Please work the problems in the white space provided and clearly box your solutions (where appropriate). If there is not enough space please indicate that you wrote additional work on the back of a sheet.

Reminder: LI is read "linear independence". Also, $W \leq V$ reads "W is a subspace of V". If in doubt about whether I mean for something to be a vector space or linear transformation please ask.

20pts **Problem 1** Consider $W = \{(x+y, y-z, x+z) \mid x, y, z \in \mathbb{R}\}$. Prove or disprove: $W \leq \mathbb{R}^3$.

$$= \{ x(1,0,1) + y(1,1,0) + z(0,-1,1) \mid x, y, z \in \mathbb{R} \}$$

$$= \text{Span}\{ (1,0,1), (1,1,0), (0,-1,1) \}$$

$\therefore$ $W \leq \mathbb{R}^3$ as W is formed by a span.

20pts **Problem 2** Prove: if $S = \{x, y\}$ is LI then $T = \{x + 2y, 3x + 4y\}$ is LI.

Assume S is LI. Consider

$$\underline{c_1(x + 2y) + c_2(3x + 4y) = 0} \quad *$$

$$\Rightarrow (c_1 + 3c_2)x + (2c_1 + 4c_2)y = 0$$

However, by LI of $\{x, y\}$ we have the coefficients $c_1 + 3c_2 = 0$ and $2c_1 + 4c_2 = 0$. Solve for $c_1 = -3c_2$ and substitute to obtain $2(-3c_2) + 4c_2 = -2c_2 = 0 \therefore \underline{c_2 = 0}$ and it follows $c_1 = -3(0) = 0$. Hence $* \Rightarrow c_1 = c_2 = 0$ and we find T is LI.//

20pts **Problem 3** Let V and W be vector spaces and suppose T and S are linear transformations from V to W. Define $M = \{v \in V \mid T(v) + S(3v) = 0\}$. Prove that $M \leq V$.

Notice $T(v) + S(3v) = T(v) + 3S(v) = (T + 3S)(v)$ and $L = T + 3S$ is a linear transformation as T & S are linear transformations. Further

$$M = \text{Ker}(L) \quad \therefore \quad M \leq V.$$

(alternatively, could use subspace theorem)

20 pts

Problem 4 Let $M \in \mathbb{R}^{2\times 2}$ and define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(X) = XM + 3X$. Prove that T is a linear transformation.

$$\begin{aligned} T(cX + Y) &= (cX+Y)M + 3(cX+Y) \\ &= c(XM+3X) + YM + 3Y \\ &= c\,T(X) + T(Y) \end{aligned}$$

thus T is additive $(c=1)$ and homogeneous $(Y=0)$ and it follows $T:\mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ is linear mapping.

20 pts

Problem 5 Let T be defined as in the previous problem and suppose $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the matrix of T with respect to the basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ for $\mathbb{R}^{2\times 2}$. That is: calculate $[T]_{\beta,\beta}$.

$$\begin{aligned} T(A) = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \left[\begin{array}{c|c} a+3b & 2a+4b \\ \hline c+3d & 2c+4d \end{array}\right] + \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} \\ &= \left[\begin{array}{c|c} 4a+3b & 2a+7b \\ \hline 4c+3d & 2c+7d \end{array}\right] \end{aligned}$$

& $[A]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ because

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Observe $[T(A)]_\beta = \begin{bmatrix} 4a+3b \\ 2a+7b \\ 4c+3d \\ 2c+7d \end{bmatrix}$

$$[T]_{\beta\beta}[A]_\beta = \begin{bmatrix} & & & \\ & & & \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4a+3b \\ 2a+7b \\ 4c+3d \\ 2c+7d \end{bmatrix} = [T(A)]_\beta \longrightarrow \boxed{[T]_{\beta\beta} = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix}}$$

20 pts

Problem 6 Let $T : V \to W$ be a linear transformation and $V = \text{span}\{v_1, v_2, v_3, \cancel{v_4}\}$ and $W = \text{span}\{w_1, w_2, w_3\}$ for some nonzero vectors $v_1, v_2, v_3, v_4, w_1, w_2, w_3$. List the possible dimensions for $\text{Ker}(T)$ and $\text{Range}(T)$. Write your answer in tabular form as indicated below (explain your reseasoning briefly before table)

use rank/nullity

$\text{nullity}(T) + \text{rank}(T) = \dim(V)$

dim (V)	Ker (T)	Range (T)
1	1	0
1	0	1
2	2	0
	1	1
	0	2
3	3	0
	2	1
	1	2
	0	3

Problem 7 Let $W = \{A \in \mathbb{R}^{2\times 2} \mid \text{trace}(A) = 0\}$. Prove that $\beta = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$ is a basis for W. Also, find $\Phi_\beta(A)$ for a generic element A in W.

