Topics: residue calculus, the usual suspects.

Problem 67 problem 4 of section 60 of Churchill (page 208).(rational function integral to ∞)

Problem 68 problem 6 of section 61 of Churchill (page 215). (infinite trig. integral)

Problem 69 problem 13 of section 61 of Churchill (page 216-217).(rectangular contour)

Problem 70 problem 4 of section 62 of Churchill (page 219).(trig. integral, non-infinite)

Problem 71 problem 1 of section 64 of Churchill (page 226).(indented contour)

Problem 72 problem 5 of section 64 of Churchill (page 226).(formal branch cut computation)

Problem 73 problem 1 of page 207 Gamelin (handout). (keyhole contour)

Problem 74 problem 8 of page 208 Gamelin (handout). (pie-wedge contour)

Problem 75 problem 2 of page 222 Gamelin (handout). (dog-bone contour)

PROBLEM SET 7 Solution

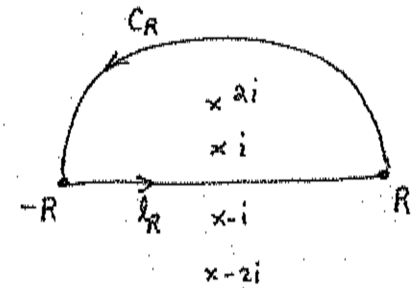
$\boxed{\text{PROBLEM 67}}$ #4 of §60 : $\displaystyle\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$ (why?)

Observe the integrand is even $\Rightarrow$ $\displaystyle P.V.\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)} = 2\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)}$

$\displaystyle f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ has simple poles at $z=\pm i$ and $z=\pm 2i$

Consider the usual half-circle contour, $R>2$,

$C_R \cup l_R = C$



By Cauchy's Residue Th^m; $\displaystyle\oint_C f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=i}(f(z)) + \operatorname*{Res}_{z=2i}(f(z))\right]$

$$\operatorname*{Res}_{z=i}(f(z)) = \frac{z^2}{(z+i)(z^2+4)}\bigg|_{z=i} = \frac{i^2}{(2i)(-1+4)} = \frac{-1}{6i} = \frac{i}{6}.$$

$$\operatorname*{Res}_{z=2i}(f(z)) = \frac{z^2}{(z^2+1)(z+2i)}\bigg|_{z=2i} = \frac{(2i)^2}{(-4+1)(4i)} = \frac{-4}{(-3)(4i)} = \frac{1}{3i} = \frac{-i}{3}.$$

Note $2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = 2\pi i\left(\frac{-i}{6}\right) = \frac{\pi}{3}$ thus,

$$\frac{\pi}{3} = \int_{l_R}\frac{z^2dz}{(z^2+1)(z^2+4)} + \int_{C_R}\frac{z^2dz}{(z^2+1)(z^2+4)}$$

As $R\to\infty$ we find, since $z=x$ on l_R,

$$\frac{\pi}{3} = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2dx}{(x^2+1)(x^2+4)} + \underbrace{\lim_{R\to\infty}\int_{C_R}\frac{z^2dz}{(z^2+1)(z^2+4)}}_{\star}$$

Once we show $\star = 0$ then the result follows since

$$\frac{\pi}{3} = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2dx}{(x^2+1)(x^2+4)} = P.V.\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)} = 2\int_0^\infty\frac{x^2dx}{(x^2+1)(x^2+4)}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}}$$

Observe, if $z\in C_R$ then $|z|=R$ then

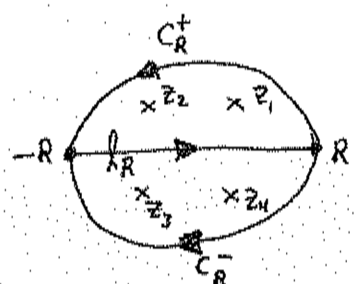
$$|f(z)| = \frac{|z|^2}{|z^2+1||z^2+4|} \leq \frac{R^2}{(R^2-1)(R^2-4)}$$

Hence, $\displaystyle\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R^3}{(R^2-1)(R^2-4)} \longrightarrow 0$ as $R\to\infty$ $\therefore$ $\star$ is true.

PROBLEM 68 #6 of §61 in Churchill; if $a>0$ then $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin(ax)\,dx}{x^4+4} = \pi e^{-a}\cos(a)$

Consider
$$f(z) = \frac{z^3 \sin(az)}{z^4+4} = \underbrace{\frac{z^3 e^{iaz}/2i}{z^4+4}}_{f_+(z)} - \underbrace{\frac{z^3 e^{-iaz}/2i}{z^4+4}}_{f_-(z)}$$

