# Lecture Notes for Complex Variables

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## introduction and motivations for these notes

A complex variable is simply a variable whose possible values are allowed to reside in the complex numbers. We're using the classic text by Churchill and Brown:

> "Complex Variables and Applications" by Churchill and Brown, 6-th Ed.

This text has been a staple of several generations of mathematicians at this time. I'll try to follow the text somewhat closely. I plan to ask you to prove certain pivotal Lemmas as we develop the material together this semester. In previous courses you may have heard me advocate a certain point of view about complex numbers but I would ask you forget all that for a time. Our goal here is to start from scratch and build complex numbers from the "ground" up. The purpose of these notes is to complement Churchill's text. I will try to add examples to expand on what is already in the text. Also, I will try to give comments about connections to other fields of mathematics where appropriate. Most of the theorems contained in these notes are likewise contained in Churchill and I will try to make a note when they are sufficiently famous. Other theorems are more the natural outgrowth of carefully chosen defintions and I probably will not source those theorems. I will try to include some historical comments to help you understand how the theory of complex variables was developed ( and is continuing to develop).

We will use a fair amount of linear algebra in portions of this course, however if you have not had math 321 you should still be able to follow along.

Approximate Lecture List:

- ✓ history of complex numbers and competing defintions.
- ✓ algebraic properties of $\mathbb{C}$.
- ✓ polar form of complex numbers.
- ✓ complex logarithms and subtletites of multiply valued functions.
- ✓ topological properties of $\mathbb{C}$.
- ✓ continuous functions of a complex variable.
- ✓ complex differentiation and the Cauchy Riemann equations.
- ✓ the conjugate variable notation, homomorphic and antiholomorphic.
- ✓ Maximum modulus theorem.
- ✓ Cauchy-Goursat theorem.
- ✓ contour integration.
- ✓ Laurent series.
- ✓ geometric series techniques.

- ✓ theory of residues.
- ✓ integration techniques.
- ✓ proof of fundamental theorem of algebra.
- ✓ conformal mapping.
- ✓ Riemann surfaces.

Before we begin, I should warn you that I assume quite a few things from the reader. These notes are intended for someone who has already grappled with the problem of constructing proofs. I assume you know the difference between $\Rightarrow$ and $\Leftrightarrow$. I assume the phrase "iff" is known to you. I assume you are ready and willing to do a proof by induction, strong or weak. I assume you know what $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ denote. I assume you know what a subset of a set is. I assume you know how to prove two sets are equal. I assume you are familar with basic set operations such as union and intersection (although we don't use those much). More importantly, I assume you have started to appreciate that mathematics is more than just calculations. Calculations without context, without theory, are doomed to failure. At a minimum theory and proper mathematics allows you to communicate analytical concepts to other like-educated individuals.

Some of the most seemingly basic objects in mathematics are insidiously complex. We've been taught they're simple since our childhood, but as adults, mathematical adults, we find the actual definitions of such objects as $\mathbb{R}$ is rather involved. I will not attempt to provide foundational arguments to build real numbers from basic set theory. I believe it is possible, I think it's well-thought-out mathematics, but we take the existence of the real numbers as an axiom for these notes. We assume that $\mathbb{R}$ exists and that the real numbers possess all their usual properties. In fact, I assume $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ all exist complete with their standard properties. In short, I assume we have numbers to work with. We leave the rigorization of real numbers to a different course. (truth is that complex numbers are relatively easy to construct once you have the starting point of $\mathbb{R}$.)
**Finally, please be warned these notes are a work in progress. I look forward to your input on how they can be improved, corrected and supplemented.**

**James Cook, January 19, 2010.**

**version 0.2**

# Contents

# Chapter 1

# complex numbers

## 1.1 foundations of complex numbers

Let's begin with the definition of complex numbers due to Gauss. We assume that the real numbers exist with all their usual field axioms. Also, we assume that $\mathbb{R}^n$ is the set of $n$-tuples of real numbers. For example, $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$.

**Definition 1.1.1.**

We define **complex multiplication** of points in $\mathbb{R}^2$ according to the rule:

$$(x, y) * (a, b) = (xa - yb,\ xb + ya)$$

for all $(x, y), (a, b) \in \mathbb{R}^2$. We define the **real part** of $(x, y)$ by $Re(x, y) = x$ and the **imaginary part** of $(x, y)$ by $Im(x, y) = y$. We define **complex addition** and **complex subtraction** by the usual operations on vectors in $\mathbb{R}^2$

$$(x, y) + (a, b) = (x + a, y + b) \qquad (x, y) - (a, b) = (x - a, y - b)$$

We say $z \in \mathbb{R}^2$ is **real** iff $Im(z) = 0$. Likewise, $z \in \mathbb{R}^2$ is said to be **imaginary** iff $Re(z) = 0$.

Notice that $*$ is a binary operation on $\mathbb{R}^2$; in other words $* : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is a function. Of course, there are many other binary operations you can imagine for the plane. What makes this one so special is that it models all the desired algebraic traits of a complex number. Since many people are unwilling to cede the existence of mathematical objects merely on the basis of algebra this construction due to Gauss is nice. It gives us an answer to the question: "what is a complex number?" The answer is: "you can view them as two dimensional vectors with a special multiplication". There are many other answers but that is the one we mostly pursue in these notes[1]. At this point you should be saying to yourself, WHAT? How in the world is $\mathbb{R}^2$ with $*$ the same as the complex numbers $\mathbb{C}$ we needed to solve quadratic equations? Let's work it out.

[1] complex numbers can also be constructed from $2 \times 2$ matrices or through field extension theory as you can study in Math 422 at LU, there are likely other ways to **construct** complex numbers.

**Proposition 1.1.2.**

Let $z \in \mathbb{R}^2$ then $z * (1,0) = z$ and $(1,0) * z = z$ therefore the vector $(1,0)$ is a multiplicative identity for complex multiplication.

**Proof:** suppose $z = (x,y) \in \mathbb{R}^2$ then $z * (1,0) = (x,y) * (1,0) = (x1 - y0, x0 + y1) = (x,y)$. Likewise, $(1,0) * z = (1,0) * (x,y) = (1x - 0y, 1y + 0x) = (x,y) = z$. $\square$

**Proposition 1.1.3.**

The equation $z * z = (-1,0)$ has solution $(0,1)$.

**Proof:** to say that $(0,1)$ solves the equation means that if we substitute it for $z$ in the given equation then the equation holds true. Note then

$$(0,1) * (0,1) = (0(0) - 1(1), 0(1) + 1(0)) = (-1,0). \qquad \square$$

In the notation of later sections $(-1,0) = 1$ and $(0,1) = i$ and we just proved that $i^2 = -1$. This funny vector multiplication gives us a way to build the imaginary number $i$.

**Theorem 1.1.4.** *Complex numbers form a field.*

Let $v, w, z \in \mathbb{R}^2$ with $z = (x,y)$ then

1. $z + w = w + z$; addition is commutative.
2. $(v + w) + z = v + (w + z)$; addition is associative.
3. $z + (0,0) = z$; additive identity.
4. $z + (-x,-y) = (0,0)$; additive inverse.
5. $z * w = w * z$; multiplication is commutative.
6. $(v * w) * z = v * (w * z)$; multiplication is associative.
7. $z * (1,0) = z$; multiplicative identity.
8. for $z \neq 0$ there exists $z^{-1}$ such that $z * z^{-1} = (1,0)$; additive inverse.
9. $v * (z + w) = v * z + v * w$; distributive property.

**Proof:** each of these is proved by simply writing it out and using the definition of the $*$ multiplication. Notice we already proved (7.). I'll prove (8.) and (9.), Some of the others are in your homework.

Begin with (9.). Let $v = (a, b), z = (x, y)$ and $w = (r, t)$. Observe by defintion of $*$ and $+$ on $\mathbb{R}^2$,

$$\begin{aligned} v * (z + w) &= (a, b) * [(x, y) + (r, t)] \\ &= (a, b) * (x + r, y + t) \\ &= (a(x + r) - b(y + t), a(y + t) + b(x + r)) \\ &= (ax + ar - by - bt, ay + at + bx + br) \\ &= (ax - by, ay + bx) + (ar - bt, at + br) \\ &= (a, b) * (x, y) + (a, b) * (r, t) \\ &= v * z + v + w. \end{aligned}$$

Therefore (9.) is true for all $v, w, z \in \mathbb{R}^2$. Notice in the calculation above I used the distributive field axioms for $\mathbb{R}$ several times.

To prove (8.) we first must search out the formula for $z^{-1}$. Set it up as an algebra problem. We're given that $z = (x, y) \neq 0$ hence either $x \neq 0$ or $y \neq 0$. We would like to find $z^{-1} = (a, b)$ such that

$$(x, y) * (a, b) = (1, 0) \quad \Rightarrow \quad (ax - by, xb + ya) = (1, 0)$$

Thus by definition of vector equality,

$$ax - by = 1 \qquad \text{and} \qquad xb + ya = 0$$

We'll need to consider several cases.

**Case 1:** $x \neq 0$ but $y = 0$ then $ax = 1$ hence $a = 1/x$ and so $ya = 0$ and it follows $xb = 0$ hence $b = 0$ and we deduce $z^{-1} = (1/x, 0)$.

**Case 2:** $x = 0$ but $y \neq 0$ then $-by = 1$ hence $b = -1/y$ and so $xb = 0$ and it follows $ya = 0$ hence $a = 0$ and we deduce $z^{-1} = (0, -1/y)$.

**Case 3:** $x \neq 0$ and $y \neq 0$ so we can divide by both $x$ and $y$ without fear,

$$xb + ya = 0 \Rightarrow b = -ya/x$$

$$ax - by = 1 \ \Rightarrow \ ax + y^2a/x = 1 \ \Rightarrow \ a(x^2 + y^2) = x \Rightarrow \ \underline{a = \frac{x}{x^2 + y^2}}$$

Substitute that into $b = -ya/x$,

$$b = \frac{-y}{x} \frac{x}{x^2 + y^2} = \frac{-y}{x^2 + y^2}$$

Note that the formulas for cases 1 and 2 are also covered by 3 despite the fact that the derivation for case 3 is nonsense in those cases, neat. To summarize:

$$\boxed{z^{-1} = \left( \frac{x}{x^2 + y^2}, \ \frac{-y}{x^2 + y^2} \right).}$$

The formula above solves $z^{-1} * z = (1, 0)$ for all $z \in \mathbb{R}^2$ such that $x^2 + y^2 \neq 0$. The proof of (8.) follows. $\square$

**Definition 1.1.5.**

We define **division of $z$ by $w$** for $z, w \in \mathbb{R}^2$ where $w \neq 0$ to be multiplication by the inverse of the reciprocal, $z/w = z * w^{-1}$.

**Example 1.1.6.** .

## 1.2 complex conjugation

**Definition 1.2.1.**

The **complex conjugate** of $(x, y) \in \mathbb{R}^2$ is denoted $\overline{(x, y)}$ where we define $\overline{(x, y)} = (x, -y)$.

The complex conjugate of a vector is the reflection of the vector about the $x$-axis. Naturally if we do two such reflections we'll get back to where we started. I don't suppose that all the properties listed in the theorem below are that easy to "see".

**Theorem 1.2.2.** *Properties of conjugation.*

Let $z, w \in \mathbb{R}^2$,

1. $\overline{z + w} = \overline{z} + \overline{w}$.
2. $\overline{z * w} = \overline{z} * \overline{w}$.
3. $\overline{z/w} = \overline{z}/\overline{w}$.
4. $\overline{\overline{z}} = z$

The properties above are easy to verify, I leave it to the reader or the test.

**Theorem 1.2.3.** *Properties of conjugation.*

Let $z \in \mathbb{R}^2$,

1. if $z = (x, y)$ then $z * \overline{z} = (x^2 + y^2, 0)$.
2. if $z = (x, y)$ then $(x, 0) = \frac{1}{(2,0)}(z + \overline{z})$
3. if $z = (x, y)$ then $(y, 0) = \frac{1}{(0,2)}(z - \overline{z})$

**Proof:** Begin with (1.),

$$z * \overline{z} = (x, y) * (x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0).$$

Now (2.),

$$z + \overline{z} = (x, y) + (x, -y) = (2x, 0) \quad \Rightarrow \quad z + \overline{z} = (x, 0) * (2, 0).$$

To see (3.) we subtract,

$$z - \overline{z} = (x, y) - (x, -y) = (0, 2y) \quad \Rightarrow \quad z - \overline{z} = (y, 0) * (0, 2).$$

The theorem follows. $\square$.

**Remark 1.2.4.**

I believe at this point we have proved enough properties of $\mathbb{R}^2$ paired with $*$ to convince you that we really can construct such a thing as $\mathbb{C}$. From this point onward I will revert to the standard notation which assumes the things we have just proved in these notes so far. In short I will omit the $*$ and write $(x, 0) = x$ and $(0, y) = yi$. The fundamental formulas are $(1, 0) = 1$ and $(0, 1) = i$. Thus we find the unit vectors in the Argand plane are precisely the number one and the imaginary number $i$. In view of this correspondence we find great logic in saying the vertical axes in the complex plane $\mathbb{R}^2$ has unit vector $i$ whereas the $x$-axes has unit vector 1. We adopt the notation $\mathbb{R}^2$ together with $*$ is $\mathbb{C}$.

Let me restate the theorem in less obtuse notation,

**Theorem 1.2.5.** *Properties of conjugation.*

Let $z \in \mathbb{C}$,

1. if $z = (x, y)$ then $z\overline{z} = x^2 + y^2$.
2. if $z = (x, y)$ then $x = \frac{1}{2}(\overline{z} + z)$
3. if $z = (x, y)$ then $y = \frac{1}{2i}(\overline{z} - z)$
4. If $z = Re(z) + iIm(z)$ then $Re(z) = \frac{1}{2}(z + \overline{z})$ and $Im(z) = \frac{1}{2i}(z - \overline{z})$.

