Topics:revised Problem 53 to make it more interesting, the other problems concern computations involving Taylor and Laurent series. I should ask you some deeper questions about the analysis we have discussed in the past few meetings, but I have not found the right problems yet. So, in the mean time, I'll let you chew on these.

Problem 53 If $f(z)$ is complex-differentiable at z_o then $L(z) = f(z_o) + f'(z_o)(z - z_o)$ gives the best affine approximation to $f(z)$ near z_o. Consider two additional points z_1, z_2 such that z_o, z_1, z_2 are distinct. We can construct vectors $\vec{a}_1 = z_1 - z_o$ and $\vec{a}_2 = z_2 - z_o$ and visualize them as vectors based at z_o. Consider the transformation of the points z_o, z_1, z_2. Define $w_j = f(z_j)$ for $j = 0, 1, 2$. These transformed points define new vectors in the w-plane; $\vec{b}_1 = w_1 - w_o$ and $\vec{b}_2 = w_2 - w_o$. **Show that the angle between $\vec{a}_1, \vec{a}_2$ is the same as the angle between $\vec{b}_1, \vec{b}_2$.** It's useful to note the identity below (since we use dot-products to find angles...):

$$X \cdot Y = X_1Y_1 + X_2Y_2 = Re((X_1 + iX_2)(Y_1 - iY_2)) = Re(X\overline{Y}).$$

Problem 58 problem 7 of section 45 of Churchill (page 149).

Problem 59 problem 2 of section 47 of Churchill (page 156).

Problem 60 problem 4 of section 47 of Churchill (page 157).

Problem 61 problem 8 of section 47 of Churchill (page 157).

Problem 62 problem 10 of section 51 of Churchill (page 174).

Problem 63 problem 11 of section 51 of Churchill (page 174).

Problem 64 problem 12 of section 51 of Churchill (page 174).

Problem 65 problem 15 of section 51 of Churchill (page 175).

Problem 66 problem 16 of section 51 of Churchill (page 175).

PROBLEM SET 6

$\boxed{\text{PROBLEM 58}}$ #7 of §45 of Churchill. Expand $f(z) = \sinh(z)$ about $z_0 = \pi i$

$$\begin{aligned}
\sinh(z) &= \sinh(z - \pi i + \pi i) \\
&= \frac{1}{2}\left(e^{z-\pi i+\pi i} - e^{-(z-\pi i+\pi i)}\right) \\
&= \frac{1}{2}\left[e^{z-\pi i} e^{\pi i} - e^{-(z-\pi i)} e^{-\pi i}\right] \\
&= \frac{-1}{2} e^{z-\pi i} + \frac{1}{2} e^{-(z-\pi i)} \\
&= -\sinh(z - \pi i) \\
&= -\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(z-\pi i)^{2n+1}
\end{aligned}$$

$\boxed{\text{PROBLEM 59}}$ #2 of §47 of Churchill: derive $\dfrac{e^z}{(z+1)^2} = \dfrac{1}{e}\left[\sum_{n=0}^{\infty}\left(\dfrac{(z+1)^n}{(n+2)!}\right) + \dfrac{1}{z+1} + \dfrac{1}{(z+1)^2}\right]$ for $0 < |z+1| < \infty$

$$e^z = e^{z+1-1} = e^{z+1}e^{-1} = \frac{1}{e}e^{z+1} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^n}{n!}$$

Thus,

$$\begin{aligned}
\frac{e^z}{(z+1)^2} &= \frac{1}{e}\cdot\frac{1}{(z+1)^2}\left[1 + (z+1) + \sum_{n=2}^{\infty}\frac{(z+1)^n}{n!}\right] \\
&= \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty}\frac{(z+1)^{n-2}}{n!}\right] \qquad \begin{array}{l} j = n-2 \\ n = j+2 \end{array} \\
&= \frac{1}{e}\left[\sum_{j=0}^{\infty}\frac{(z+1)^j}{(j+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right]. //
\end{aligned}$$

$\boxed{\text{PROBLEM 60}}$ #4 of §47: Expand $f(z) = \dfrac{1}{z^2(1-z)}$ about $z=0$ in two ways and explain domain of convergence for each.

If $|z|<1$ then $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ by geom. series result.

Thus $f(z) = \dfrac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \boxed{\dfrac{1}{z^2} + \dfrac{1}{z} + \sum_{n=0}^{\infty} z^n}$

for $0 \le |z| < 1$.

