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MATH 331: SPRING 2013, DUE 2-22 BEFORE 4:30PM. PROBLEM SET 4

Topics: Cauchy's Integral formula, estimating theorems.

Problem 41 problem 13 of section 33 of Churchill (page 103).

Problem 42 problem 17 of section 33 of Churchill (page 103). 2nd.

Problem 43 problem 5 of section 38 of Churchill (page 120).

Problem 44 problem 6 of section 38 of Churchill (page 120).

Problem 45 problem 9 of section 38 of Churchill (page 120).

Problem 46 problem 1 of section 40 of Churchill (page 128).

Problem 47 problem 9 of section 40 of Churchill (page 128).

Problem 48 Prove the lemma on page 86 of my posted notes. In particular, show why

merely

$$\left|\int_a^b [u(t)+iv(t)]dt\right| \leq \int_a^b \left|u(t)+iv(t)\right|dt.$$

You are free to use the theorem from calculus I which states $|\int_a^b g(t)dt| \leq \int_a^b |g(t)|dt$. I think the proof is mostly just that theorem filtered through the involved definitions.

Dial **Problem 49** Complete the proof of Theorem 4 on page 95 of my notes. (This was the claim which is similar to the proof (c.) implies (a.) from page 92, Theorem 4 is Cauchy's Theorem for star-shaped regions)

Problem 50 Let $P(z) \in \mathbb{C}[z]$ be a nonconstant polynomial. Suppose C is a simple-closed-contour. Study $\frac{1}{2\pi i}\int_C \frac{P'(z)dz}{P(z)}$ and explain what this integral detects. Hint: derive the first half of Problem 8 on page 100 of Freitag and use that algebraic result paired with Cauchy's integral formula.

nonconstant polynomial – a polyn

$\boxed{\text{PROBLEM 41}}$ #13 of §33

Let C_0: $z = z_0 + Re^{i\theta}$, $-\pi \le \theta \le \pi$ desribe the CCW circle $|z - z_0| = R$.

a.) $\displaystyle\int_{C_0} \frac{dz}{z-z_0} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} d\theta = i\int_{-\pi}^{\pi} d\theta = \boxed{2\pi i.}$

b.) $\displaystyle\int_{C_0} (z-z_0)^{n-1} dz = \int_{-\pi}^{\pi} (Re^{i\theta})^{n-1} Rie^{i\theta} d\theta$ (where $n \in \mathbb{Z}$.)

$$= i\int_{-\pi}^{\pi} R^n e^{ni\theta} d\theta$$

$$= iR^n \left.\frac{e^{ni\theta}}{n}\right|_{-\pi}^{\pi}$$

$$= \frac{iR^n}{n}\left(e^{ni\pi} - e^{-ni\pi}\right)$$

$$= \frac{iR^n}{n}\left(\cancel{\cos n\pi} + i\underbrace{\sin n\pi}_{0} - \left(\cancel{\cos(-n\pi)} + i\underbrace{\sin(-n\pi)}_{0}\right)\right)$$

$$= \boxed{0.}$$

c.) $\displaystyle\int_{C_0} (z-z_0)^{a-1} dz = \int_{-\pi}^{\pi} \exp\left((a-1)\operatorname{Log}(Re^{i\theta})\right) Rie^{i\theta} d\theta$

$$= iR\int_{-\pi}^{\pi} \exp\left[(a-1)(\ln R + i\theta)\right] e^{i\theta} d\theta$$

$$= iR\int_{-\pi}^{\pi} e^{\ln(R^{a-1})} e^{i(a-1)\theta} e^{i\theta} d\theta$$

$$= iRR^{a-1}\int_{-\pi}^{\pi} e^{ia\theta} d\theta$$

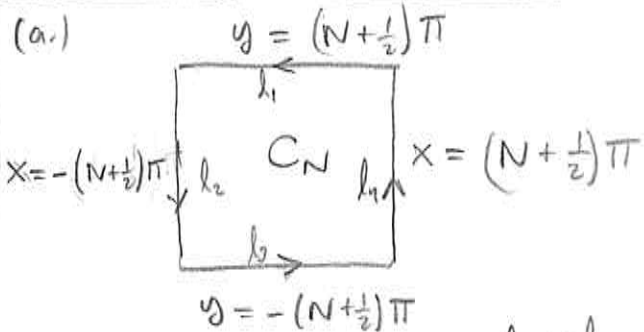
$$= iR^a \frac{1}{ia}\left(e^{ia\pi} - e^{-ia\pi}\right)$$