We can also restate the field axioms with the $*$ omitted. Our custom will be the usual one throught the remainder of the course, we use *juxtaposition* to denote multiplication. At this point I have covered what I am likely to cover from §1&2 of Churhill.

## 1.3 modulus and reality

The modulus of a complex number is the length of the corresponding vector in $\mathbb{R}^2$.

**Definition 1.3.1.**

The **modulus** of $z \in \mathbb{C}$ is denoted $|z|$ where we define $|z| = \sqrt{z\bar{z}}$.

Notice that item (1.) of Theorem 1.2.5 shows that $z\bar{z}$ is a non-negative quantity therefore the squareroot will return a real, non-negative, quantity. We also can calculate the distance between complex numbers via the modulus as follows:

**Definition 1.3.2.**

Let $z, w \in \mathbb{C}$. The **distance between** $z$ **and** $w$ is denoted $d(z, w)$ and we define $d(z, w) = |z - w|$.

Let's pause to contemplate the geometrical meaning of a few complex equations.

**Example 1.3.3.** .

**Example 1.3.4.** .

Notice that we cannot write inequalities for complex numbers with nonzero imaginary parts. We have no definition for $z < w$ given arbitrary $z, w \in \mathbb{C}$. However, the modulus of a complex number is a real number so we can write various inequalities. These will be important to limit arguments in upcoming sections.

**Theorem 1.3.5.** *Properties of the modulus.*

Let $z, w \in \mathbb{C}$,

1. $|z|^2 = Re(z)^2 + Im(z)^2$
2. $Re(z) \leq |Re(z)| \leq |z|$
3. $Im(z) \leq |Im(z)| \leq |z|$
4. $|zw| = |z||w|$
5. $|z^{-1}| = 1/|z|$

**Proof:** follows from Theorem 1.2.3.

**Theorem 1.3.6.** *Inequalities of the modulus.*

Let $z, w \in \mathbb{C}$,

1. $|z + w| \leq |z| + |w|$
2. $|z + w| \geq |z| - |w|$

**Proof:** item (1.) is geometrically obvious. We'll prove it algebraically for the sake of logical completeness.

## 1.4 polar form of complex numbers

Given a point $z = (x, y) = x + iy$ in the complex plane we can find the **polar coordinates** in the same way we did in calculus II or III. Recall that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ so

$$x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$$

However, we insist that $r \geq 0$ in this course and the value for the angle requires some discussion. The trouble with angles is that one direction geometrically corresponds to infinitely many angles. This makes the angle a multiply-valued function (a contradiction in terms if you want to be critical!). To give a careful account of the ambiguity of choosing the angle we have to invent some notation to summarize these concerns. This is the reason for "*arg*" and "*Arg*". Be warned I am more careful than Churchill in my use of *arg* however I probably agree with his use of *Arg*.

**Definition 1.4.1.**

Let $z = (x, y) \in \mathbb{C}$. We define the **polar radius of** $z$ to be the modulus of $z$; $r = |z| = \sqrt{x^2 + y^2}$. The **argument** of $z$ is the **set** of values below:

$$arg(z) = \{\theta \in \mathbb{R} \mid z = r(\cos(\theta) + i\sin(\theta)\}$$

The **principal argument** of $z$ is the **single** value defined below:

$$Arg(z) = \theta \in arg(z) \text{ such that } -\pi < \theta \leq \pi.$$

We may also use the notation $Arg(z) = \Theta$.

We should probably pause and appreciate that the following set of equations does define the angle up to an integer multiple of $2\pi$, if $z = (x, y) = x + iy$ then

$$x = |z|\cos(\theta) \qquad y = |z|\sin(\theta).$$

The set of equations above does not suffer the ambiguity of the tangent.

**Example 1.4.2.** .

**Example 1.4.3.** .

## 1.5 complex exponential notation

There are various approaches to this topic. I'll get straight to the point here.

**Definition 1.5.1.**

Let $z = (x, y) \in \mathbb{C}$, we define the **complex exponential function** by

$$\boxed{e^{x+iy} = e^x(\cos(y) + i\sin(y))}$$

where $e^x$ is the usual exponential function as defined in elementary calculus and sine and cosine are likewise the standard trigonometric functions defined in elementary trigonometry.

I wanted to emphasize that the definition of the complex exponential has been given purely in terms of things that you already know from calculus and trig. Notice that an immediate consequence of this definition is Euler's formula:

**Definition 1.5.2.**

Let $\theta \in \mathbb{R}$ then $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Churchill says this *defines* the *imaginary exponential function*[2]. Then later through a few sections 6 and 23 he eventually arrives at the definition I just gave. I give the definition now so we can avoid heuristic calculuations. We should pause to appreciate the geometric genius of the formula above. We prove on the next page that $e^{z+w} = e^z e^w$, let's look at the special case of imaginary numbers $z = \imath\theta$ and $w = i\beta$:

[2] see page 13 equation (3)

**Theorem 1.5.3.**

Let $z, w \in \mathbb{C}$ then

1. $e^0 = 1$
2. $e^{z+w} = e^z e^w$
3. $(e^z)^{-1} = e^{-z}$

**Proof:** This is one of my favorite proofs. I need to assume you know the adding angles formulas for sine and cosine and also the ordinary law of exponents for the exponential function.

**Theorem 1.5.4.**

Let $z \in \mathbb{C}$ and define $(e^z)^n$ inductively by $(e^z)^0 = 1$ and $(e^z)^n = (e^z)^{n-1}e^z$ for all $n \in \mathbb{N}$. Likewise define $(e^z)^{-n} = (e^{-z})^n$ for all $n \in \mathbb{N}$.

1. $(e^z)^n = e^{nz}$ for all $n \in \mathbb{Z}$
2. if $z = |z|e^{i\theta}$ then $z^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$ for $n \in \mathbb{N}$.

**Proof:** Notice that if we know (1.) holds for all $z \in \mathbb{C}$ then we can use it to prove (2.). Observe that $z^n = (|z|e^{i\theta})^n = (|z|e^{i\theta})^{n-1}|z|e^{i\theta}$ and you can prove by induction that $z^n = |z|^n(e^{i\theta})^n$. Apply (1.) to $(e^{i\theta})^n$ and we find $z^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$. I encourage the reader to supply the induction argument omitted in the paragraph above. Incidentally, the formula

$$\boxed{(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)}$$

is called *de Moivre's formula*. Let us prove (1.):

**Example 1.5.5.** *Show how to use de Moivre's formula to obtain nontrivial trig. identities. .*

**Theorem 1.5.6.**

If $z_1 = r_1 e^{i\theta_1}$ and $z_1 = r_2 e^{i\theta_2}$ are nonzero then

$$arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

where the sum of the sets is defined by

$$arg(z_1) + arg(z_2) = \{\theta_1 + \theta_2 \mid \theta_1 \in arg(z_1),\ \theta_2 \in arg(z_2)\}$$

The practical meaning of Theorem 1.5.6 is that when we are faced with solving equations such as $e^z = e^w$ we must be careful to consider a multitude of possible cases. The complex exponential function is far from one-one.

### 1.5.1 trigonmetric identities from the imaginary exponential

Now that we have a few of the basics settled let's do a few interesting calculations. I probably didn't cover these in lecture.

**Example 1.5.7.** .

**Example 1.5.8.** .

## 1.6 complex roots of unity

In this section we examine the meaning of fractional exponent of a complex number. It turns out that we cannot expect a single value. Instead we'll learn that $z^{\frac{m}{n}}$ is a set of values. The complex roots of unity are used to generate the set of values. There is a neat connection between rotations by $\theta = 2\pi/n$ and $e^{i\theta}$ and $\mathbb{Z}_n$.

**Definition 1.6.1.**

Let $z_o \in \mathbb{C}$ be nonzero. The **$n$-th roots of** $z_o$ is the set of values defined below:

$$z_o^{1/n} = \{z \in \mathbb{C} \mid \ z^n = z_o\}$$

Suppose that $z_o = r_o e^{i\theta_o}$ and $z = re^{i\theta}$ then the requirement $z^n = z_o$ yields

$$r^n e^{in\theta} = r_o e^{i\theta_o}$$

It follows that $r^n = r_o$ and $n\theta_o = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Therefore, if we denote the **positive $n$-th root of the real number** $r_o$ **by** $\sqrt[n]{r_o}$ **then** $r = \sqrt[n]{r_o}$. Moreover, we may write the set of roots as follows:

$$z_o^{1/n} = \{ \ \sqrt[n]{r_o} exp\Big[ \ \frac{i(\theta+2\pi k)}{n}\Big] \mid k \in \mathbb{Z} \ \}$$

For example,

$$1^{1/2} = \{exp(i2\pi k/2) \mid k \in \mathbb{Z}\}$$

where I identified that $\theta = 0$ and $r_o = 1$ since $z_o = 1e^{i0}$. Great, but what is this set $1^{1/2}$? Notice that

$$exp(i2\pi k/2) = \cos(\pi k) + i\sin(\pi k)$$

If $k \in 2\mathbb{Z}$ then $k$ is an even integer and $\cos(\pi k) = 1$. However, if $k \in 2\mathbb{Z}+1$ then $k$ is an odd integer and $\cos(\pi k) = -1$. In all cases the sine term vanishes. We find,

$$1^{1/2} = \{1, -1\}$$

To find the cube roots of 1 we'd examine the values of $exp(i2\pi k/3) = \cos(2\pi k/3) + i\sin(2\pi k/3)$. We'd soon learn that $k \in 3\mathbb{Z}$ give $exp(i2\pi k/3) = 1$ whereas $k \in 3\mathbb{Z}+1$ give $exp(i2\pi k/3) = exp(2\pi/3) = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and finally $k \in 3\mathbb{Z}+2$ give $exp(i2\pi k/3) = exp(4\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We denote these by

$$1^{1/3} = \{1, \omega_3, \omega_3^2\}$$

here $\omega_3 = exp(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is called the **prinicpal cube root of unity**. Naturally we can do this for any $n \in \mathbb{N}$ and it is not hard to show that the $n$-th roots of unity are generated from powers of $\omega_n = exp(2\pi/n)$. Indeed we could show that

$$1^{1/n} = \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$$

The correspondence with $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ is provided by the mapping $\Phi(\omega_n^k) = \bar{k}$. You can check that $\Phi(zw) = \Phi(z) + \Phi(w)$. It is a homomorphism between the multiplicative group of units and the additive group $\mathbb{Z}_n$.

**Theorem 1.6.2.**

If $z_o = r_o exp(i\theta_o)$ then the $n$-th roots of $z_o$ are generated from the $n$-th roots of unity as follows:

$$z_o^{1/n} = \{c, c\omega_n, c\omega_n^2, \dots, c\omega_n^{n-1}\}$$

where $c$ is a particular $n$-th root of $z_o$; $c^n = z_o$. Notice that $|c| = \sqrt[n]{r_o}$ and in the case that $0 < z_o \in \mathbb{R}$ we may choose $c = \sqrt[n]{r_o}$ where $\sqrt[n]{r_o}$ denotes the positive $n$-th root of the positive real number $r_o$. In the formula above I am using our standard notation that $\omega_n$ is the principal $n$-th root of unity which is given by the formula:

$$\omega_n = exp(i2\pi/n).$$

Geometrically this theorem is very nice. It gives us a way to find the vectors which point to the vertices of a regular polygon with $n$-sides. Moreover, we can rotate the polygon by using a $z_o \neq 1$.

**Example 1.6.3.** .

**Example 1.6.4.** .

**Example 1.6.5.** .

## 1.7 complex numbers and factoring

In this section we examine a few examples of the **factor theorem**. This theorem states that every zero of a complex polynomial corresponds to a factor. Don't mind the definitions if you're not interested, just skip to the examples:

**Definition 1.7.1.**

A **polynomial in $x$ with coefficients in $S$** is an expression

$$p(x) = c_0 + c_1 x + \cdots + c_k x^k = \sum_{j=0}^{\infty} c_j x^j$$

where $c_j \in S$ for all $j \in \mathbb{N} \cup \{0\}$ and only finitely many of these coefficients are nonzero. The $deg(p) = k$ if $c_k$ is the nonzero coefficient with the largest index $k$. We say that $p(x) \in S(x)$. The set of polynomials in $z$ with coefficients in $\mathbb{C}$ is denoted $\mathbb{C}\,(z)$. The set of polynomials in $z$ with coefficients in $\mathbb{R}$ is denoted $\mathbb{R}\,(z)$.

**Remark 1.7.2.**

In the definition above I am thinking of polynomials as abstract expressions. Notice we can add, subtract and multiply polynomials provided we can perform the same operations in $S$. This makes $S(x)$ a vector space over $S$ if $S$ is a field. However, if $S$ is only a ring then the set of polynomials forms what is known as a **module**. Polynomials can be used to build number systems through an algebraic construction called **field extension**. This material is discussed in some depth in Math 422 at LU.

Obviously we are primarily interested in either $\mathbb{C}\,(z)$ or $\mathbb{R}\,(x)$ in most undergraduate mathematics. These are precisely the objects we learned to factor in highschool and so forth. Let me give a precise definition of factoring. Since we can view $\mathbb{R}\,(z) \subset \mathbb{C}\,(z)$ we will focus on $\mathbb{C}\,(z)$ in remainder of this section.

**Definition 1.7.3.**

Suppose $f(z), g(z), h(z) \in \mathbb{C}\,(z)$. Suppose $deg(h), deg(g) \geq 1$. If $f(z) = h(z)g(z)$ then we say that $g(z)$ and $h(z)$ **factor** $f(z)$. If $f(z)$ has no factors then we say that $f$ is **irreducible**. If $deg(f) = 1$ then we say $f(z)$ is a **linear factor**.