If $|z|>1$ then $\dfrac{1}{|z|} < 1$ and so,

$$f(z) = \frac{1}{z^2}\left[\frac{-1}{z(1-1/z)}\right] = \frac{-1}{z^3}\left[\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right] = -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+3}$$

$$\boxed{f(z) = -\sum_{j=3}^{\infty}\frac{1}{z^j} \quad \text{for } |z|>1}$$

$\boxed{\text{PROBLEM 61}}$ #8 of §47, a.) Show $\dfrac{a}{z-a} = \sum_{n=1}^{\infty}\dfrac{a^n}{z^n}$

b.) write $z = e^{i\theta}$ and derive formulas for $\sum a^n \cos n\theta$ & $\sum a^n \sin n\theta$

(a.) Assume $-1 < a < 1$. ∴ Consider, $-1<a<1 \Rightarrow |a|<1$ thus

$$\frac{a}{z-a} = \frac{+1}{z}\,\frac{a}{(1-a/z)} = \frac{1}{z}\sum_{n=0}^{\infty} a\left(\frac{a}{z}\right)^n$$

provided $\left|\frac{a}{z}\right| < 1$

$|<|a|<|z|$

Hence $f(z) = \dfrac{a}{z-a} = \sum_{n=0}^{\infty}\dfrac{a^{n+1}}{z^{n+1}} = \sum_{j=1}^{\infty}\left(\dfrac{a}{z}\right)^j$. //

(b.) $z = e^{i\theta}$ then $z^n = e^{ni\theta}$ hence,

$$\sum_{n=1}^{\infty}\left(\frac{a}{e^{i\theta}}\right)^n = \sum_{n=1}^{\infty}\frac{a^n}{e^{ni\theta}} = \sum_{n=1}^{\infty}\frac{a^n}{\cos(n\theta)+i\sin(n\theta)}\left[\frac{\cos n\theta - i\sin(n\theta)}{\cos n\theta - i\sin(n\theta)}\right]$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[a^n\cos(n\theta) - ia^n\sin(n\theta)\right] = \frac{a}{e^{i\theta}-a} = \frac{a}{\cos\theta + i\sin\theta - a}$$

$$\Rightarrow \sum_{n=1}^{\infty} a^n\cos(n\theta) - \sum_{n=1}^{\infty} a^n\sin(n\theta) = \frac{a(\cos(\theta)-a) - ia\sin(\theta)}{(\cos\theta - a)^2 + \sin^2\theta}$$

But, $(\cos\theta - a)^2 + \sin^2\theta = \cos^2\theta - 2a\cos\theta + a^2 + \sin^2\theta = 1 - 2a\cos\theta + a^2$ hence,

$$\boxed{\sum_{n=1}^{\infty} a^n\cos(n\theta) = \frac{a\cos\theta - a^2}{1-2a\cos\theta + a^2}} \quad \& \quad \boxed{\sum_{n=1}^{\infty} a^n\sin(n\theta) = \frac{a\sin\theta}{1-2a\cos\theta + a^2}}$$

$\boxed{\text{PROBLEM 62}}$ # 10 of § 51 of Churchill
Multiply series to show $\dfrac{e^z}{z(z^2+1)} = \dfrac{1}{z} + 1 - \dfrac{1}{2}z - \dfrac{5}{6}z^2 + \cdots$ for $0<|z|<1$

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n \quad \text{for } |z^2|<1 \Rightarrow |z|<1. \quad (\star)$$

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots$$

Hence, as $e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$ we find,

$$e^z\left[\frac{1}{z(z^2+1)}\right] = e^z\left[\frac{1}{z}\left(1 - z^2 + z^4 - z^6 + \cdots\right)\right]$$

$$= e^z\left[\frac{1}{z} - z + z^3 - z^5 + \cdots\right]$$

$$= \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)\left(\frac{1}{z} - z + z^3 - z^5 + \cdots\right)$$

$$= \frac{1}{z} - z + 1 - z^2 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots$$

$$= \underline{\frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots}$$

holds for $0<|z|<1$ by $(\star)$.

Remark: I kept just enough terms to make sure I obtained all terms of order z^2 or less.