$$= \frac{2iR^a}{a} \frac{1}{2i}\left(e^{ia\pi} - e^{-ia\pi}\right)$$

$$= \boxed{\frac{2iR^a}{a} \sin(a).}$$

PROBLEM 42 #17 of §33

(a.)



We can show that
$|\sin z| \geq |\sin x|$
$|\sin z| \geq |\sinh y|$.

Show $|\sin z| \geq 1$ on $l_2 \cup l_4$ and show $|\sin z| > \sinh(\frac{\pi}{2})$ for $l_1 \cup l_3$.
Thus show $\exists A > 0$ independent of N such that $|\sin z| \geq A \ \forall z \in C_N$

(b.) Show $\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)\pi A}$

(a.) If $z \in l_2 \cup l_4$ then $z = x + iy$ with $x = \pm (N + \frac{1}{2})\pi$ and $-(N+\frac{1}{2})\pi \leq y \leq (N + \frac{1}{2})\pi$. Consider,

$$|\sin z| \geq |\sin(x)| = \left|\sin\left(\pm(N+\tfrac{1}{2})\pi\right)\right| = 1.$$

Likewise, for $z \in l_1 \cup l_3$,

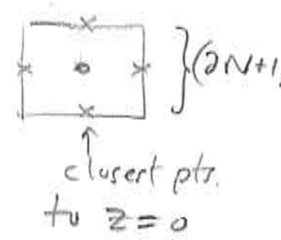
$$|\sin z| \geq |\sinh(y)| = \left|\sinh\left(\pm(N+\tfrac{1}{2})\pi\right)\right| = \sinh((N+\tfrac{1}{2})\pi).$$

But, $\sinh(x)$ is an everywhere increasing function hence

$$\sinh(N\pi + \tfrac{\pi}{2}) > \sinh(\tfrac{\pi}{2})$$

Thus $|\sin z| \geq \sinh(\pi/2)$ for $z \in l_1 \cup l_3$. Let $A = \min\{1, \sinh \pi/2\}$ and observe for $z \in C_N$ we have $|\sin z| \geq A$. Let $z \in C_N$ and consider,

(b.)
$$\left| \frac{1}{z^2 \sin z} \right| \leq \frac{1}{A|z|^2} \leq \frac{1}{A\left(\underbrace{\tfrac{1}{2}(2N+1)\pi}_{\text{smallest distance to } C_N \text{ from } z=0.}\right)^2}$$



Note also,

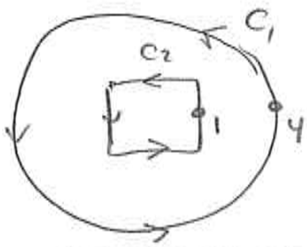
$$l(C_N) = 4(2N+1)\pi$$

Thus,

$$|I| = \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{4(2N+1)\pi}{A\left(\frac{1}{4}(2N+1)^2\pi^2\right)} = \frac{16}{\pi A(2N+1)}$$

clearly $|I| \longrightarrow 0$ as $N \longrightarrow \infty$ hence $\lim_{N\to\infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$.

PROBLEM 43 | #5 of §38



why does $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for:

a.) $f(z) = \dfrac{1}{3z^2+1} = \dfrac{1}{(\sqrt{3}\,z)^2 - i^2} = \dfrac{1}{(\sqrt{3}z - i)(\sqrt{3}z + i)}$

$f(z)$ is not analytic at $\sqrt{3}\,z = \pm i \hookrightarrow z = \frac{\pm i}{\sqrt{3}}$

But, $\frac{1}{\sqrt{3}} < 1$ so both these singularities fall inside C_2. Thus $f(z)$ is analytic between and on C_1 & C_2.