**Example 1.7.4.** .

**Example 1.7.5.** .

**Example 1.7.6.** .

In the next chapter we discuss the concept of a complex function. Once we take that viewpoint we can *evaluate* polynomials at complex numbers. It's worth noticing that if $(z-r)$ is a factor of $f(z)$ then it follows $f(c)=0$. The converse is also true; if $f(r)=0$ for some $r \in \mathbb{C}$ then $f(z)=(z-r)g(z)$ where $g(z)$ is some other polynomial (the proof of the converse is less obvious). In any event, if you believe me, then we have the following: (here I mean for $c_j, b_j$ to denote complex constants)

$$\boxed{c_0 + c_1 z + \cdots + c_n z^n = 0 \text{ for } z = r \quad \Leftrightarrow \quad c_0 + c_1 z + \cdots + c_k z^k = (z-r)(b_0 + b_1 z + \cdots + b_m z^m)}$$

I sometimes refer to the calculation above as the **fundmental theorem of algebra**. We'll probably prove that theorem sometime this semester.

# Chapter 2

# topology and mappings

Mathematics is built with functions and sets for the most part. In this chapter we learn what a complex function is and we examine a number of interesting features. Mappings are also studied and contrasted with functions. Since a complex function is a real mapping we begin with a brief overview of what is known about real mappings. Continuity of complex functions is then discussed in some depth. We then define connected sets, domains and regions. Next the extended complex plane as modeled by the Riemann sphere is introduced as a convenient device to capture limits at $\infty$. We then examine a number of transformations and introduce the idea of the $w$-plane. Branch-cuts are defined to extract functions from multiply-valued functions. In particular, $n$-th root functions is defined. The complex logarithm is defined as a local inverse to the complex exponential. We discover many of the standad examples in this chapter. Notable exceptions are sine, cosine and hyperbolic sine or cosine etc... We focus on algebraic functions and the complex exponential.

## 2.1 open, closed and continuity in $\mathbb{R}^n$

In this section we describe the *metric topology* for $\mathbb{R}^n$. The topology is built via the **Euclidean norm** which is denoted by $||\cdot|| : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ where $||x|| = \sqrt{x \cdot x}$ and $x \cdot x$ denotes the dot-product where $x \cdot y = x_1 y_1 + \cdots x_n y_n$ for all $x, y \in \mathbb{R}^n$. Once we're done with this section I will recapitulate many of the definitions given in this section in the special case of $\mathbb{R}^2 = \mathbb{C}$ where we have the familar formula $|z| = \sqrt{\bar{z}z}$ and this is in fact the same idea of length; $|z| = ||z||$. These notes are borrowed from my advanced calculus notes which in turn mirror the excellent text by Edwards on the subject.

In the study of functions of one real variable we often need to refer to open or closed intervals. The definition that follows generalizes those concepts to $n$-dimensions.

**Definition 2.1.1.**

An **open ball** of radius $\epsilon$ centered at $a \in \mathbb{R}^n$ is the subset all points in $\mathbb{R}^n$ which are less than $\epsilon$ units from $a$, we denote this open ball by $B_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| < \epsilon\}$.
The **closed ball** of radius $\epsilon$ centered at $a \in \mathbb{R}^n$ is likewise defined by
$\overline{B}_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| \leq \epsilon\}$.

Notice that in the $n = 1$ case we observe an open ball is an open interval: let $a \in \mathbb{R}$,

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid ||x - a|| < \epsilon\} = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

In the $n = 2$ case we observe that an open ball is an open disk: let $(a, b) \in \mathbb{R}^2$,

$$B_\epsilon((a,b)) = \left\{(x,y) \in \mathbb{R}^2 \mid ||\,(x,y) - (a,b)\,|| < \epsilon\right\} = \left\{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\right\}$$

For $n = 3$ an open-ball is a sphere without the outer shell. In contrast, a closed ball in $n = 3$ is a solid sphere which includes the outer shell of the sphere.

**Example 2.1.2.** . .

**Definition 2.1.3.**

Let $D \subseteq \mathbb{R}^n$. We say $y \in D$ is an **interior point** of $D$ iff there exists some open ball centered at $y$ which is completely contained in $D$. We say $y \in \mathbb{R}^n$ is a **limit point** of $D$ iff every open ball centered at $y$ contains points in $D - \{y\}$. We say $y \in \mathbb{R}^n$ is a **boundary point** of $D$ iff every open ball centered at $y$ contains points not in $D$ and other points which are in $D - \{y\}$. We say $y \in D$ is an **isolated point** or **exterior point** of $D$ if there exist open balls about $y$ which do not contain other points in $D$. The set of all interior points of $D$ is called the **interior of** $D$. Likewise the set of all boundary points for $D$ is denoted $\partial D$. The **closure** of $D$ is defined to be $\overline{D} = D \cup \{y \in \mathbb{R}^n \mid y \text{ a limit point}\}$

If you're like me the paragraph above doesn't help much until I see the picture below. All the terms are aptly named. The term "limit point" is given because those points are the ones for which it is natural to define a limit.

**Example 2.1.4.** . .

**Definition 2.1.5.**

Let $A \subseteq \mathbb{R}^n$ is an **open set** iff for each $x \in A$ there exists $\epsilon > 0$ such that $x \in B_\epsilon(x)$ and $B_\epsilon(x) \subset A$. Let $B \subseteq \mathbb{R}^n$ is an **closed set** iff it contains all of its boundary points.

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. The limit of a mapping is likewise defined via deleted open balls:

**Definition 2.1.6.**

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. We say that $f$ has limit $b \in \mathbb{R}^m$ at limit point $a$ of $U$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathbb{R}^n$ with $0 < ||x - a|| < \delta$ implies $||f(x) - b|| < \epsilon$. In such a case we can denote the above by stating that $\lim_{x \to a} f(x) = b$.

**Definition 2.1.7.**

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. If $a \in U$ is a limit point of $f$ then we say that $f$ is **continuous at** $a$ iff

$$\lim_{x \to a} f(x) = f(a)$$

If $a \in U$ is an isolated point then we also say that $f$ is continuous at $a$. The mapping $f$ is **continuous on** $S$ iff it is continuous at each point in $S$. The **mapping** $f$ **is continuous** iff it is continuous on its domain.

Notice that in the $m = n = 1$ case we recover the definition of continuous functions from calc. I. It turns out that most of the theorems for continuous functions transfer over to appropriately generalized theorems on mappings. The proofs can be found in Edwards.

**Proposition 2.1.8.**

Suppose that $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a mapping with component functions $f_1, f_2, \dots, f_m$. Let $a \in U$ be a limit point of $f$ then $f$ is continous at $a$ iff $f_j$ is continuous at $a$ for $j = 1, 2, \dots, m$. Moreover, $f$ is continuous on $S$ iff all the component functions of $f$ are continuous on $S$. Finally, a mapping $f$ is continous iff all of its component functions are continuous. .

**Proposition 2.1.9.**

Let $f$ and $g$ be mappings such that $f \circ g$ is well-defined. The composite function $f \circ g$ is continuous for points $a \in dom(f \circ g)$ such that the following two conditions hold:

1. $g$ is continuous at $a$
2. $f$ is continuous at $g(a)$.

The proof of the proposition is in Edwards, it's his Theorem 7.2. I'll prove this theorem in a particular context in this chapter.

**Proposition 2.1.10.**

Assume $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ are continuous functions at $a \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at $a$.
2. $fg$ is continuous at $a$
3. $cf$ is continuous at $a$.

Moreover, if $f, g$ are continuous then $f + g$, $fg$ and $cf$ are continuous.

## 2.2 open, closed and continuity in $\mathbb{C}$

The definitions of the preceding section remain unaltered except that we specialize to two dimensions and use appropriate complex notation in this section. Trade the word "ball" for "disk" and "norm" for "modulus". Just to remind you the connection between the modulus and norm is simply the following:

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = ||(x, y)||.$$

**Definition 2.2.1.**

An **open disk** of radius $\epsilon$ centered at $z_o \in \mathbb{C}$ is the subset all complex numbers which are less than an $\epsilon$ distance from $z_o$, we denote this open ball by

$$D_\epsilon(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < \epsilon\}.$$

The **deleted-disk** with radius $\epsilon$ centered at $z_o$ is likewise defined

$$D^o_\epsilon(z_o) = \{z \in \mathbb{C} \mid 0 < |z - z_o| < \epsilon\}.$$

The **closed disk** of radius $\epsilon$ centered at $z_o \in \mathbb{C}$ is defined by

$$\overline{D}_\epsilon(z_o) = \{z \in \mathbb{C} \mid |z - z_o| \leq \epsilon\}$$

.

The following definition is nearly unchanged from the preceding section.

**Definition 2.2.2.**

Let $S \subseteq \mathbb{C}$. We say $y \in S$ is an **interior point** of $S$ iff there exists some open disk centered at $y$ which is completely contained in $S$. We say $y \in \mathbb{C}$ is a **limit point** of $S$ iff every open disk centered at $y$ contains points in $S - \{y\}$. We say $y \in \mathbb{C}$ is a **boundary point** of $S$ iff every open disk centered at $y$ contains points not in $S$ and other points which are in $S - \{y\}$. We say $y \in S$ is an **isolated point** or **exterior point** of $S$ if there exist open disks about $y$ which do not contain other points in $S$. The set of all interior points of $S$ is called the **interior of** $S$. Likewise the set of all boundary points for $S$ is denoted $\partial S$. The **closure** of $S$ is defined to be $\overline{S} = S \cup \{y \in \mathbb{C} \mid y \text{ a limit point of } S\}$

Perhaps the following picture helps clarify these definitions: .

**Definition 2.2.3.**

Let $S \subseteq \mathbb{C}$ is an **open set** iff for each $z \in S$ there exists $\epsilon > 0$ such that $D_\epsilon(z) \subset S$. If $B \subseteq \mathbb{C}$ then $B$ is a **closed set** iff it contains all of its boundary points. In other words, a closed set $S$ has $\partial S \subset S$.

A **complex function** is simply a function whose domain and codomain are subsets of $\mathbb{C}$.

**Definition 2.2.4.**

Let $f : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ be a complex function. We say that $f$ has limit $w_o \in \mathbb{C}$ at limit point $z_o$ of $U$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $z \in \mathbb{C}$ with $0 < |z - z_o| < \delta$ implies $|f(z) - w_o| < \epsilon$. In such a case we can denote the above by stating that $\lim_{z \to z_o} f(z) = w_o$. In other words, we say $\lim_{z \to z_o} f(z) = w_o$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $f(D^o_\delta(z_o)) \subset D_\epsilon(w_o)$.

**Example 2.2.5.** . . .

We should also note that $z_o$ need not be inside the domain of $f$ in the limit. In the special case that $f(z_o)$ is defined and $f(z_o) = w_o$ we say the complex function is continuous at $z_o$.

**Definition 2.2.6.**

Let $f : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ be a complex function. If $z_o \in U$ is a limit point of $f$ then we say that $f$ is **continuous at** $a$ iff

$$\lim_{z \to z_o} f(z) = f(z_o)$$

If $z_o \in U$ is an isolated point then we also say that $f$ is continuous at $z_o$. The function $f$ is **continuous on** $S$ iff it is continuous at each point in $S$. The **function** $f$ **is continuous** iff it is continuous on its domain.

**Example 2.2.7.** . . .

We postpone the proof of the proposition below until the end of this section. In short, most of the limit theorems for real-valued functions generalize naturally to the context of $\mathbb{C}$.

**Proposition 2.2.8.**

Assume $f, g : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ are functions with limit point $z_o \in U$ where the limits of $f$ and $g$ exist at $z_o$.

1. $\lim_{z \to z_o} \big( f(z) + g(z) \big) = \lim_{z \to z_o} f(z) + \lim_{z \to z_o} g(z)$.
2. $\lim_{z \to z_o} \big( f(z) g(z) \big) = \Big( \lim_{z \to z_o} f(z) \Big) \Big( \lim_{z \to z_o} g(z) \Big)$.
3. if $c \in \mathbb{C}$ then $\lim_{z \to z_o} \big( c f(z) \big) = c \Big( \lim_{z \to z_o} f(z) \Big)$.
4. if $\lim_{z \to z_o} g(z) \neq 0$ then $\lim_{z \to z_o} \Big[ \frac{f(z)}{g(z)} \Big] = \frac{\lim_{z \to z_o} f(z)}{\lim_{z \to z_o} g(z)}$.
5. if $h : dom(h) \to \mathbb{C}$ is continuous at $\lim_{z \to z_o} f(z)$ then
$$\lim_{z \to z_o} h(f(z)) = f(\lim_{z \to z_o} f(z)).$$

An immediate consequence of the theorem above is that the sum, product, quotient and composite of continuous complex functions is again continuous. Moreover, induction can be used to extend these results to power functions of $z$ and arbitrary finite sums. It then follows that complex polynomials are continuous on $\mathbb{C}$. A complex **rational function** is defined pointwise as the quotient of two complex polynomials. Rational functions in $\mathbb{C}$ are continuous for points where the denominator polynomial is nonzero.

**Example 2.2.9.** . .

**Example 2.2.10.** .

### 2.2.1 complex functions are real mappings

If the complex function is as simple as the last example then the direct computation of limits via the modulus is not too difficult. However, in general it is nice to be able to apply the calculus of many real variables. Notice that $f : dom(f) \subseteq \mathbb{C} \rightarrow \mathbb{C}$ then we can split each output of the function into its real and imaginary part. We define:

$$\boxed{u(z) = Re(f(z)) \qquad v(z) = Im(f(z))}$$

Therefore, since $Re(f(z)), Im(f(z)) \in \mathbb{R}$ for all $z \in dom(f)$ there exist real-valued functions $u, v : dom(f) \rightarrow \mathbb{R}$ such that

$$f(z) = u(z) + iv(z).$$

Moreover, since or convention is to write $z = x + iy = (x, y)$ we can view a complex function as a mapping from $dom(f) \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$ where

$$\boxed{f(x + iy) = u(x, y) + iv(x, y)}$$

This is a standard notation in most texts.