$\boxed{\text{PROBLEM 63}}$ # 11 of § 51 of Churchill: obtain power series for $\csc(z)$ and $\dfrac{1}{e^z-1}$ by division

Try something different.

$$\csc(z) = g(z) = \frac{1}{\sin z} \Rightarrow \underbrace{g(z)\sin(z) = 1}$$

$$\left[\frac{a_{-1}}{z} + \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)\right]\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right) = 1$$

Const: $a_{-1} \cdot 1 = 1 \quad \therefore \quad \underline{a_{-1} = 1}$.

z: $a_0 = 0 \quad \therefore \quad \underline{a_0 = 0}$.

z^2: $a_{-1}\left(\frac{-1}{3!}\right) + a_1(1) = 0 \Rightarrow a_1 = \frac{1}{6}a_{-1} = \underline{\frac{1}{3!}}$

z^3: $a_2 \cdot 1 = 0 \quad \therefore \underline{a_2 = 0}$.

z^4: $\frac{a_{-1}}{5!} - \frac{a_1}{3!} + a_3 \cdot 1 = 0 \quad \therefore \quad a_3 = \underline{\frac{1}{(3!)^2} - \frac{1}{5!}}$.

Remark: need to use Laurent Series to find fractions of power series!

Therefore, $\boxed{\csc(z) = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots}$

PROBLEM 63 continued

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12} z - \frac{1}{720} z^3 + \cdots \quad \text{for } 0 < |z| < 2\pi$$

Note $e^z - 1 = 1 + z + \frac{1}{2}z^2 + \cdots - 1 = z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots$

$$\begin{array}{r|l}
 & \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \\
z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots \; (+\frac{1}{24}z^4 +) \; (+\frac{1}{120}z^5) & 1 \\
 & 1 + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \frac{1}{120}z^4 \\
 & -\frac{1}{2}z - \frac{1}{6}z^2 - \frac{1}{24}z^3 - \frac{1}{120}z^4 \\
 & -\frac{1}{2}z - \frac{1}{4}z^2 - \frac{1}{12}z^3 - \frac{1}{2}\left(\frac{1}{24}\right)z^4 \\
 & \frac{1}{12}z^2 + \frac{1}{24}z^3 + \frac{1}{80}z^4 \\
 & \frac{1}{12}z^2 + \frac{1}{24}z^3 + \frac{1}{72}z^4 \\
 & -\frac{1}{720}z^4
\end{array}$$

Thus $\boxed{\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots}$

- I suppose long-division is faster if you anticipate how many terms you need from the outset. You can see how I didn't realize I needed z^4 & z^3 until mid-computation.

PROBLEM 64 #12 of §51 of Churchill

Given $\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z + \cdots$ for $0 < |z| < \pi$

Calculate $\int_C \frac{dz}{z^2 \sinh(z)} = \frac{-\pi i}{3}$ (hints about Ex 1. of §47)

Ex. 1 of §47 essentially says to look at the residue of the pole inside the contour.

$$\int_C \frac{dz}{z^2 \sinh(z)} = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2 \sinh(z)}\right)$$

$$= 2\pi i \left[\frac{-1}{6}\right]$$

$$= \boxed{\frac{-\pi i}{3}}$$

Remark: technically this problem predates Cauchy's Res. Th^m, but that development is more notation on top of concept in Ex. 1 of §47 so this is fair use.

PROBLEM 65 #15 of §51 of Churchill

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be entire.

(a.) Let $g(z) = f(f(z))$ and find MacClaurin series for $g(z)$ by Taylor's Th$^{\underline{m}}$.

(b.) obtain f-la for $g(z)$ by formally manipulating

$$g(z) = f(z) + a_2 (f(z))^2 +$$

(c.) use part. a. to find $\sin(\sin(z)) = z - \frac{1}{3} z^3 + \cdots$

(a.) $g(0) = f(f(0)) = f(0) = 0.$

$g'(z) = f'(f(z))f'(z) \Rightarrow g'(0) = f'(f(0))f'(0) = [f'(0)]^2 = 1$

I used $f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots$

$\underline{f'(0) = 1}$.

$f''(z) = 2a_2 + 6a_3 z + \cdots$

$\underline{f''(0) = 2a_2}$.

$f'''(z) = 6a_3 + \cdots$

$\underline{f'''(0) = 6a_3}$.

$$g''(z) = f''(f(z))[f'(z)]^2 + f'(f(z))f''(z)$$

$$g''(0) = f''(f(0))[f'(0)]^2 + f'(f(0))f''(0) = f''(0) + f'(0)\cdot 2a_2 = \underline{4a_2}.$$

$$g'''(z) = f'''(f(z))[f'(z)]^3 + f''(f(z))\,2f'(z)f''(z) + f''(f(z))f'(z)f''(z) + f'(f(z))f'''(z)$$

$$g'''(0) = f'''(0) + f''(0)\cdot 2\cdot f''(0) + f''(0)f''(0) + f'(0)f'''(0)$$

$$g'''(0) = 6a_3 + 2(2a_2)^2 + (2a_2)^2 + 6a_3$$

$$\underline{g'''(0) = 12a_3 + 12a_2^2}.$$

(a.) $g(z) = g(0) + g'(0)z + \frac{1}{2}g''(0)z^2 + \frac{1}{6}g'''(0)z^3 + \cdots$

$$\Rightarrow \boxed{g(z) = z + 2a_2 z^2 + 2(a_3 + a_2^2)z^3 + \cdots}$$

(b.) $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_2(z + a_2 z^2 + a_3 z^3 + \cdots)^2 + \cdots$