(b.) $f(z) = \dfrac{z+2}{\sin(z/2)}$ need to be sure no zeros of $\sin z/2$ fall between C_1 & C_2. Observe

$$\sin(z/2) = 0 \Rightarrow \frac{z}{2} = n\pi,\ n \in \mathbb{Z}$$

(see pg. 70 of Churchill, this is not immediately obvious remember, you can't just assume properties of the real sine function for $\sin : \mathbb{C} \to \mathbb{C}$.)

Hence,
$z = 2n\pi,\ n \in \mathbb{Z}$
give divergent pts. for
$f(z) = \dfrac{z+2}{\sin(z/2)}$

Consider
$n = 0,\quad z = 0$ (inside C_2)
$n = \pm 1,\quad z = \pm 2\pi$ (outside C_1)

c.) $f(z) = \dfrac{z}{1-e^z}$. Observe $1 - e^z = 0$
$\Rightarrow 1 = e^z$
$\Rightarrow z = \log(1) = i\arg(1)$

$i\arg(1) = \{2\pi i k \mid k \in \mathbb{Z}\}$ but again closest points are $-2\pi i,\ 0,\ 2\pi i$

outside C_1 — inside C_2.

PROBLEM 44 | #6 of §38

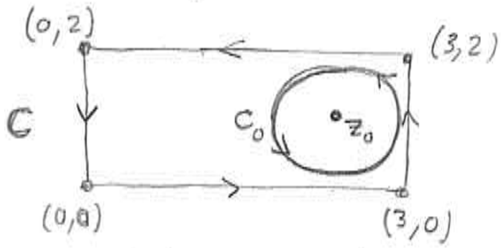
Let C_0 : $|z - z_0| = R$ oriented CCW. We saw in #13 of §33

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{when } n = 0 \end{cases}$$

Use the above result and Cor. 2 of §38 (aka the deformation Thm) to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$ oriented CCW then

$$\int_C (z - 2 - i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{when } n = 0. \end{cases}$$

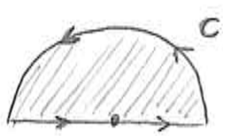
Mostly the argument here is the following picture,



$z_0 = 2 + i$

choose R such that C_0 is inside C. Observe $f(z) = (z - 2 - i)^{n-1}$ is analytic between and on C_0 and C hence $\int_{C_0} f(z)dz = \int_C f(z)dz$, the result follows.

PROBLEM 45 | #9 of §38



$f(0) = 0$

$\underbrace{f(z) = \sqrt{r}\, e^{i\theta/2}, \ r > 0, \ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}}_{\text{particular branch of } z^{1/2}}$

(also explain why Cauchy/Goursat does not apply here)

Show $\int_C f(z) dz = 0$ by separately evaluating C_1 and C_2

$$\int_{C_1} f(z)dz = \int_0^{\pi} f(re^{i\theta}) \frac{d}{d\theta}(re^{i\theta})\, d\theta = \int_0^{\pi} \sqrt{r}\, e^{i\theta/2}\, rie^{i\theta} d\theta = \frac{ir^{3/2} e^{\frac{3i}{2}\theta}}{3i/2}\Bigg|_0^{\pi}$$

$$= \frac{2}{3} r^{3/2} \left(e^{3i\pi/2} - e^0\right)$$

$$= \frac{2}{3} r^{3/2} (-i - 1)$$

$$\int_{C_2} f(z)dz = \int_0^1 f(-r + 2rt)\, 2r\, dt \quad \left(\begin{array}{l} C_2: z = -r + t(2r) \\ \text{for } 0 \le t \le 1 \end{array}\right)$$

$$= \int_0^{1/2} 2r\sqrt{|r(2t-1)|}\, e^{i\pi/2} dt + \int_{1/2}^1 2r\sqrt{|r(2t-1)|}\, e^{i(0)/2} dt$$

$$= 2r\sqrt{r}\left(i\int_0^{1/2} \sqrt{1-2t}\, dt + \int_{1/2}^1 \sqrt{2t-1}\, dt\right)$$

$$= 2r^{3/2}\left(i\left(\frac{2}{3}\right)\left(\frac{1}{-2}\right)(1-2t)^{3/2}\Big|_0^{1/2} + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(2t-1)^{3/2}\Big|_{1/2}^1\right)$$