30pts

<u>LI of β ?</u>/ Suppose $c_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

it follows $\left[\begin{array}{c|c} c_3 & c_1 + c_3 \\ \hline c_2 + c_3 & -c_3 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_3 = 0 \\ c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{array}$

But $c_3 = 0 \Rightarrow c_1 + c_3 = \underline{c_1 = 0}$ and $c_2 + c_3 = \underline{c_2 = 0}$.

Thus β is LI.

<u>Spanning?</u>/ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ then $\text{trace}(A) = a + d = 0$ hence $d = -a$ and we have $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. Suppose

$$c_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} c_3 & c_1 + c_3 \\ \hline c_2 + c_3 & -c_3 \end{array}\right] = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

Thus choose $\underline{c_3 = a}$, $c_1 + c_3 = b \Rightarrow c_1 = b - c_3 = b - a$ hence choose $\underline{c_1 = b - a}$ and we also wish to solve $c_2 + c_3 = c \therefore c_2 = c - c_3 = c - a$. Hence choose $c_2 = c - a$.

In summary,

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = (b-a) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (c-a) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \text{span}\,\beta$$

But, $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is an arbitrary element of W and it follows that $\text{span}\,\beta \supseteq W$. ~~(The fact span $\beta \subseteq W$ is immediate from def^n of~~

Conversely, it is clear $\beta \subseteq W$ and hence $\text{span}\,\beta \subseteq W \therefore \text{span}\,\beta = W$. Last, we have shown $\boxed{\Phi_\beta \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = (b-a,\ c-a,\ a)}$

15pts

Problem 8 Let $\beta = \{1, x, f(x)\}$ form a basis for P_2. Furthermore, $v = x^2 + x - 3$ has $[v]_\beta = (2, 3, 4)$. Calculate $f(x)$.

$[v]_\beta = (2,3,4) \Rightarrow v = 2 + 3x + 4f(x)$. But, $v = x^2 + x - 3$

thus $2 + 3x + 4f(x) = x^2 + x - 3$ and we derive,

$$\boxed{f(x) = \frac{x^2 - 2x - 5}{4}}$$

Problem 9 Given that $V = V_1 \oplus V_2$. **Prove:** $V/V_2 \approx V_1$.

15pts

Note, as $V = V_1 \oplus V_2$ for each $x \in V$, $\exists!\ x_1 \in V_1$ and $x_2 \in V_2$ s.t. $x = x_1 + x_2$. Consider then,

$\Pi_1 : V_1 \oplus V_2 \longrightarrow V_1$ (this is clearly linear ☺)

defined by $\Pi_1(x) = x_1$ (or $\Pi(x_1 + x_2) = x_1$ if you prefer)

Observe Π_1 is surjective as $x_1 \in V_1$ has $\Pi_1(x_1) = x_1$.

Moreover, $\ker(\Pi_1) = \{x_1 + x_2 \mid \Pi_1(x_1 + x_2) = x_1 = 0\}$
$= \{x_2 \mid x_2 \in V_2\} = V_2$.

Then, by First Isomorphism Th^{m}, $\dfrac{V_1 \oplus V_2}{\ker \Pi_1} = \Pi_1(V_1 \oplus V_2) \Rightarrow \dfrac{V_1 \oplus V_2}{V_2} \approx V_1$ //

Problem 10 Let V be a n-dimensional vector space over $\mathbb{R}$. Suppose there exists a basis $\beta = \{f_1, f_2, \cdots, f_n\}$ for which $T(f_j) = j^2 f_j$ for $j = 1, 2, \ldots, n$. Prove that T is an isomorphism of V.