**Example 2.2.11.** . .

**Example 2.2.12.** . .

**Example 2.2.13.** . .

**Example 2.2.14.** . .

### 2.2.2 proofs on continuity of complex functions

To begin note that if $f = u + iv$ is a complex function then we may as well identify $f = (u, v)$ as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ with component functions $f_1 = u$ and $f_2 = v$. Therefore, by Proposition 2.1.8 we find:

**Proposition 2.2.15.**

If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is a complex function with $f(x + iy) = u(x, y) + iv(x, y)$ for all $(x, y) \in D$ then $f$ is continuous at $z_o \in D$ as a complex function iff $u$ and $v$ are continuous at $z_o = (x_o, y_o) \in D$ as real functions from $dom(f) \subseteq \mathbb{R}^2$ to $\mathbb{R}$. Moreover, a complex function is continuous iff its component functions are continuous real functions.

**Proof:** it is interesting that the proof in Edwards is similar, just it uses norms instead of modulus. In any event, since you may not have had advanced calculus it's probably best for me to include this proof here:

In view of Proposition 2.2.15 we can easily deduce that Examples 2.2.11 - 2.2.14 give complex functions that are mostly continuous. I assume you recall the definition of continuous functions of two variables from calculus III, remember the function $g(x, y)$ is continuous iff the limit of the function $g(\vec{r}(t)) \to g(p)$ as $t \to 0$ for all curves $t \to \vec{r}(t)$ with $\vec{r}(0) = p$. Typically we only employ this definition directly for the purpose of finding a contradiction. If you can show the limit is different along two different paths then the limit does not exist. Of course, to be rigorous one should consult the $\epsilon - \delta$ definition of continuity offered in the preceding section.

**Example 2.2.16.** . . .

The proof of the following proposition is identical to the proof given in Edwards for the general case. I leave the proof as an exercise for the reader.

**Proposition 2.2.17.**

Let $f$ and $g$ be complex functions such that $f \circ g$ is well-defined. The composite function $f \circ g$ is continuous for points $a \in dom(f \circ g)$ such that the following two conditions hold:

1. $g$ is continuous at $a$
2. $f$ is continuous at $g(a)$.

The proof of part (1.) the following theorem is identical to the proof given in Edwards for the general case. I will show that (2.) and (3.) also follow from the corresponding proposition for sums and products of real functions. Then I give a second proof which does not borrow from the theory of real mappings.

**Proposition 2.2.18.**

Assume $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ are continuous functions at $z_o \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at $z_o$.
2. $fg$ is continuous at $z_o$
3. $cf$ is continuous at $z_o$.

Moreover, if $f, g$ are continuous then $f + g, fg$ and $cf$ are continuous.

**Proof:** We begin with the proof of (2.). Suppose $f = u + iv$ and $g = a + ib$ are continuous at $z_o$ and note, omitting the $z$-dependence,

$$fg = (u + iv)(a + ib) = ua - vb + i(ub + va).$$

In terms of real notation we have $fg = (ua - vb, ub + va)$. But, we know $u, v, a, b$ are continuous at $z_o$ because they are the component functions of continuous functions $f, g$. Moreover, we find $fg$ is continuous at $z_o$ since it has component functions $(fg)_1 = ua - vb$ and $(fg)_2 = ub + va$ which are the sum or difference of products of continuous functions at $z_o$.

To prove (3.) just take the constant function $g(z) = c$. I leave the proof that the constant function is continuous as an exercise for the reader. $\square$

Hopefully you've noticed that the heart of the proofs given above were stolen from the corresponding theorems of real mappings. I did this purposefully because I want to draw a clear distinction between these results on continuity and the later results we'll find for complex differentiability. The proof that follows is self-contained.

**Proof:** Suppose $\lim_{z\to z_o} f(z) = f_o$ and $\lim_{z\to z_o} g(z) = g_o$. Let $\epsilon > 0$. Since the limit of $f$ at $z_o$ exists we can find $\delta_f > 0$ such that $0 < |z - z_o| < \delta_f$ implies $|f(z) - f_o| < \epsilon/2$. Likewise, as the limit of $g$ at $z_o$ exists we can find $\delta_g > 0$ such that $0 < |z - z_o| < \delta_g$ implies $|g(z) - g_o| < \epsilon/2$. Suppose that $\delta = min(\delta_f, \delta_g)$ and assume that $z \in D^o_\delta(z_o)$. It follows that

$$|(f + g)(z) - (f_o + g_o)| = |f(z) + g(z) - f_o - g_o| \leq |f(z) - f_o| + |g(z) - g_o| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $0 < |z - z_o| < \delta$ implies $|(f + g)(z) - (f_o + g_o)| < \epsilon$. Therefore,

$$\boxed{\lim_{z\to z_o} (f(z) + g(z)) = \lim_{z\to z_o} f(z) + \lim_{z\to z_o} g(z)}$$

Part (1.) of the proposition follows immediately.

**Preparing for the proof of (2):** We need to study $|f(z)g(z) - f_og_o|$. Consider that

$$|f(z)g(z) - f_og_o| = |f(z)g(z) - f(z)g_o + f(z)g_o - f_og_o| = \big|f(z)\big(g(z) - g_o\big) + \big(f(z) - f_o\big)g_o\big|$$

Then we can use properties of the modulus to find:

$$|f(z)g(z) - f_og_o| \leq |f(z)||g(z) - g_o| + |g_o||f(z) - f_o|$$

Note that we can choose a $\delta > 0$ such that if $z \in D^o_\delta(z_o)$ then both $|g(z) - g_o|$ and $|f(z) - f_o|$ are as small as we'd like. Furthermore, if $|f(z) - f_o| < \beta$ then $|f(z)| < |f_o| + \beta$. Consider then that if $|f(z) - f_o| < \beta$ and $|g(z) - g_o| < \beta$ it follows that

$$|f(z)g(z) - f_og_o| < (|f_o| + \beta)|\beta + |g_o|\beta = \beta^2 + \beta\big(|f_o| + |g_o|\big)$$

Our goal is to find a $\delta$ such that $z \in D^o_\delta(z_o)$ implies $|f(z)g(z) - f_og_o| < \epsilon$. In view of our calculations up to this point we see that this can be accomplished if we could choose $\beta$ such that

$$\beta^2 + \beta\big(|f_o| + |g_o|\big) = \epsilon.$$

Apply the quadratic equation to find

$$\beta = \frac{-|f_o| - |g_o| \pm \sqrt{(|f_o| + |g_o|)^2 + 4\epsilon}}{2}$$

Note that it is clear that the $(+)$ solution does yield $\beta > 0$.

**Proof:** Let $\epsilon > 0$. Define

$$\beta = \frac{-|f_o| - |g_o| + \sqrt{(|f_o| + |g_o|)^2 + 4\epsilon}}{2}.$$

Since the limits of $f$ and $g$ exist it follows that we choose $\delta > 0$ such that $z \in D^o_\delta(z_o)$ implies both $|g(z) - g_o| < \beta$ and $|f(z) - f_o| < \beta$. The following calculations were justified in the paragraph preceding the proof:

$$|f(z)g(z) - f_og_o| < \beta^2 + \beta\big(|f_o| + |g_o|\big) = \epsilon.$$

Therefore,

$$\boxed{\lim_{z \to z_o} f(z)g(z) = \left(\lim_{z \to z_o} f(z)\right)\left(\lim_{z \to z_o} g(z)\right)}$$

Part (2.) of the proposition follows immediately. $\Box$.

## 2.3 connected sets, domains and regions

To avoid certain pathological cases we often insist that the set considered is a **domain** or a **region**. These are technical terms in this context and we should be careful not to confuse them with their previous uses in mathematical discussion.

**Definition 2.3.1.**

If $a, b \in \mathbb{C}$ then we define the **directed line segment from $a$ to $b$ to be the set**

$$[a, b] = \{a + t(b - a) \mid t \in [0, 1]\}$$

**Definition 2.3.2.**

A **polygonal path** $\gamma$ from $a$ to $b$ in $\mathbb{C}$ is the union of finitely many line segments which are placed end to end;

$$\gamma = [a, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-2}, z_{n-1}] \cup [z_{n-1}, b]$$

**Definition 2.3.3.**

A set $S \subseteq \mathbb{C}$ is **connected** iff there exists a polygonal path contained in $S$ between any two points in $S$. That is for all $a, b \in S$ there exists a polygonal path $\gamma$ from $a$ to $b$ such that $\gamma \subseteq S$.

Incidentally, the definitions just offered for $\mathbb{C}$ apply equally well to $\mathbb{R}^n$.

**Definition 2.3.4.**

An open connected set is called a **domain**. We say $R$ is a **region** if $R = D \cup S$ where $D$ is a domain $D$ and $S \subseteq \partial D$.

**Example 2.3.5.** . .

## 2.4 Riemann sphere and the point at $\infty$

The Riemann sphere sets up a correspondence between the sphere $x^2+y^2+z^2=1$ and the complex plane. In short, the stereographic projection maps each point on the sphere to a particular point on the complex plane. The one exception is the North Pole $(0,0,1)$. It is natural to identify the North Pole with $\infty$ for the complex plane. This is primarily a topological construction, all sense of distance is lost in the mapping.

As far as this course is concerned the point at infinity is simply a convenient concept to describe a limit where the value of the modulus gets arbitrarily large. The complex numbers together with $\infty$ is called the *extended complex plane*.

**Definition 2.4.1.**

We say that $\lim_{z \to z_o} f(z) = \infty$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |z - z_o| < \delta$ implies $|f(z)| > 1/\epsilon$. We define a neighborhood of $\infty$ as follows:

$$D_\epsilon(\infty) = \{w \in \mathbb{C} \mid |w| > 1/\epsilon\}$$

For each $\epsilon > 0$ we need to find $\delta > 0$ such that $f(D_\delta(z_o)) \subset D_\epsilon(\infty)$ if we wish to prove $\lim_{z \to z_o} f(z) = \infty$. Limits "at" infinity are likewise defined:

**Definition 2.4.2.**

We say that $\lim_{z \to \infty} f(z) = w_o$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|z| > 1/\delta$ implies $|f(z) - w_o| < \epsilon$.

**Example 2.4.3.** . . .

**Proposition 2.4.4.**

Suppose $f : S \to \mathbb{C}$ is a complex function and $z_o$ is a limit point of $S$ then,

$$(1.)\ \lim_{z\to z_o} f(z) = \infty \qquad \text{iff} \qquad \lim_{z\to z_o} \frac{1}{f(z)} = 0$$

$$(2.)\ \lim_{z\to \infty} f(z) = w_o \qquad \text{iff} \qquad \lim_{z\to 0} f(1/z) = w_o$$

$$(3.)\ \lim_{z\to \infty} f(z) = \infty \qquad \text{iff} \qquad \lim_{z\to 0} \frac{1}{f(1/z)} = 0$$

I leave the proof to the reader. Let's see how to use these. We will need these later in places.

**Example 2.4.5.** . .

**Example 2.4.6.** . .

## 2.5 transformations and mappings

The examples given in this section are by no means comprehensive. Mostly this section is just for fun. Notice that most of the transformations are given by functions with the exception of the square root transformation. The transformation $z \to w = z^{1/2}$ is called a **multiply-valued function**. We could say it is a 1 to 2 function, technically this means it is not a function in the strict sense of the term common to modern mathematics. We ought to say it is a **relation**. However, it is customary to refer to such relations as multiply-valued functions. We begin with a few simple transformations: in each case we picture the domain and range as separate complex planes. The domain is called the $z$-plane whereas the range is in the $w$-plane.

### 2.5.1 translations

**Example 2.5.1.** . .

### 2.5.2 rotations

**Example 2.5.2.** . .

### 2.5.3 magnifications

**Example 2.5.3.** . . .

### 2.5.4 linear mappings

**Example 2.5.4.** . . .

### 2.5.5 the $w = z^2$ mapping

**Example 2.5.5.** . .

### 2.5.6 the $w = z^{1/2}$ mapping

**Example 2.5.6.** . .

### 2.5.7 reciprocal mapping

**Example 2.5.7.** . . .

### 2.5.8 exponential mapping

**Example 2.5.8.** . . .

## 2.6 branch cuts

The inverse mappings of $w = z^n$ and $w = e^z$ are $w = z^{1/n}$ or $w = log(z)$. Technically these are not functions since the mappings $w = z^n$ and $w = e^z$ are not injective. If we cut down the domain of $w = z^n$ or $w = e^z$ then we can gain injectivity. The process of selecting just one of the many values of a multiply-valued function is called a **branch cut**. If a particular point is common to all the branch cuts for a particular mapping then the point is called a **branch point**. I don't attempt a general definition here. We'll see how the branch cuts work for the root and logarithm in this section.

### 2.6.1 the principal root functions

### 2.6.2 logarithms

# Chapter 3

# complex differentiation

The concept of complex differentiation is the natural analogue of real differentiation.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

The interesting feature is that there are many complex functions which have simple formulas and yet fail to be complex-differentiable. For example $f(z) = \bar{z}$. Such functions are usually real-differentiable. The Cauchy-Riemann equations for $f = u + iv$ are

$$u_x = -v_y \qquad u_y = v_x \qquad \text{Cauchy-Riemann (CR)-equations.}$$

We'll see the CR-equations at a point are necessary conditions for differentiability of a complex function at a point. However, they are not sufficient. This is not surprising since the same is true in multivariate real calculus. We all should have learned in calculus III that the derivative of a mapping exists at some point iff the partial derivatives exist and are continuous in some neighborhood of a point. What is interesting is that the rather unrestrictive condition that the partial derivatives of the component functions exist is replaced with the technical condition that the Cauchy Riemann equations are satisfied. But again, that is not enough to insure complex differentiability. We need continuity of the partial derivatives in some neighborhood of the point.

In this chapter we also discuss the polar form of the CR-equations as well as the concept of analytic functions and entire functions. We introduce a few new functions which are natural extensions of their real counterparts.