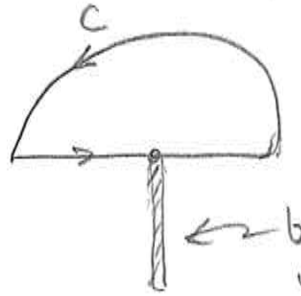
$$= 2r^{3/2}\left[\frac{i}{3} + \frac{1}{3}\right]$$

thus $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$.

Cauchy - Goursat does not apply because f is not analytic at the origin.

PROBLEM 45 Comment

We must be able to find a simply connected domain D on which f is analytic. Then if we can place a closed contour C inside D then for such contours $\int_C f(z)\,dz = 0$.



← any domain containing C must contain 0 and hence point(s) where f is not analytic.

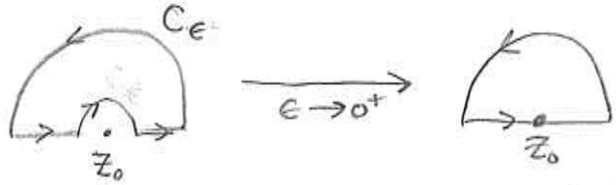
← branch-cut of $z^{1/2}$ used in this problem.

However, we could think of this like

little half-circle radius ϵ

Can apply Cauchy - Goursat to (I) and (II) and the cross-cuts cancel. As $\epsilon \to 0^+$ we obtain C.

Question: can we just argue $\epsilon \to 0^+$ and the ordinary Cauchy - Goursat $\Rightarrow \int_C f(z)\,dz = 0$? What's the danger in this? Consider



$$\int_{C_\epsilon} \frac{dz}{z - z_0} = 0 \quad \text{BUT (!)} \quad \int_C \frac{dz}{z - z_0} = \pi i$$

Apparently, the non-analyticity of $\sqrt{z}$ and $\frac{1}{z}$ are rather different.

$\boxed{\text{PROBLEM 46}}$ #1 of §40

Let C be ccw-oriented square formed by $x = \pm 2$, $y = \pm 2$ that is $C = \partial([-2,2]\times[-2,2])$. Consider: (each by Cauchy's Integral-formula)

a.) $$\int_C \frac{e^{-z}\,dz}{z - \pi i/2} = 2\pi i\, e^{-\frac{\pi i}{2}} = 2\pi i(-i) = \boxed{2\pi}$$

Cauchy's-∫-formula since $z_0 = \frac{\pi i}{2}$ is interior to C.

b.) $$\int_C \underbrace{\left(\frac{\cos z}{z^2+8}\right)}_{f(z)} \frac{dz}{z} = 2\pi i f(0) = \frac{2\pi i \cos(0)}{0^2+8} = \boxed{\frac{\pi i}{4}}$$

Note f analytic inside and on the square C since $2\sqrt{2} > 2$.

c.) $$\int_C \frac{z\,dz}{2z+1} = \int_C \frac{z}{2}\underbrace{\left(\frac{dz}{z+1/2}\right)}_{z_0 = -1/2} = 2\pi i \left.\frac{z}{2}\right|_{z=-\frac{1}{2}} = 2\pi i\left(\frac{-1}{4}\right) = \boxed{\frac{-\pi i}{2}}$$

analytic inside and on C.

Cauchy's Gen. ∫-formula.

d.) $$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = \int_C \frac{f(z)\,dz}{(z-x_0)^2} = 2\pi i f'(x_0) = \boxed{i\pi \sec^2(x_0/2)}$$

$-2 < x_0 < 2$

(Note $\tan(z/2) = \frac{\sin z/2}{\cos z/2}$ has singularities where $\cos(z/2) = 0$ which means $z/2 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \Rightarrow z = \pm\pi, \pm 3\pi, \ldots$ fortunately those fall outside C.)