20pts

$\ker(T) = \{x \in V \mid T(x) = 0\}$

However, $x \in V \Rightarrow x = \sum_{i=1}^{n} x_i f_i$ thus,

$T(x) = 0 \Rightarrow \sum_{i=1}^{n} x_i\, T(f_i) = 0 \Rightarrow \sum_{i=1}^{n} x_i\, i^2 f_i = 0$

But, by LI of β we deduce $i^2 x_i = 0$ for $i = 1, 2, \ldots, n$ hence $x_i = 0$ $\forall i = 1, 2, \ldots, n$. Thus as $T(0) = 0$ we find $\ker(T) = \{0\}$. But, surely $\dim V = \dim V$ hence $T : V \longrightarrow V$ is an isomorphism as surjectivity is immediate from rank nullity Th^{m}

$\dim(\underbrace{\ker(T)}_{0}) + \dim(\text{Range}(T)) = \underbrace{\dim(V)}_{n} \Rightarrow \text{Range}(T) = V$.

Thus T is an isomorphism from V to V

(this makes T an <u>automorphism</u> of V)

Problem 11 Relate $[T]_{\beta,\gamma}$ and $[T]_{\bar\beta,\bar\gamma}$ given that $T : V \to W$ is a linear transformation and $\beta, \bar\beta$ are bases for V whereas $\gamma, \bar\gamma$ are bases for W. Please include a diagram which motivates your coordinate change formula.

10pts

$$[T]_{\bar\beta,\bar\gamma} = \left[\Phi_{\bar\gamma} \circ \Phi_{\gamma}^{-1} \circ L_{[T]_{\beta,\gamma}} \circ \Phi_{\beta} \circ \Phi_{\bar\beta}^{-1} \right]$$

$$\begin{tikzcd}
 & V \arrow[r, "T"] \arrow[dl, "\Phi_\beta"'] \arrow[d, "\Phi_{\bar\beta}"] & W \arrow[d, "\Phi_{\bar\gamma}"'] \arrow[dr, "\Phi_\gamma"] & \\
\mathbb{R}^n \arrow[rrr, bend right, "{[T]_{\beta,\gamma}}"'] & \mathbb{R}^n \arrow[l, "①"'] \arrow[r, "{[T]_{\bar\beta,\bar\gamma}}"] & \mathbb{R}^m & \mathbb{R}^m \arrow[l, "②"']
\end{tikzcd}$$

① $\Phi_\beta \circ \Phi_{\bar\beta}^{-1}$

② $\Phi_{\bar\gamma} \circ \Phi_\gamma^{-1}$

$$\therefore \boxed{[T]_{\bar\beta,\bar\gamma} = [\Phi_{\bar\gamma} \circ \Phi_\gamma^{-1}][T]_{\beta,\gamma}[\Phi_\beta \circ \Phi_{\bar\beta}^{-1}]}$$

Problem 12 Suppose V is a finite dimensional vector space over $\mathbb{R}$. Let $x \neq 0$ be a vector in V and suppose $T : V \to V$ is a linear transformation. Let

$$W = \text{span}\{x, T(x), T^2(x), \dots\}.$$

First, prove $W \leq V$. Second, show that $T(W) \subseteq W$.

10pts

Observe W is formed by the span of a set of vectors in a finite dim'l vector space $\therefore W \leq V$.

Observe $\dim(V) = n \Rightarrow \dim(W) \leq n$ thus

$\beta = \{x, T(x), T^2(x), \dots, T^n(x)\}$ is a set of $(n+1)$-vectors which must be linearly dependent. Let

~~$\gamma = \{x, T(x), \dots, T^{k-1}(x)\}$ be the largest LI subset of this form. It~~

It follows $W = \text{span}\,\beta$. Consider then

$v = c_1 x + c_2 T(x) + c_3 T^2(x) + \dots + c_n T^n(x)$,

Note $T(v) = c_1 T(x) + c_2 T^2(x) + \dots + c_n T^{n+1}(x) \in W$

$\therefore T(W) \subseteq W.$ //

Problem 13 Suppose V is a finite dimensional vector space over $\mathbb{R}$. Let $x \neq 0$ be a vector in V and suppose $T : V \to V$ is a linear transformation. Let

$$W = \text{span}\{x, T(x), T^2(x), \dots\}.$$

Prove (or find for part (b.)) the following:

(a.) if $\dim(W) = k$ and we define the T-cyclic basis generated by x to be:

$$\beta(x) = \{x, T(x), T^2(x), \dots, T^{k-1}(x)\}.$$

Then $\beta(x)$ is a basis for W.

(b.) the matrix of the restriction of T to W with respect to $\beta(x)$.

(c.) if there exists $y \notin W$ then show $W_2 = \text{span}(\beta(y)) \cap W = \{0\}$.

(d.) Finally, explain why V can be written as the direct sum of T-invariant subspaces like W and W_1.