## 3.1 theory of differentiation for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$

I give a short account here. You can read more in the advanced calculus notes if you wish for motivations and examples etc... Our goal here is to briefly describe how to differentiate $f(x, y) = (u(x, y), v(x, y))$. The derivative is the matrix of the linear transformation which gives the best linear approximation to the change in the transformation near some point.

**Definition 3.1.1.**

Suppose that $U$ is open and $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping the we say that $f$ is **differentiable** at $p_o = (x_o, y_o) \in U$ iff there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - L(h)}{||h||} = 0.$$

In such a case we call the linear mapping $L$ the **differential at** $p_o$ and we denote $L = df_{p_o}$. If $f = (u, v)$ then the matrix of the differential is called the **Jacobian of** $f$ **at** $p_o$ and it has the form

$$J_f(p_o) = \begin{bmatrix} u_x(p_o) & v_x(p_o) \\ v_x(p_o) & v_y(p_o) \end{bmatrix} \qquad L(v) = J_f(p_o)v$$

.

**Example 3.1.2.** . .

If we were given that the partial derivatives of $u$ and $v$ exist at $p_o$ then we could not say for certain that the derivative of $f = (u, v)$ exists at $p_o$. It could be that strange things happen along directions other than the coordinate axes. We need another concept to be able to build differentiability from partial derivatives.

**Definition 3.1.3.**

A mapping $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is **continuously differentiable** at $p_o \in U$ iff the partial derivative mappings $u_x, u_y, v_x, v_y$ are continuous on an open set containing $p_o$.

The condition of continuity is key.

**Theorem 3.1.4.**

If $f$ is continuously differentiable at $p_o$ then $f$ is differentiable at $p_o$

You can find the proof in Edwards on pages 72-73. This is not a trivial theorem.

**Example 3.1.5.** . .

## 3.2 complex linearity

Finally, note that we have $L(cv) = cL(v)$ for all $c \in \mathbb{R}$ in the context of the definitions and theorems thus far in this section. The linearity is with respect to $\mathbb{R}$. In contrast, if we has some function $T : \mathbb{C} \to \mathbb{C}$ such that

(1.) $T(v + w) = T(v) + T(w)$ for all $v, w \in \mathbb{C}$ (2.) $T(cv) = cT(v)$ for all $c, v \in \mathbb{C}$

then we would say that $T$ is **complex-linear**. Condition (1.) is **additivity** whereas condition (2.) is **homogeneity**. Note that complex linearity implies real linearity however the converse is not true.

**Example 3.2.1.** . .

Suppose that $L$ is a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. It is known from linear algebra that there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $L(v) = Av$ for all $v \in \mathbb{R}^2$.

**Theorem 3.2.2.**

The linear mapping $L(v) = Av$ is complex linear iff the matrix $A$ will have the special form below:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

To be clear, we mean to identify $\mathbb{R}^2$ with $\mathbb{C}$ as before. Thus the condition of complex homogeneity reads $L((a, b) * (x, y)) = (a, b) * L(x, y)$

**Proof:** assume $L$ is complex linear. Define the matrix of $L$ as before:

$$L(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This yields,

$$L(x + iy) = ax + by + i(cx + dy)$$

We can gain conditions on the matrix by examining the special points $1 = (1, 0)$ and $i = (0, 1)$

$$L(1, 0) = (a, c) \qquad L(0, 1) = (b, d)$$

Note that $(c_1, c_2) * (1, 0) = (c_1, c_2)$ hence $L((c_1 + ic_2)1) = (c_1 + ic_2)L(1)$ yields

$$(ac_1 + bc_2) + i(cc_1 + dc_2) = (c_1 + ic_2)(a + ic) = c_1a - c_2c + i(c_1c + c_2a)$$

We find two equations by equating the real and imaginary parts:

$$ac_1 + bc_2 = c_1a - c_2c \qquad cc_1 + dc_2 = c_1c + c_2a$$

Therefore, $bc_2 = -c_2c$ and $dc_2 = c_2a$ for all $(c_1, c_2) \in \mathbb{C}$. Suppose $c_1 = 0$ and $c_2 = 1$. We find $b = -c$ and $d = a$. We leave the converse proof to the reader. The proposition follows. $\square$

**Example 3.2.3.** . .

## 3.3 complex differentiability and the Cauchy Riemann equations

In analogy with the real case we could define $f'(z)$ as the slope of the best **complex-linear** approximation to the change in $f$ near $z$. This is equivalent to the following definition:

**Definition 3.3.1.**

Suppose $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ and $z \in dom(f)$ then we define $f'(z)$ by the limit below:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

The derivative function $f'$ is defined pointwise for all such $z \in dom(f)$ that the limit above exists.

Note that $f'(z) = \lim_{h\to 0} \frac{f'(z)h}{h}$ hence

$$\lim_{h\to 0} \frac{f'(z)h}{h} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \quad \Rightarrow \quad \lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

Note that the limit above simply says that $L(v) = f'(z)v$ gives the is the best complex-linear approximation of $\Delta f = f(z+h) - f(z)$.

**Proposition 3.3.2.**

If $f$ is a complex differentiable at $z_o$ then linearization $L(h) = f'(z_o)h$ is a complex linear mapping.

**Proof:** let $c, h \in \mathbb{C}$ and note $L(ch) = f'(z_o)(ch) = cf'(z_o)h = cL(h)$. $\square$

The difference between the definitions of $L(h) = f'(z_o)h$ and $L(v) = J_f(p_o)v$ ( see Definition 3.1.1) is that in the complex derivative we divide by a small complex number whereas in the derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$ we divided by the norm of a two-dimensional vector[1].

**Proposition 3.3.3.**

If $f$ is a complex differentiable at $z_o$ then $f$ is (real) differentiable at $z_o$ with $L(h) = f'(z_o)h$.

**Proof:** note that $\lim_{h\to 0} \frac{f(z+h)-f(z)-f'(z)h}{h} = 0$ implies

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{|h|} = 0$$

but then $|h| = ||h||$ and we know $L(h) = f'(z_o)h$ is real-linear hence $L$ is the best linear approximation to $\Delta f$ at $z_o$ and the proposition follows. $\square$

[1]note that the definition of the derivative for $f : \mathbb{R}^n \to \mathbb{R}^m$ is the same but $J_f(p_o)$ is then a $m \times n$ matrix of partials in the general case. Each row is the gradient vector of a component function, in the case $n = 1$ the Jacobian matrix gives us the gradient of the function; $J_f = (\nabla f)^T$.

Let's summarize what we've learned: if $f : dom(f) \to \mathbb{C}$ is complex differentiable at $z_o$ and $f = u + iv$ then,

1. $L(h) = f'(z_o)h$ is complex linear.
2. $L(h) = f'(z_o)h$ is the best real linear approximation to $f$ viewed as a mapping on $\mathbb{R}^2$.

The Jacobian matrix for $f = (u, v)$ has the form

$$J_f(p_o) = \begin{bmatrix} u_x(p_o) & u_y(p_o) \\ v_x(p_o) & v_y(p_o) \end{bmatrix}$$

Theorem 3.2.2 applies to $J_f(p_o)$ since $L$ is a complex linear mapping. Therefore we find the Cauchy Riemann equations: $u_x = v_y$ and $u_y = -v_x$. We have proved the following theorem:

**Theorem 3.3.4.**

If $f = u + iv$ is a complex function which is differentiable at $z_o$ then the partial derivatives of $u$ and $v$ exist at $z_o$ and satisfy the Cauchy-Riemann equations at $z_o$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Example 3.3.5.** . . .

The converse of Theorem 3.3.4 is not true in general. It is possible to have functions $u, v : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR-equations at $z_o \in U$ and yet $f = u+iv$ fails to be complex differentiable at $z_o$. Indeed, this is the case even if we weakened our demand and simply requested real differentiability of $f = (u, v)$.

**Example 3.3.6.** *Counter-example to converse of Theorem 3.3.4.*

**Theorem 3.3.7.**

If $u, v, u_x, u_y, v_x, v_y$ are continuous functions in some open disk of $z_o$ and $u_x(z_o) = v_y(z_o)$ and $u_y(z_o) = -v_x(z_o)$ then $f = u + iv$ is complex differentiable at $z_o$.

**Proof:** we are given that a function $f : D_\epsilon(z_o) \subset \mathbb{R}^2 \to \mathbb{R}^2$ is continuous with continuous partial derivatives of its component functions $u$ and $v$. Therefore, by Theorem 3.1.4 we know $f$ is (real) differentiable at $z_o$. Therefore, we have a best linear approximation to the change in $f$ near $z_o$ which can be induced via multiplication of the Jacobian matrix:

$$L(v_1, v_2) = \begin{bmatrix} u_x(z_o) & u_y(z_o) \\ v_x(z_o) & v_y(z_o) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Note then that the given CR-equations show the matrix of $L$ has the form

$$[L] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = u_x(z_o)$ and $b = v_x(z_o)$. Consequently we find $L$ is complex linear and it follows that $f$ is complex differentiable at $z_o$ since we have a complex linear map $L$ such that

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - L(h)}{||h||} = 0$$

note that the limit with $h$ in the denominator is equivalent to the limit above which followed directly from the (real) differentiability at $z_o$. (the following is not needed for the proof of the theorem, but perhaps it is interesting anyway) Moreover, we can write

$$\begin{aligned} L(h_1, h_2) &= \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} u_x h_1 + u_y h_2 \\ -u_y h_1 + u_x h_2 \end{bmatrix} \\ &= u_x h_1 + u_y h_2 + i(-u_y h_1 + u_x h_2) \\ &= (u_x - i u_y)(h_1 + i h_2) \end{aligned}$$

Therefore we find $f'(z_o) = u_x - i u_y$ gives $L(h) = f'(z_o)z$. $\square$

### 3.3.1 how to calculate $df/dz$ via partial derivatives of components

If the partials exist and are continuous near a point $z_o$ and satisfy the CR-equations then we have a few nice formulas to calculate $f'(z)$:

$$\boxed{f'(z) = u_x + iv_x} \quad \boxed{f'(z) = v_y - iu_y} \quad \boxed{f'(z) = u_x - iu_y} \quad \boxed{f'(z) = v_y + iv_x}$$

**Example 3.3.8.** . .

**Example 3.3.9.** . .

**Example 3.3.10.** . .

### 3.3.2 Cauchy Riemann equations in polar coordinates

If we use polar coordinates to rewrite $f$ as follows:

$$f(x(r,\theta),y(r,\theta)) = u(x(r,\theta),y(r,\theta)) + iv(x(r,\theta),y(r,\theta))$$

we use shorthands $F(r,\theta) = f(x(r,\theta),y(r,\theta))$ and $U(r,\theta) = u(x(r,\theta),y(r,\theta))$ and $V(r,\theta) = v(x(r,\theta),y(r,\theta))$. We derive the CR-equations in polar coordinates via the chain rule from multivariate calculus,

$$U_r = x_r u_x + y_r u_y = \cos(\theta)u_x + \sin(\theta)u_y \quad \text{and} \quad U_\theta = x_\theta u_x + y_\theta u_y = -r\sin(\theta)u_x + r\cos(\theta)u_y$$

Likewise,

$$V_r = x_r v_x + y_r v_y = \cos(\theta)v_x + \sin(\theta)v_y \quad \text{and} \quad V_\theta = x_\theta v_x + y_\theta v_y = -r\sin(\theta)v_x + r\cos(\theta)v_y$$

We can write these in matrix notation as follows:

$$\begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Multiply these by the inverse matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix}$ to find

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)U_r - \frac{1}{r}\sin(\theta)U_\theta \\ \sin(\theta)U_r + \frac{1}{r}\cos(\theta)U_\theta \end{bmatrix}$$

A similar calculation holds for $V$. To summarize:

$$u_x = \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta \qquad v_x = \cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta$$

$$u_y = \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta \qquad v_y = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta$$

Another way to derive these would be to just apply the chain-rule directly to $u_x$,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. I leave it to the reader to show you get the same formulas from that approach. The CR-equation $u_x = v_y$ yields:

$$(A.) \quad \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta$$

Likewise the CR-equation $u_y = -v_x$ yields:

$$(B.) \quad \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta = -\cos(\theta)V_r + \tfrac{1}{r}\sin(\theta)V_\theta$$

Multiply (A.) by $r\sin(\theta)$ and $(B.)$ by $r\cos(\theta)$ and subtract (A.) from (B.):

$$U_\theta = -rV_r$$

Likewise multiply (A.) by $r\cos(\theta)$ and $(B.)$ by $r\sin(\theta)$ and add (A.) and (B.):

$$rU_r = V_\theta$$

Finally, recall that $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ hence

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= (\cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta) + i(\cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta) \\ &= (\cos(\theta)U_r + \sin(\theta)V_r) + i(\cos(\theta)V_r - \sin(\theta)U_r) \\ &= (\cos(\theta) - i\sin(\theta))U_r + i(\cos(\theta) - i\sin(\theta))V_r \\ &= e^{-i\theta}(U_r + iV_r) \end{aligned}$$

**Theorem 3.3.11.**

If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is a complex function written in polar coordinates $r, \theta$ then the Cauchy Riemann equations are written $U_\theta = -rV_r$ and $rU_r = V_\theta$. If $f'(z_o)$ exists then the CR-equations in polar coordinates hold. Likewise, if the CR-equations hold in polar coordinates and all the polar component functions and their partial derivatives with respect to $r, \theta$ are continuous on an open disk about $z_o$ then $f'(z_o)$ exists and $f'(z) = e^{-i\theta}(U_r + iV_r)$.

**Example 3.3.12.** . .

**Example 3.3.13.** . .

## 3.4 analytic functions

In the preceding section we found necessary and sufficient conditions for the component functions $u, v$ to construct an complex differentiable function $f = u + iv$. The definition that follows is the next logical step: we say a function is analytic[2] at $z_o$ if it is complex differentiable at each point in some open disk about $z_o$.

