beta = \{f_1, f_2, \ldots, f_n\}$ for which $T(f_j) = j^2 f_j$ for $j = 1, 2, \ldots, n$. Prove that T is an isomorphism of V.

Problem 11 Relate $[T]_{\beta,\gamma}$ and $[T]_{\bar{\beta},\bar{\gamma}}$ given that $T : V \to W$ is a linear transformation and $\beta, \bar{\beta}$ are bases for V whereas $\gamma, \bar{\gamma}$ are bases for W. Please include a diagram which motivates your coordinate change formula.

Problem 12 Suppose V is a finite dimensional vector space over $\mathbb{R}$. Let $x \neq 0$ be a vector in V and suppose $T : V \to V$ is a linear transformation. Let

$$W = \text{span}\{x, T(x), T^2(x), \dots\}.$$

First, prove $W \leq V$. Second, show that $T(W) \subseteq W$.

Problem 13 Suppose V is a finite dimensional vector space over $\mathbb{R}$. Let $x \neq 0$ be a vector in V and suppose $T : V \to V$ is a linear transformation. Let

$$W = \text{span}\{x, T(x), T^2(x), \dots\}.$$

Prove (or find for part (b.)) the following:

(a.) if $\dim(W) = k$ and we define the T-cyclic basis generated by x to be:

$$\beta(x) = \{x, T(x), T^2(x), \dots, T^{k-1}(x)\}.$$

Then $\beta(x)$ is a basis for W.

(b.) the matrix of the restriction of T to W with respect to $\beta(x)$.

(c.) if there exists $y \notin W$ then show $W_2 = \text{span}(\beta(y)) \cap W = \{0\}$.

(d.) Finally, explain why V can be written as the direct sum of T-invariant subspaces like W and W_1.