$= z + a_2 z^2 + a_3 z^3 + \cdots + a_2(z^2 + 2a_2 z^3 + \cdots) + \underbrace{a_3(z + a_2 z^2 \cdots)^3}_{\text{almost forgot! need it!}}$

$= z + 2a_2 z^2 + (a_3 + 2a_2^2)z^3 + a_3 z^3 + \cdots$

$= \boxed{z + 2a_2 z^2 + 2(a_3 + a_2^2)z^3 + \cdots}$

(c.) left to reader & grader (observe $a_2 = 0$, $a_3 = \frac{-1}{3!} = \frac{-1}{6}$ and win.)

PROBLEM 66 | #16 of §51: The Euler #'s E_n ($n=0,1,2,\ldots$) are the #'s in the Maclaurin series rep. $\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$ for $|z| < \pi/2$. Explain why this rep. is valid in said disk and why $E_{2n+1} = 0$ for $n = 0,1,2,\ldots$. Then show

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61$$

Observe $\cosh(z) = \frac{1}{2}(e^z + e^{-z}) = 0 \Rightarrow e^z = -e^{-z}$ — complex calculation

$$\Rightarrow e^{x+iy} = -e^{-x-iy}$$

$$\Rightarrow e^{2x+iy} = -e^{-iy}$$

$$\Rightarrow e^{2x+2iy} = -1 = e^{\pi i} \qquad e^{2z} = -1$$

$\cosh(z) = 0$ if $e^{2z} = -1 \Rightarrow e^{2z - \pi i} = 1$

poles for $\frac{1}{\cosh z}$ at $2z - \pi i = 2\pi i k$ for $k = 0, \pm 1, \pm 2, \ldots$

$k=0$ | $2z = \pi i \longrightarrow z = \pi i/2$

$k=1$ | $2z = 3\pi i \rightarrow z = 3\pi i/2$

$k=-1$ | $2z = \pi i - 2\pi i = -\pi i \Rightarrow z = -\pi i/2$

Clearly $z = \pm \pi i/2$ give poles for $\frac{1}{\cosh z}$ hence the power series will converge up to the open disk $|z| < \pi/2$.

QUESTION: why the whole disk? How do we know the series expansion at $z=0$ extends that far?

(can you justify this with the theory we discussed? How?)

Next, observe $f(z) = \frac{1}{\cosh(z)}$ is an even function as $f(-z) = f(z)$

But, $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (-z)^n = f(-z)$

$$\Rightarrow a_n = (-1)^n a_n \quad \text{hence} \quad \begin{array}{ll} a_{2k} = a_{2k} & (n \text{ even}) \\ a_{2k+1} = -a_{2k+1} & (n \text{ odd}) \end{array}$$

Consequently, a_{2k} free, but $a_{2k+1} = 0. \Rightarrow \boxed{E_{2k+1} = 0}$

$\left(a_n = \frac{E_n}{n!}\right) (n! \neq 0)$

(this remark is general, odd fnct $\Rightarrow$ odd series at $z=0$
even fnct $\Rightarrow$ even series at $z=0$)

PROBLEM 66: $\cosh(z) = 1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots$

$$
\begin{array}{r l}
 & 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \\
\cosh(z) \,\big) & 1 \\
 & \underline{1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots} \\
 & -\frac{1}{2}z^2 - \frac{1}{24}z^4 - \frac{1}{720}z^6 + \cdots \\
 & \underline{-\frac{1}{2}z^2 - \frac{1}{4}z^4 - \frac{1}{48}z^6 + \cdots} \\
 & \frac{5}{24}z^4 + \frac{7}{360}z^6 + \cdots \\
 & \underline{\frac{5}{24}z^4 + \frac{5}{48}z^6 + \cdots} \\
 & -\frac{61}{720}z^6 + \cdots
\end{array}
$$

Thus $\dfrac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots$

However, $\dfrac{1}{\cosh z} = E_0 + \frac{1}{2}E_2 z^2 + \frac{1}{24}E_4 z^4 + \frac{1}{720}E_6 + \cdots$

Comparing we derive,

$$\boxed{E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61}$$