**Definition 3.4.1.**

Let $f = u + iv$ be a complex function. If there exists $\epsilon > 0$ such that $f$ is complex differentiable for each $z \in D_\epsilon(z_o)$ then we say that $f$ is **analytic** at $z_o$. If $f$ is analytic for each $z_o \in U$ then we say $f$ is analytic on $U$. If $f$ is not analytic at $z_o$ then we say that $z_o$ is a **singular point**. Singular points may be outside the domain of the function. If $f$ is analytic on the entire complex plane then we say $f$ **is entire. Analytic functions are also called holomorphic functions**

The theorem below shows that the sum, difference, quotient, product and composite of analytic functions is again analytic provided that there is no division by zero in the expresssion. This means that polynomials will be analytic everywhere, rational functions will be analytic at points where the denominator is nonzero and similar comments apply to algebraic functions of a complex variable. For the most part singular points will arise from division by zero in later examples.

**Theorem 3.4.2.**

Suppose $f, g$ are complex differentiable at $z \in \mathbb{C}$ and $c \in \mathbb{C}$ then

1. $\frac{d}{dz}(f(z) + g(z)) = \frac{df}{dz} + \frac{dg}{dz}$
2. $\frac{d}{dz}(f(z)g(z)) = \frac{df}{dz}g(z) + f(z)\frac{dg}{dz}$
3. if $g(z) \neq 0$ then $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$
4. $\frac{d}{dz}(cf) = c\frac{df}{dz}$
5. if $h$ is differentiable at $f(z)$ then $\frac{d}{dz}(h(f(z))) = \frac{dh}{dz}\frac{df}{dz} = h'(f(z))f'(z)$
6. $\frac{d}{dz}(c) = 0$
7. $\frac{d}{dz}(z^n) = nz^{n-1}$ for $n \in \mathbb{N}$
8. $\frac{d}{dz}(e^z) = e^z$

[2]you may recall that a function on $\mathbb{R}$ was analyic at $x_o$ if its Talyor series at $x_o$ converged to the function in some neighborhood of $x_o$. This terminology is consistent but it'll be while before we make the connection explicit

**Proof:** I use Proposition 2.2.8 to simplify limits throughout the argument below. That proposition helps us avoid direct $\epsilon - \delta$ argumentation. Assume $f, g$ are differentiable at $z$ then

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} + \lim_{h \to 0} \frac{g(z+h) - g(z)}{h} \\ &= f'(z) + g'(z). \end{aligned}$$

This proves (1.). I leave the of the other parts (2-7) as exercises for the reader. To prove (8.) recall that $e^z = e^x \cos(y) + ie^x \sin(y)$ for $z = x + iy$. Note that the Cauchy Riemann equations are indeed satisfied by $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$ since

$$u_x = u, \;\; u_y = -v, \;\; v_x = v, \quad v_y = u$$

gives $u_x = v_y$ and $u_y = -v_x$. Moreover, $u, v, u_x, u_y, v_x, v_y$ are clearly continuous on $\mathbb{C}$ thus we find $f(z) = e^z$ is differentiable at each $z \in \mathbb{C}$. Moreover,

$$f'(z) = u_x + iv_x = u + iv \;\Rightarrow\; \frac{d}{dz}(e^z) = e^z. \qquad \square$$

**Example 3.4.3.** . . .

**Example 3.4.4.** . . .

**Example 3.4.5.** . . .

**Example 3.4.6.** . . .

**Theorem 3.4.7.**

If $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_o$ then $f$ is continuous at $z_o$. Moreover, if $f$ is analytic at $z_o$ then there exists an open disk $D_\epsilon(z_o)$ on which $f$ is continuous.

**Proof:** We seek to show that $\lim_{h\to 0} f(z_o+h) = f(z_o)$. Consider that

$$\begin{aligned}
\lim_{h\to 0} f(z_o+h) = f(z_o) \quad &\Leftrightarrow \quad \lim_{h\to 0}(f(z_o+h) - f(z_o)) = 0 \\
&\Leftrightarrow \quad \lim_{h\to 0} \frac{h(f(z_o+h) - f(z_o))}{h} = 0 \\
&\Leftrightarrow \quad \lim_{h\to 0}(h) \lim_{h\to 0} \frac{f(z_o+h) - f(z_o)}{h} = 0 \\
&\Leftrightarrow \quad 0 \cdot f'(z_o) = 0
\end{aligned}$$

The last statement is clearly true and the limit properties clearly hold because all the limits stated in the calculation exist. Finally, if $f$ is analytic at $z_o$ then it follows that there exists an open disk $D_\epsilon(z_o)$ such that $f$ is differentiable at each point in the disk. But then $f$ is continuous at each point as well by the first part of the theorem. $\square$

The contrapositive of the theorem above indicates that if there does not exist at least one open disk on which the function is continuous then the function is not analytic at that point. It follows that we have to throw out some of the domain of the branches we've used for the root function or the principal argument. To avoid discontinuity we must throw out the branch entirely.

**Example 3.4.8.** *The princpal square root function is defined by $f_1(z) = |z| exp(iArg(z)/2)$. The domain of $f_1$ is governed by the princpal argument; $dom(f) = \{z \in \mathbb{C} \mid z \neq 0\}$ However, $Arg(z) = \pi$ gives points of discontinuity since*

$$f_1(z) = |z| exp(iArg(z)/2) = |z|(cos(Arg(z)/2) + isin(Arg(z)/2))$$

*has $f_1(z) \to |z|\sin(\pi/2) = |z|$ for paths with $Arg(z) \to \pi$ whereas $f_1(z) \to |z|\sin(-\pi/2) = -|z|$ for paths with $Arg(z) \to -\pi$. We must remove $Arg(z) = \pi$ from the domain if we wish $f_1$ to be analytic.*

**Example 3.4.9.** *Note, $f(z) = Arg(z)$ is analytic if we restrict to*

$$dom(f) = \{z \in \mathbb{C} \mid \textit{if } Im(z) = 0 \textit{ then } Re(z) \nleq 0\}.$$

*In other words, $dom(f) = \mathbb{C} -$ negative real axis and origin.*

Similar comments apply to various branches of the logarithm and the $n-th$ root mapping. The key is that continuity is required for an analytic function. However, continuity is not a sufficient condition for analyticity.

**Theorem 3.4.10.**

If $f$ is analytic on a domain $D$ and $f'(z) = 0$ for all $z \in D$ then $f$ is constant on $D$.

**Proof:** let $a, b \in D$ and, by connectedness of $D$, consider the line segment $[a, b] \subset D$ parametrized by $\gamma(t) = a + t(b - a)$ for $0 \leq t \leq 1$. Note, $f \circ \gamma : [0, 1] \to [a, b] \to \mathbb{C}$. The generalized chain rule states that the differential of the composite of two functions is the composite of the differentials,

$$d_t(f \circ \gamma) = d_{\gamma(t)} f \circ d_t \gamma$$

But, $f'(z) = 0$ for all $z \in D$ implies $d_{\gamma(t)} f(h) = f'(\gamma(t))h = 0$ for all $h \in \mathbb{C}$. Thus $d_{\gamma(t)} f = 0$ which gives us $d_t(f \circ \gamma) = 0$. It follows that, if $f = u + iv$ then $df = (df/dt)dt = (du/dt + idv/dt)dt = 0$ thus,

$$\tfrac{d}{dt}(u(\gamma(t))) = 0 \ \text{ and } \ \tfrac{d}{dt}(v(\gamma(t))) = 0$$

for all $t \in [0, 1]$. But then $u([a, b]) = \{u_o\}$ and $v([a, b]) = \{v_o\}$ and we find that $f([a, b]) = u_o + iv_o$ so the function is constant along the line segment in $D$. But, if $D$ is connected then we can connect any two points $p, q$ by a sequence of line segments and each line segment remains in $D$ hence the value of the function is constant on each line segement. It follows that the function has $f(p) = f(q)$ for all $p, q \in D$ thus $f(D) = \{u_o + iv_o\}$. □

## 3.5 differentiation of complex valued functions of a real variable

Perhaps some of the concepts in the proofs of Theorem 3.4.10 above seemed bizarre. In this section we take some time to derive the chain rule for complex-valued functions of a real variable with an analytic complex function. We'll conclude this section with a few comments which reflect on the logical necessity of the complex exponential function[3]. Let me point out that differentiation of a complex-valued function of a real variable is nothing more than differentiation of a two-dimensional space curves in calculus III. We just use a complex notation for two-dimensional real vectors in this course. Let $\vec{f}(t) =< u(t), v(t) >$ for $t \in \mathbb{R}$ then

$$\frac{d}{dt}[\ \vec{f}(t)\ ] =< \tfrac{du}{dt},\ \tfrac{dv}{dt} >$$

In complex notation, $f = u + iv$ and $\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$. The criteria for the existence of $df/dt$ for $f : \mathbb{R} \to \mathbb{C}$ is much weaker than the criteria for the existence of $df/dz$ for $f : \mathbb{C} \to \mathbb{C}$.

**Example 3.5.1.** *Note, $f(z) = \bar{z}$ is not analytic so $f'(z)$ is not defined. However, if $\gamma(t) = t + it^2$ then $(f \circ \gamma)(t) = f(t + it^2) = t - it^t$ and*

$$\tfrac{d}{dt}[\ (f \circ \gamma)(t)\ ] = \tfrac{d}{dt}[\ t - it^2\ ] = \tfrac{dt}{dt} - i\tfrac{dt^2}{dt} = 1 - 2it.$$

[3] you could take this section as motivation for the complex exponential we defined earlier, this section is not logically necessary to earlier calculations however it might give you some idea of **why** the complex exponential was defined as it was. Another motivation comes from the extension of power series to the complex setting, we'll see that later on

Suppose $f = u + iv$ is analytic and and let $\gamma(t) = a(t) + ib(t)$ for each $t \in \mathbb{R}$ where $a, b : \mathbb{R} \to \mathbb{R}$. If we compose $f$ with $\gamma$ then $f \circ \gamma : \mathbb{R} \to \mathbb{C}$ and we can calculate

$$\begin{aligned}
\tfrac{d}{dt}[\ (f \circ \gamma)(t)\ ] &= \tfrac{d}{dt}[\ u(\gamma(t)) + iv(\gamma(t))\ ] \\
&= \tfrac{d}{dt}[\ u(a(t), b(t)) + iv(a(t), b(t))\ ] \\
&= \tfrac{d}{dt}[\ u(a(t), b(t))\ ] + i\tfrac{d}{dt}[\ v(a(t), b(t))\ ] \\
&= u_x \tfrac{da}{dt} + u_y \tfrac{db}{dt} + i[\ v_x \tfrac{da}{dt} + v_y \tfrac{db}{dt}\ ] \\
&= (u_x + iv_x)\tfrac{da}{dt} - i(u_y + iv_y) i \tfrac{db}{dt} \\
&= \tfrac{\partial f}{\partial x}\tfrac{da}{dt} - i\tfrac{\partial f}{\partial y} i \tfrac{db}{dt} \\
&= f'(\gamma(t))\tfrac{da}{dt} + f'(\gamma(t)) i \tfrac{db}{dt} \\
&= f'(\gamma(t))(\tfrac{da}{dt} i \tfrac{db}{dt}) \\
&= f'(\gamma(t))\tfrac{d\gamma}{dt}
\end{aligned}$$

We could omit the arguments as is often done in the statement of a chain rule and simply say that

$$\boxed{\tfrac{d}{dt}[\ f(z(t)\ ] = \tfrac{df}{dz}\tfrac{dz}{dt}}$$

I remind the reader that the formula above holds for *analytic* functions.

**Theorem 3.5.2.**

Let $f(t) = exp(\lambda t)$ for all $t \in \mathbb{R}$ then $df/dt = \lambda exp(\lambda t)$.

**Proof:** Observe that $\gamma(t) = \lambda t = Re(\lambda)t + iIm(\lambda)t$ has $d\gamma/dt = Re(\lambda) + iIm(\lambda)$ and $f(z) = exp(z)$ has $df/dz = exp(z)$ therefore, by the calculation preceding the theorem, $d/dt(exp(\lambda t)) = \lambda exp(\lambda t)$.

Some authors might motivate the definition of the complex exponential function by assuming it should satisfy the theorem above. However you choose the starting point we should all agree that the complex exponential function should reduce to the real exponential function when restricted to the real-axis and it should maintain as many properties of the real exponential function as is reasonably possible in the complex setting. Indeed this is how all complex functions are typically defined. We want two main things: to extend $f : \mathbb{R} \to \mathbb{R}$ to $\tilde{f} : \mathbb{C} \to \mathbb{C}$ we expect

1. $\tilde{f}|_{\mathbb{R}} = f$
2. interesting properties of $f$ generalize to properties of $\tilde{f}$.

Item (2.) is where the fun is. We'll see how to define complex trigonmetric and hyperbolic functions in the upcoming sections. I suspect it's worth noting that one problem that naturally suggests the definition of the complex exponential is the problem of $2^{nd}$ order ordinary-constant-coefficient differential equations: that is, suppose you want to solve:

$$ay'' + by' + cy = 0$$

Since this analagous to $y' = \alpha y$ which has solution $y = e^{\alpha t}$ it's natural to guess the solution has the form $y = e^{\lambda t}$. Clearly, $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$ hence

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c\lambda e^{\lambda t} = (a\lambda^2 + b\lambda + c)e^{\lambda t} = 0$$

Therefore we find a necessary condition on $\lambda$ is that it satisfy the **characteristic equation:**

$$\boxed{y = e^{\lambda t} \ \text{ solves } \ ay'' + by' + cy = 0 \quad \Rightarrow \quad a\lambda^2 + b\lambda + c = 0.}$$

Apparently, solving the differential equation $ay'' + by' + cy = 0$ reduces to the problem of solving a corresponding algebra equation. Notice that we are tempted to answer the question of what a complex exponential is in this setting. Whether or not we began this discussion with complex things in mind the math has brought us an equation which necessarily includes complex cases. Moreover, it's easy to see that $y'' + y = 0$ has $y = \sin(t)$ and $y = \cos(t)$ as solutions. Note that $y'' + y = 0$ gives $\lambda^2 + 1 = 0$ which has solutions $\lambda = \pm i$. We then must suspect that the complex exponential function has something to do with sine and cosine. The founders of complex analysis were well aware of these sort of differential equations and it is likely that many of the complex functions first found their home inside some differential equation where they naturally arise as part of some general *ansatz*.