Cauchy's Gen. ∫-formula

e.) $$\int_C \frac{\cosh z\,dz}{z^4} = (2\pi i)(3!)\sinh(0) = \boxed{0.}$$

$f(z) = \cosh z$ is entire!

$f'(z) = \sinh z$

$f''(z) = \cosh z$

$f'''(z) = \sinh z$

Problem 47 #9 of 540

Let $C: z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$ and show for any $a \in \mathbb{R}$,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then derive the real integral $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$ as a consegence of the above contour-integral.

Observe, $f(z) = e^{az}$ has $f'(z) = ae^{az}$ $\forall z \in \mathbb{C}$ hence f is entire. Clearly f is analytic on and inside C hence Cauchy's $\int$-fla applies:

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i f(0) = 2\pi i.$$

Now, I'll calculate $\int_C \frac{1}{z}e^{az}\,dz$ explicitly from def$^{\underline{n}}$ of contour integral to find: letting $g(z) = e^{az}/z$,

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} g(e^{i\theta})\,\frac{de^{i\theta}}{d\theta}\,d\theta$$

$$= \int_{-\pi}^{\pi} \frac{\exp(ae^{i\theta})}{\cancel{e^{i\theta}}}\,i\cancel{e^{i\theta}}\,d\theta$$

$$= \int_{-\pi}^{\pi} i\exp(a\cos\theta + ia\sin\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi} ie^{a\cos\theta}\big(\cos(a\sin\theta) + i\sin(a\sin\theta)\big)\,d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta$$

But, we have $\int_C \frac{e^{az}}{z}\,dz = 2\pi i$ from Cauchy's Th$^{\underline{m}}$ hence

$$-\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta = 0$$

obvious w/o $\mathbb{C}$-analysis. Follows merely due to oddness of integrand

$$\underbrace{\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\cos\theta)\,d\theta}_{\text{even fnct. of }\theta} = 2\pi i$$

$$\Rightarrow \boxed{\int_0^{\pi} e^{a\cos\theta}\cos(a\cos\theta)\,d\theta = \pi i}$$

$\boxed{\text{PROBLEM 48}}$ Show $\left| \int_a^b [u(t) + i\,v(t)]\,dt \right| \leq \int_a^b |u(t) + i v(t)|\,dt$.

$$\int_a^b (u+iv)\,dt = \int_a^b u\,dt + i\int v\,dt \quad \text{by def}^{\text{n}} \text{ of } \mathbb{R} \longrightarrow \mathbb{C} \text{ integral.}$$

Apply Δ-inequality,

$$\left| \int_a^b (u+iv)\,dt \right| \leq \left| \int_a^b u\,dt \right| + \left| i \int_a^b v\,dt \right| = \left| \int_a^b u\,dt \right| + \left| \int_a^b v\,dt \right|$$

$$\leq \int_a^b |u|\,dt + \int_a^b |v|\,dt$$

$$= \underbrace{\int_a^b (|u| + |v|)\,dt}$$

This is actually all I need for the step on 86 where I invoke the Lemma.

I should have claimed

$$\left| \int_a^b (u+iv)\,dt \right| \leq \int_a^b |u|\,dt + \int_a^b |v|\,dt$$

Or better yet:

$$\left| \int_a^b \sum_{j=1}^n f_j\,dt \right| \leq \sum_{j=1}^n \int_a^b |f_j|\,dt$$

This is what was needed to complete the proof on 86 for the $M\ell(c)$-Theorem.

$\boxed{\text{PROBLEM 49}}$ Given the set-up on pg. 95 of my notes, define

$$F(z) = \int_{z_*}^{z} f(\zeta)\, d\zeta$$

Consider z, z_o in the ☆-shaped region where $z \in D(\delta, z_o)$ and hence the line-segment from $z \to z_o$ lies inside the ☆-shaped region. Moreover it follows $\Delta(z_*, z_o, z)$ is interior the ☆-shaped region and we apply the Cauchy-Goursat for Δ-Th$^{\underline{m}}$ (Pringsheim)

$$0 = \int_{\Delta} f(\zeta)d\zeta = \underbrace{\int_{z_*}^{z} f(\zeta)d\zeta}_{F(z)} + \int_{z}^{z_o} f(\zeta)d\zeta + \underbrace{\int_{z_o}^{z_*} f(\zeta)d\zeta}_{-F(z_o)}$$