## 3.6 analytic continuations

We do not yet have the tools to prove the following statement. I postpone the proof for now.

**Conjecture 3.6.1.**

If $f$ is analytic on a disk $D_1$ and $f(z) = 0$ for all $z \in S$ where $S$ is either a line-segment or another disk contined in $D_1$ then $f(z) = 0$ for all $z \in D_1$.

Note we can extend this to a domain without too much trouble.

**Theorem 3.6.2.**

If $f$ is analytic on a domain $D$ and if $f(z) = 0$ for all $z \in S$ where $S$ is either a line-segment or another disk contined in $D$ then $f(z) = 0$ for all $z \in D$.

**Proof:** Let $z_o \in S$ and pick $w \in D - S$. Since $D$ is connected there exists a polygonal path $\gamma = [z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, w]$ where each of the line segments lies inside $D$. Let $\delta$ be the smallest distance between a point on $L$ and the boundary of $D$. Construct disks of radius $\delta$ with centers separated by a distance $\delta$ all along $L$. Notice that $D$ is open so even the closest open disk will not get to the edge of $D$ (which is not contained in the open set $D$). Moreover, the rest of the disks also remain in $D$. By our conjecture we find that the disk which is partially in $S$ must have $f$ identically zero since we can find a smaller disk totally in $S$ so the conjecture gives us $f$ zero on the first disk partly outside $S$. Then we can continue this process to the next disk. We simply take a smaller disk in the intersection of the two disks and because we already know it is zero from the last step of the argument it follows by the conjecture that $f$ is zero on the second disk. Let me sketch a picture of the argument above:

As you can see, the argument can be repeated until we reach the disk containing $w$. Thus we find $f(w) = 0$ for arbitrary $w \in D$ hence $f(D) = \{0\}$. $\square$

**Theorem 3.6.3.**

Let $\Omega$ be a domain or a line segment. If $f$ is analytic on a domain $D$ which contains $\Omega$ then $f$ is uniquely determined by its values on $\Omega$.

**Proof:** Suppose $f$ and $g$ are analytic on $D$ and $f(z) = g(z)$ for all $z \in \Omega$. Notice that $h : D \to \mathbb{C}$ defined by $h = f - g$ is identically zero on $\Omega$ since $h(z) = f(z) - g(z) = 0$ for all $z \in \Omega$. But then by 3.6 we find $h(z) = 0$ for all $z \in D$. It follows that $f = g$. $\square$

We say that $f$ is the **analytic continuation** of $f|_\Omega$. There is more to learn and say about analytic continuations in general, however we have what we need for our purposes at this point. Let's get to the point:

**Theorem 3.6.4.**

If $f : \mathbb{R} \to \mathbb{R} \subset \mathbb{C}$ is a function and $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is an extension of $f$ which is analytic then $\tilde{f}$ is unique. In particular, if there is an analytic extension of sine, cosine, hyperbolic sine or hyperbolic cosine then those extensions are unique.

This means if we demand analyticity then we actually had no freedom in our choice of the exponential. If we find a complex function which matches the exponential function on a line-segment ( *in particular a closed interval in* $\mathbb{R}$ *viewed as a subset of* $\mathbb{C}$ *is a line-segment* ) then there is just one complex function which agrees with the real exponential and is complex differentiable everywhere.

$$f(x) = e^x \qquad \text{extends uniquely to} \qquad \tilde{f}(z) = e^{Re(z)}(\cos(Im(z)) + i\sin(Im(z))).$$

Note $\tilde{f}(x + 0i) = e^x(\cos(0) + i\sin(0)) = e^x$ thus $\tilde{f}|_{\mathbb{R}} = f$. Naturally, analyiticity is a desireable property for the complex-extension of known functions so this concept of analytic continuation is very nice. Existence aside, we should first construct sine, cosine etc... then we have to check they are both analytic and also that they actually agree with the real sine or cosine etc... If a function on $\mathbb{R}$ has vertical asymptotes, points of discontinuity or points where it is not smooth then the story is more complicated.

## 3.7 trigonometric and hyperbolic functions

Recall we found that for $\theta \in \mathbb{R}$ the formulas $\cos(\theta) = \frac{1}{2}\big(e^{i\theta} + e^{-i\theta}\big)$ and $sin(\theta) = \frac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big)$ were useful for deriving trigonmetric identities. We now extend to complex arguments.

**Definition 3.7.1.**

We define $cos(z) = \frac{1}{2}\big(e^{iz} + e^{-iz}\big)$ and $sin(z) = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big)$ for all $z \in \mathbb{C}$

Note that $cos(z)$ and $sin(z)$ are sums of composites of analytic functions since the function $g(z) = cz$ is clearly analytic and $h(z) = e^z$ is analytic. Moreover, it is clear that $\tilde{f}(z) = cos(z)$ restricts to the usual real cosine function along the real axis. This is a consequence of Euler's formula:

$$\cos(x + i0) = \frac{1}{2}\big(e^{ix} + e^{-ix}\big) = \frac{1}{2}\big(\cos(x) + i\sin(x) + \cos(x) + i\sin(-x)\big) = \cos(x)$$

Don't get lost in the notation here, the "cos" on the left is the newly defined complex cosine whereas the "cos" on the right is the cosine you know and love from the study of circular functions. The fact that the complex cosine is the unique analytic continuation of the real cosine function makes this notation reasonable. Similar comments apply to the sine function.

**Proposition 3.7.2.**

Let $z \in \mathbb{C}$,

1. $\sin(z)$ and $\cos(z)$ are unbounded.
2. $\sin(z)$ is an odd function of $z$
3. $\cos(z)$ is an even function of $z$

Notice that $z = iy$ gives $\cos(z) = \cos(iy) = \frac{1}{2}\left(e^{-y} + e^{y}\right)$. Thus the complex cosine can assume arbitrarily large values. Likewise, $\sin(iy) = \frac{1}{2i}\left(e^{-y} - e^{y}\right)$ has $|\sin(iy)|$ take arbitrarily large values as we range over the complex plane. Items (2.) and (3.) are immediate from the definition.

**Proposition 3.7.3.**

Let $z, w \in \mathbb{C}$,

1. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
2. $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
3. $\sin^2(z) + \cos^2(z) = 1$
4. $\sin(2z) = 2\sin(z)\cos(z)$
5. $\sin^2(z) = \frac{1}{2}(1 - \cos(2z))$
6. $\cos^2(z) = \frac{1}{2}(1 + \cos(2z))$

I leave the proof to the reader. I think some of these are homeworks in Churchill, some I may have assigned.

**Definition 3.7.4.**

We define the **hyperbolic cosine** $cosh(z) = \frac{1}{2}\left(e^{z} + e^{-z}\right)$ and **the hyperbolic sine** $sinh(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right)$ for all $z \in \mathbb{C}$.

You may recall that we **defined** hyperbolic cosine and sine to be the even and odd parts of the exponential function respective,

$$e^x = \underbrace{\frac{1}{2}\left(e^x + e^{-x}\right)}_{\cosh(x)} + \underbrace{\frac{1}{2}\left(e^x - e^{-x}\right)}_{\sinh(x)}$$

Clearly the complex hyperbolic functions restrict to the real exponential functions and they are also entire since they are the sum and composite of entire functions $e^z$ and $-z$. It follows that the complex hyperbolic functions defined above are the unique analytic continuation of the real hyperbolic functions. You could probably fill a small novel with interesting formulas which are known for hyperbolic functions. We will content ourselves to notice these three items:

**Proposition 3.7.5.**

1. $i\sinh(iz) = \sin(z)$
2. $\cosh(iz) = \cos(z)$
3. $\cosh^2(z) - \sinh^2(z) = 1$

Finally, I should mention that the other elementary trigonometric and hyperbolic functions are likewise defined. For example,

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad sech(z) = \frac{1}{\cosh(z)}.$$

Inverse functions may also be defined for suitably restricted functions. Of course this should not be surprising, even in the real case we have to restrict sine, cosine and tangent in order to obtain standard inverse functions. This is essentially the same issue as the one we were forced to deal with in our discussion of branch cuts.

**Proposition 3.7.6.**

1. $d/dz(\sin(z)) = \cos(z)$
2. $d/dz(\cos(z)) = -\sin(z)$
3. $d/dz(\tan(z)) = \sec^2(z)$
4. $d/dz(\sinh(z)) = \cosh(z)$
5. $d/dz(\cosh(z)) = \sinh(z)$
6. $d/dz(\tanh(z)) = sech^2(z)$

I leave the proof of these to the reader. Also, it might be interesting to study the geometry of the mapping $w = \sin(z)$ and so forth. Many complex variables texts have nice pictures of the geometry, I may put some up on the projecter in lecture. The links on the webpage point you to several sites which explore the geometry of mappings.

## 3.8 harmonic functions

We've discussed in some depth how to determine if a given function $f = u + iv$ is in fact analytic. In this section we study another angle on the story. We learn that the component functions $u, v$ of an analytic function $f = u + iv$ are *harmonic conjugates* and they satisfy the phyically significant Laplace's equation $\nabla^2\phi = 0$ where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In addition we'll learn that if we have one solution of Laplace's equation then we can consider it to be the "$u$" of some yet undetermined analytic function $f = u + iv$. The remaining function $v$ is then constructed through some integration guided by the CR-equations. The construction is similar to the problem of construction of a potential function for a given conservative force in calculus III.

**Proposition 3.8.1.**

If $f = u + iv$ is analytic on some domain $D \subseteq \mathbb{C}$ then $u$ and $v$ are solutions of Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ on $D$.

**Proof:** since $f = u + iv$ is analytic we know the CR-equations hold true; $u_x = v_y$ and $u_y = -v_x$. Moreover, $f$ is continuously differentiable so we may commute partial derivatives by a theorem from multivariate calculus. Consider

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

Likewise,

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0$$

Of course these relations hold for all points inside $D$ and the proposition follows. $\square$

**Example 3.8.2.** *Note $f(z) = z^2$ is analytic with $u = x^2 - y^2$ and $v = 2xy$. We calculate,*

$$u_{xx} = 2, \quad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

*Note $v_{xx} = v_{yy} = 0$ so $v$ is also a solution to Laplace's equation.*

Now let's see if we can reverse this idea.

**Example 3.8.3.** *Let $u(x, y) = x + c_1$ notice that $u$ solves Laplace's equation. We seek to find a harmonic conjugate of $u$. We need to find $v$ such that,*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$$

*Integrate these equations to deduce $v(x, y) = y + c_2$ for some constant $c_2 \in \mathbb{R}$. We thus construct an analytic function $f(x, y) = x + c_1 + i(y + c_2) = x + iy + c_1 + ic_2$. This is just $f(z) = z + c$ for $c = c_1 + ic_2$.*

**Example 3.8.4.** *Suppose $u(x, y) = e^x \cos(y)$. Note that $u_{xx} = u$ whereas $u_{yy} = -u$ hence $u_{xx} + u_{yy} = 0$. We seek to find $v$ such that*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin(y)$$

*Integrating $v_y = e^x \cos(y)$ with respect to $y$ and $v_x = e^x \sin(y)$ with respect to $x$ yields $v(x, y) = e^x \sin(y)$. We thus construct an analytic function $f(x, y) = e^x \cos(y) + ie^x \sin(y)$. Of course we should recognize the function we just constructed, it's just the complex exponential $f(z) = e^z$.*

Notice we cannot just construct an analytic function from any given function of two variables. We have to start with a solution to Laplace's equation. This condition is rather restrictive. There is much more to say about harmonic functions, especially where applications are concerned. My goal here was just to give another perspective on analytic functions. Geometrically one thing we could see without further work at this point is that for an analytic function $f = u + iv$ the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. Note $grad(u) =< u_x, u_y >$ and $grad(v) =< v_x, v_y >$ have

$$grad(u) \cdot grad(v) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

This means the normal lines to the level curves for $u$ and $v$ are orthogonal. Hence the level curves of $u$ and $v$ are orthogonal.

# Chapter 4

# complex integration

In this chapter I collect many interesting theorems about contour integrals. The proofs can be found in Churchill. I will focus my efforts on adding a few examples.

## 4.1 integrals of a complex-valued function of a real variable

Integration of functions from $\mathbb{R}$ into $\mathbb{C}$ follows the rule you may recall from calculus III, we integrate componentwise.

**Definition 4.1.1.**

Let $f : U \subset \mathbb{R} \to \mathbb{C}$ have component functions $u, v : U \subset \mathbb{R} \to \mathbb{R}$ where $f(t) = u(t) + iv(t)$. If $[a, b] \subseteq U$ then we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Indefinte integrals similarly.

**Example 4.1.2.** *Calculate the integral,*

$$\begin{aligned}\int [\ \cos(t) + i\sin(t)\ ]dt &= \int \cos(t)\,dt + i\int \sin(t)\,dt \\ &= \sin(t) + c_1 + i(-\cos(t) + c_2) \\ &= -i[\cos(t) + i\sin(t)] + c_1 + ic_2\end{aligned}$$

*Thus, following the same steps just with bounds,*

$$\int_0^\pi [\ \cos(t) + i\sin(t)\ ]dt = \sin(t)\big|_0^\pi - i\cos(t)\big|_0^\pi = -2i$$

*Notice that we can also summarize this calculation in terms of the imaginary exponential,*

$$\int e^{it}\,dt = \tfrac{1}{i}e^{it} + c \quad\Rightarrow\quad \int_0^\pi e^{it}\,dt = -ie^{it}\big|_0^\pi = -ie^0 + ie^{i\pi} = -2i$$

The preceding example illustrates a more general pattern in these sort of integrals.

**Proposition 4.1.3.**

Suppose $f$ and $F$ are complex-valued functions of a real variable such that $\frac{dF}{dt} = f(t)$ then

$$\int_a^b f(t)\,dt = F(b) - F(a) \qquad \text{also} \qquad \int f(t)\,dt = F(t) + c$$

The proof of the proposition follows directly from the Fundamental Theorem of Calculus. Note that if $f = u + iv$ and $F = U + iV$ and $dF/dt = f$ it follows $dU/dt = u$ and $dV/dt = v$.

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^b u\,dt + i\int_a^b v\,dt \\ &= U(b) - U(a) + i(V(b) - V(a)) \\ &= F(b) - F(a)\end{aligned}$$

The notation $\int f(t)\,dt = F(t) + c$ is just notation to indicate $F$ is the antidervative of $f$ meaning that $\frac{dF}{dt} = f$. Naturally other properties of definite integrals likewise transfer to our current context:

**Proposition 4.1.4.**

Suppose $f, g$ are complex-valued functions of a real variable and $a, b, c, c_1 \in \mathbb{R}$ such that $a < b < c$

1. $\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$
2. $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$
3. $\int_a^b [\ f(t) + g(t)\ ]dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$
4. $\int_a^b c_1 f(t)\,dt = c_1 \int_a^b f(t)\,dt$

here I have assumed that $f, g$ are continuous or possibly piecewise continuous so that the integrals above are well-defined. Properties (3.) and (4.) also hold for indefinite integrals.

The proposition below is less trivial. The chain-rule $\frac{d}{dt}(F(\phi(t))) = \frac{dF}{dz}\frac{d\phi}{dt}$ where $\phi : \mathbb{R} \to \mathbb{C}$ is path and $F : \mathbb{C} \to \mathbb{C}$ is analytic at $\phi(t)$ justifies the proposition below:

**Proposition 4.1.5.**

Suppose $f$ and $F$ are complex functions, meaning $f : dom(f) \to \mathbb{C}$ and $F : dom(F) \to \mathbb{C}$, such that $dF/dz = f$. Futhermore, suppose $\phi : I \subseteq \mathbb{R} \to \mathbb{C}$ such that $\phi'(t) \neq 0$ then

$$\int_a^b f(\phi(t))\frac{d\phi}{dt}\,dt = F(\phi(t))\Big|_{t=a}^{t=b}$$

A slight twist on the preceding proposition gives us a u-substitution theorem in our context,

$$\int_a^b f(\phi(t))\frac{d\phi}{dt}\,dt = \int_{\phi(a)}^{\phi(b)} f(\phi)d\phi = \int_{\phi(a)}^{\phi(b)} \frac{dF}{dz}dz = F(\phi(b)) - F(\phi(a)).$$

**Example 4.1.6.** *Calculate the integral,*

$$\begin{aligned}\int \Big[\ \sin(t) + it^2\ \Big]^{13} \big(\cos(t) + 2it\big)\,dt &= \int u^{13}\,du \\ &= \frac{1}{14}u^{14} + c \\ &= \frac{1}{14}\Big[\ \sin(t) + it^2\ \Big]^{14} + c\end{aligned}$$

*Don't believe me? Just differentiate the answer and see if our antiderivative returns the integrand once differentiated.*

**Example 4.1.7.** *Calculate the integral, for $\lambda \in \mathbb{C}$ ( a constant)*

$$\begin{aligned}\int \exp(\lambda t)\,dt &= \int \exp(\lambda t)\,dt \\ &= \int \exp(u)\,\frac{du}{\lambda} \\ &= \frac{1}{\lambda}\exp(u) + c \\ &= \frac{1}{\lambda}\exp(\lambda t) + c\end{aligned}$$

Suppose $z(t) = x(t) + iy(t)$ then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$, it is therefore formally reasonable to write $dz = \frac{dx}{dt}dt + i\frac{dy}{dt}dt$. This is the how we implement the substitution.

**Example 4.1.8.** *Calculate the integral,*

$$\begin{aligned}\int \sin(t + ie^t)(2 + i2e^t)\,dt &= \int \sin(z)\,\frac{dz}{2} \qquad \textit{:let } z = t + ie^t \textit{ thus } dz = (1 + ie^t)dt \\ &= -\tfrac{1}{2}\cos(z) + c \\ &= -\tfrac{1}{2}\cos(t + ie^t) + c\end{aligned}$$

Finally, the following property (proved on page 88-89 of Churchill) is important to many arguments later: (note, $w$ below is chosen such that the integrals exist)

**Proposition 4.1.9.**

Let $w(t) = u(t) + iv(t)$ be a complex valued function of a real variable,

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt \qquad \text{and} \qquad \left|\int_a^\infty w(t)\,dt\right| \leq \int_a^\infty |w(t)|\,dt$$

## 4.2 contour integrals

A contour is a curve in the complex plane which can be broken into a finite number of "smooth arcs". Contours can self-intersect, they can even have corners. However, the arcs which comprise the contour we assume to be smooth (meaning the derivative of the arc map is nonzero) on their interiors.

**Definition 4.2.1.**

An **arc** is a continuous mapping $\gamma$ from $[a, b]$ into $\mathbb{C}$ . A **smooth arc** is an arc $\gamma$ such that $\gamma'(t) \neq 0$ for all $t \in (a, b)$. Suppose $C \subset \mathbb{C}$ such that $C = C_1 \cup C_2 \cup \cdots \cup C_n$ and suppose there exist smooth arcs $\gamma_j : [a_j, b_j] \to C_j$ where $\gamma_1(a_1) = p$ and $\gamma_n(b_n) = q$ and $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ for each $j = 1, 2, \dots n-1$ then we say $C$ is a **contour** from **terminal point** $p$ to **terminal point** $q$. If $C$ is a contour which begins and ends at the same point then we say $C$ is a **closed contour**. A **simple arc** is an arc which is given by an injective arc map, it has no "self-intersections". Likewise, a contour is called **simple** if it has no self-intersections except possibly the terminal points of the contour. A **simple closed contour** only intersects itself at the terminal points. Finally, for a given contour many arcs may be used to cover the contour, we call each a particular **parametrization** of the contour.

The Jordon curve theorem applied to the complex plane states that a simple closed contour divides the complex plane into a it simply connected interior and an unbounded exterior. The term **simply connected** has the same meaning as when it was discussed in your calculus III course. In a nutshell, a set is simply connected if any loop in the set can be continuously deformed to a point. For a two-dimensional set this means it has no holes. In three dimensions, you can have a hole in a simply connected set.

**Definition 4.2.2.** *integral along an arc*

Let $\gamma : [a, b] \to C \subset \mathbb{C}$ and suppose $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ is continuous on $C$ then

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}dt$$

Moreover, we may sometimes denote $\int_C f(z)dz = \int_C f$.

I should mention that $f(z)$ is just short for $f(x, y)$ in the definition above. For example, we do allow for integration of $f(z) = \bar{z}$.

**Example 4.2.3.** *Suppose $C$ is the line segment from $p$ to $q$ then it is an arc and we may use the map $\gamma(t) = p + t(q - p)$ for $0 \leq t \leq 1$ to parameterize $C$. Generically, $\int_C f(z)dz = \int_0^1 f(p + t(q - p))(p - q)dt$. For example, if $f(z) = z^2$, $p = 1$, $q = 1 + 2i$ then*

$$\int_C z^2\, dz = \int_0^1 (1 + 2it)^2(2i)dt = \tfrac{1}{3}(1 + 2it)^3\Big|_0^1 = \tfrac{1}{3}\big[(1 + 2i)^3 - 1\big] = -\tfrac{2}{3}\big[i + 6\big]$$

You might notice in the last example the parametrization seems superfluous, couldn't we just write the following instead?

$$\int_C z^2\,dz = \int_1^{1+2i} z^2\,dz = \tfrac{1}{3}z^3\Big|_1^{1+2i} = \tfrac{1}{3}\big[(1+2i)^3 - 1\big] = -\tfrac{2}{3}\big[i+6\big]$$

The answer: no. Not without a lot of fine print. Notice that the bounded integral was never defined, it begs the question what about another curve from 1 to $1+2i$, what notation would we use for that? Would another path give the same result? Let's look at another example before thinking too much about the curious (formal) calculation above.

**Example 4.2.4.** *Suppose $C$ is the quarter circle from 1 to $i$ which is an arc with map $\gamma(t) = e^{it}$ for $0 \leq t \leq \pi/2$ to parameterize $C$.*

$$\int_C \bar{z}\,dz = \int_0^{\frac{\pi}{2}} \overline{e^{it}}\big(ie^{it}\big)dt = \int_0^{\frac{\pi}{2}} i\,dt = \pi/2.$$

*Likewise, if we let $C_2$ be the full unit circle then the same arc map applies and we simply need to extend the bounds to range from 0 to $2\pi$ and we could calculate $\int_C \bar{z}\,dz = 2\pi$.*

Note that if there was a formal calculation that could take the place of the contour integration above then it would have to return a result of zero since the contour above begins and ends at the same point. What's the difference between the preceding pair of examples? What's the difference between $f(z) = z^2$ and $g(z) = \bar{z}$? I'll let you think about it, the answer is contained in the Theorem on page 105 of Churchill.

**Definition 4.2.5.** *contour integration*

If $C = C_1 \cup C_2 \cup \cdots \cup C_n$ is a contour comprised of $n$-arcs $C_j$ then the **contour integral** of a continuous complex function $f : dom(f) \to \mathbb{C}$ such that $C \subseteq dom(f)$ is defined as the sum of the integrals over the arcs

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz$$

Often the notation $C = C_1 + C_2 + \cdots C_n$ is used in the context above, it simply means that the arcs are placed end to end and in total they cover the contour $C$. Also, we can weaken the criteria of continuity to piecewise continuity and the custom is just to add up the results of the pieces. Various texts cut this definition more or less finely. I hope I've given you enough detail as for it to make sense. It is possible to define it from a complex Riemann sum or in terms of the correspondance to line-integrals.

**Example 4.2.6.** *Suppose $S = C_1 \cup C_2 \cup C_3 \cup C_4$ is a square which is oriented in the CCW fashion* *with corners* $(a, b), (a+h, b), (a+h, b+h), (a, b+h)$*. The contour integral of* $f(z) = z$ *works out to*

$$\begin{aligned}\int_S z dz &= \int_{C_1} z dz + \int_{C_2} z dz + \int_{C_3} z dz + \int_{C_4} z dz \\ &= \int_{C_1} (x+ib) dz + \int_{C_2} (a+h+iy) dz + \int_{C_3} (x+i(b+h) dz + \int_{C_4} (a+iy) dz \\ &= \int_0^h \big(a+t+ib\big) dt + i\int_0^h \big(a+h+i(b+t)\big) dt \\ &\quad - \int_0^h \big(a+h-t+i(b+h)\big) dt - i\int_0^h \big(a+i(b+h-t)\big) dt \\ &= \int_0^h \big(2t-2t-h+h\big) dt \\ &= 0\end{aligned}$$

*The notation is very natural to calculate* $dz$ *just use* $dz = dx + idy$*, this notation is equivalent to our parametric mapping definition.*

1. $C_1 = [a+ib, a+h+ib]$ *has* $x = a+t$ *and* $y = b$ *so* $dx = dt$ *and* $dy = 0$ *hence* $dz = dt$*.*
2. $C_2 = [a+h+ib, a+h+i(b+h)]$ *has* $x = a+h$ *and* $y = b+t$ *so* $dx = 0$ *and* $dy = dt$ *and* $dz = idt$
3. $C_3 = [a+h+i(b+h), a+i(b+h)]$ *has* $x = a+h-t$ *and* $y = b+h$ *so* $dx = -dt$ *and* $dy = 0$ *and* $dz = -dt$
4. $C_4 = [a+i(b+h), a+ib]$ *has* $x = a$ *and* $y = b+h-t$ *so* $dx = 0$ *and* $dy = -dt$ *and* $dz = -idt$

*Since this was a square I was able to take* $0 \leq t \leq h$ *on each side. For a rectangle it would be natural to follow almost the same calculation except the vertical legs and horizontal legs would have a different h-value. Essentially the same cancellation will again occur. Why? (again see Theorem on page 105 and think about what makes* $f(z) = z$ *special.*

**Proposition 4.2.7.** *basic properties of contour integration*

1. let $C$ be a contour and $-C$ be the contour in the reverse direction
$$\int_{-C} f(z) dz = -\int_C f(z) dz.$$
2. Let $L$ be the arclength of the contour $C$, if $|f(z)| \leq M$ for $z \in C$ then
$$\left| \int_C f(z)\, dz \right| \leq ML$$

I gave additional examples in lecture, if you missed class please get notes from a classmate. One thing I might ask you to prove is that the contour integral is well-defined. In particular, you should be able to show it is invariant under a reparametrization of the contour. (if you can prove it for an arc that would be sufficient, it's what I did in lecture.).

**Apology: I would very much like to have notes written corresponding to sections 34-42 of Churchill however, I don't believe I will have time before your test, or perhaps this semester. The text is good, we did cover these section before spring break, and if you want good notes I would advise you to ask a student who attended all the lectures. I do hope to provide more notes but until that happens please make use of the text and perhaps my notes from when I took this course from my advisor Dr. Fulp**

## 4.3 antiderivatives and analytic functions

## 4.4 Cauchy Goursat and the deformation theorems

## 4.5 Cauchy's Integral Formula

## 4.6 Lioville's Theorem and the Fundmental Theorem of Algebra

## Chapter 5

# Taylor and Laurent series

sections 43-52

# Chapter 6

# residue theory

sections 53-59

# Chapter 7

# residue theory

sections 60-64