Consider then,

$$\left| \frac{F(z_o) - F(z)}{z_o - z} - f(z) \right| = \left| \frac{1}{z_o - z}\int_{z}^{z_o} f(\zeta)d\zeta - f(z)\overbrace{\frac{1}{z_o - z}\int_{z}^{z_o} d\zeta}^{1} \right|$$

$$= \left| \frac{1}{z_o - z}\int_{z}^{z_o} \big(f(\zeta) - f(z)\big)d\zeta \right|$$

$$\leq \frac{1}{|z_o - z|} \, |f(\zeta) - f(z)| \, |z_o - z|$$

$$\leq |f(\zeta) - f(z)|$$

By continuity of f we can choose z, z_o sufficiently close to make $|f(\zeta) - f(z)| < \epsilon$ since $\zeta, z \in D(\delta, z_o)$. It follows

$$\lim_{z_o \to z} \left(\frac{F(z_o) - F(z)}{z_o - z} \right) = f(z) \Rightarrow F'(z) = f(z).$$

<u>Geometric Leap</u>: any point in star-shaped region plays role of z_o for some z hence to show $F'(z_o)$ exists also works. (perhaps you mimicked my notes on 92 that's fine also.)

PROBLEM 50

Let $P(z) \in \mathbb{C}[z]$ with $P'(z) \neq 0$ then $\exists\, c_0, c_1, \dots, c_n$ s.t.

$$P(z) = c_0 + c_1 z + \cdots + c_n z^n = \sum_{j=0}^{n} c_j z^j.$$

$$P'(z) = c_1 + 2c_2 z + \cdots + n c_n z^{n-1} = \sum_{j=1}^{n} j c_j z^{j-1}$$

We wish to derive a nice identity for $\frac{P'(z)}{P(z)}$.

Suppose $n=1$, then $P(z) = c_0 + c_1 z$ and $P'(z) = c_1$ thus

$$\frac{P'(z)}{P(z)} = \frac{c_1}{c_0 + c_1 z} = \frac{1}{c_0/c_1 + z}$$

Let us assume $P(z)$ is <u>monic</u> in what follows. This means we assume $c_n = 1$. In the $n=1$ case,

$$\frac{P'(z)}{P(z)} = \frac{1}{c_0 + z}$$

Suppose $n=2$, $P(z) = c_0 + c_1 z + z^2 \Rightarrow P'(z) = c_1 + 2z$

$$\frac{P'(z)}{P(z)} = \frac{c_1 + 2z}{c_0 + c_1 z + z^2}$$

$$= \frac{c_1 + 2z}{\left(z + \frac{c_1}{2}\right)^2 + c_0 - \frac{c_1^2}{4}} = \frac{A}{z + \frac{c_1}{2} + \sqrt{c_0 - \frac{c_1^2}{4}}} + \frac{B}{z + \frac{c_1}{2} - \sqrt{c_0 - c_1^2/4}}$$

$$c_1 + 2z = A\left(z + \frac{c_1}{2} - \sqrt{c_0 - c_1^2/4}\right) + B\left(z + \frac{c_1}{2} + \sqrt{c_0 - c_1^2/4}\right)$$

Hence, $A + B = 2$ whereas $\underbrace{(A+B)\frac{c_1}{2} + (B-A)\sqrt{c_0 - c_1^2/4} = c_1}$

$$c_1 + (B-A)\sqrt{c_0 - c_1^2/4} = c_1$$

$$\Rightarrow \underline{B = A}. \Rightarrow \underline{A = B = 1}.$$

$$\frac{P'(z)}{P(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2}$$

where $P(r_1) = P(r_2) = 0$.

PROBLEM 50 continued

Note, $n=1$ gives $\dfrac{1}{c_0+z} = \dfrac{1}{z-(-c_0)}$ and for $P(z) = z + c_0$ we have $P(-c_0) = 0$ thus the pattern for $n = 1, 2$ is the same!

$$\boxed{\begin{array}{l}\text{Claim: If } P(z) \text{ is monic and of order } n \\ \text{with zeros } r_1, r_2, \dots, r_n \in \mathbb{C} \text{ then,} \\ \qquad \dfrac{P'(z)}{P(z)} = \dfrac{1}{z-r_1} + \dfrac{1}{z-r_2} + \dots + \dfrac{1}{z-r_n} \\ \text{for all } n \in \mathbb{N} \quad \text{((why is assumption of monic not an undue simplification here?)}\end{array}}$$

<u>Proof</u>: we've shown $n=1$ true already. Assume the Claim true for n, consider $n+1$. Let $P(z) = z^{n+1} + c_n z^n + \dots + c_1 z + c_0$. Consequent to the Fundamental th^{m} of algebra & factor therem $P(r_1) = P(r_2) = \dots = P(r_n) = 0$ yields,

$$P(z) = \prod_{j=1}^{n} (z - r_j)$$

$$P'(z) = \underbrace{\sum_{j=1}^{n} \prod_{k \neq j}^{n} (z - r_j)}_{\substack{n\text{-fold} \\ \text{product rule.}}} = (z-r_2)(z-r_3)\cdots(z-r_n) + (z-r_1)(z-r_3)\cdots(z-r_n) + \dots + (z-r_2)\cdots(z-r_n)$$

Observe that $P'(r_k) = \prod_{j \neq k} (-r_j) = (-r_1)(-r_2)\cdots\underset{\substack{\uparrow \\ \text{deleted.}}}{\widehat{(-r_k)}}\cdots(-r_n)$

(this may or may not be useful
we'll see soon)

PROBLEM 50 continued

$$\frac{P'(z)}{P(z)} = \frac{\sum_{j=1}^{n} \prod_{k\neq j}^{n} (z-r_j)}{(z-r_1)(z-r_2)\cdots(z-r_n)}$$

$$= \frac{1}{z-r_1} + \frac{1}{z-r_2} + \cdots + \frac{1}{z-r_n}.$$

Well, guess I didn't need induction explicitly here. It's just the n-fold product rule paired with FTA. //

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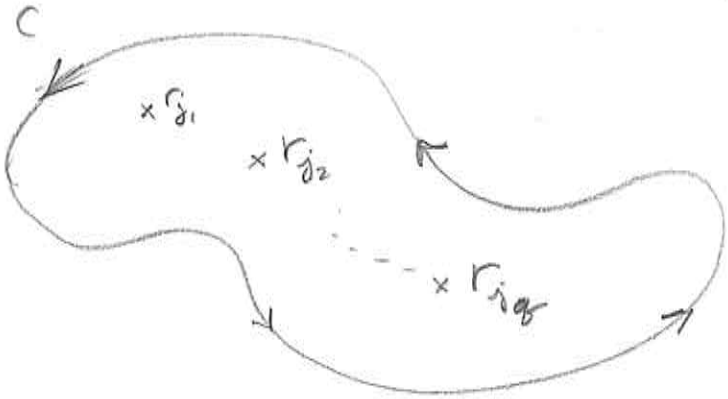
Assume $P(z)$ is monic, non constant, polynomial and let C be a simple closed contour. Let $r_1, r_2, \ldots, r_n$ be the zeros of P. Let $r_{j_1}, r_{j_2}, \ldots, r_{j_p}$ fall inside C



$$\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-r_1} + \frac{1}{z-r_2} + \cdots + \frac{1}{z-r_n}\right)dz$$

$$= \frac{1}{2\pi i}\int_C \left(\sum_{m=1}^{q} \frac{1}{z-r_{j_m}}\right)dz$$ since the other parts integrate to zero by Cauchy's Thm.

$$= \frac{1}{2\pi i}\left(\sum_{m=1}^{q}(2\pi i)\right)$$

$$= q.$$

(it counts the # of zeros of $P(z)$ inside C)

Remark: If $Q(z)$ nonmonic then same result follows easily.