Both $f_\pm(z)$ have simple poles at sol$^{n's}$ to $z^4+4=0$. In particular
$$z^4+4 = (z^2-2i)(z^2+2i)$$
$$z^4+4 = (\underbrace{z-\sqrt{2}e^{i\pi/4}}_{z_1})(\underbrace{z+\sqrt{2}e^{i\pi/4}}_{z_3})(\underbrace{z-\sqrt{2}e^{\frac{3\pi i}{4}}}_{z_2})(\underbrace{z+\sqrt{2}\,e^{\frac{3\pi i}{4}}}_{z_4})$$
Let's use some labels to save writing,



$$z_1 = \sqrt{2}e^{i\pi/4} = 1+i$$
$$z_2 = \sqrt{2}\,e^{3\pi i/4} = -1+i$$
$$z_3 = -\sqrt{2}e^{i\pi/4} = -1-i$$
$$z_4 = -\sqrt{2}\,e^{3\pi i/4} = 1-i$$
$$z^4+4 = (z-1-i)(z+1+i)(z+1-i)(z-1+i)$$

Observe $f_+(z)$ is bounded in upper half-plane, $f_-(z)$ bounded in lower half-plane. We'll need the residues at z_1, z_2, z_3, z_4, let's calculate them,

$$\operatorname*{Res}_{z=z_1}\left(f_+(z)\right) = \frac{1}{2i}\left[\frac{z_1^3 e^{iaz_1}}{(z_1+z_1)(z_1^2+2i)}\right] = \frac{1}{2i}\left[\frac{\cancel{(\sqrt{2})^3}\exp(ia(1+i))(1+i)^3}{2(1+i)[(1+i)^2+2i]}\right]$$
$$= \frac{1}{2i}\left[\frac{(1+i)^2\exp(ia-a)}{2[1+2i-1+2i]}\right]$$
$$= \frac{1}{2i}\left[\frac{(1+i)^2\exp(ia-a)}{2(4i)}\right]$$
$$= \frac{(1+i)^2\exp(ia-a)}{-16} = R_+^1$$

$$\operatorname*{Res}_{z=z_2}\left(f_+(z)\right) = \frac{1}{2i}\left[\frac{(i-1)^3\exp(ia(i-1))}{+2(-1+i)[(-1+i)^2-2i]}\right] \qquad (i-1)^2-2i = -1-2i+1-2i = -4i$$
$$= \frac{1}{2i}\left[\frac{(i-1)^2\exp(-a-ia)}{2(-4i)}\right]$$
$$= \frac{(i-1)^2\exp(-ia-a)}{+16} = R_+^2$$

PROBLEM 68 continued

$$\operatorname*{Res}_{z=z_3}\left(f_-(z)\right) = \frac{1}{2i}\left[\frac{\left(-(1+i)\right)^3 \exp(-ia(-1-i))}{-2(1+i)\left[\left(-(1+i)\right)^2 + 2i\right]}\right]$$

$(1+i)^2+2i = 1+2i-1+2i$
$= 4i$

$$= \frac{1}{2i}\left[\frac{(1+i)^2 \exp(ia-a)}{2\,(4i)}\right]$$

$$= \frac{(1+i)^2 \exp(ia-a)}{-16} = R_-^3$$

$$\operatorname*{Res}_{z=z_4}\left(f_-(z)\right) = \frac{1}{2i}\left[\frac{(1-i)^3 \exp(-ia(1-i))}{2(1-i)\left[(1-i)^2 - 2i\right]}\right]$$

$(1-i)^2 - 2i = 1-2i-1-2i$
$= -4i$

$$= \frac{1}{2i}\left[\frac{(1-i)^2 \exp(-ia-a)}{2\,(-4i)}\right]$$

$$= \frac{(1-i)^2 \exp(-ia-a)}{16} = R_-^4$$

Using the labels $R_+^1, R_+^2, R_-^3, R_-^4$ and applying Cauchy's Residue Th$^{\underline{m}}$ to $f_+(z)$ on $l_R \cup C_R^+$ and $f_-(z)$ on $l_R \cup C_R^-$ we obtain (for $R > \sqrt{2}$)

$$2\pi i\left(R_+^1 + R_+^2\right) = \int_{-R}^{R} \frac{x^3 e^{iax}/2i}{x^4+4}\,dx + \int_{C_R^+} \frac{z^3 e^{iaz}/2i}{z^4+4}\,dz$$

$$-2\pi i\left(R_-^3 + R_-^4\right) = \int_{-R}^{R} \frac{x^3 e^{-iax}/2i}{x^4+4}\,dx + \int_{C_R^-} \frac{z^3 e^{-iaz}/2i}{z^4+4}\,dz$$

As $R \longrightarrow \infty$ the integrals over $C_R^{\pm}$ vanish and we find $\left(\sin(ax) = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)\right)$,

$$P.V.\int_{-\infty}^{\infty} \frac{x^3 \sin(ax)\,dx}{x^4+4} = 2\pi i\left(R_+^1 + R_+^2 + R_-^3 + R_-^4\right)$$

$$= \frac{2\pi i}{16}\left(-(1+i)^2 e^{ia} + (i-1)^2 e^{-ia} - (1+i)^2 e^{ia} + (1-i)^2 e^{-ia}\right) e^{-a}$$

$$= \frac{\pi i}{8} e^{-a}\left(\left[-(1+2i-1) - (1+2i-1)\right] e^{ia} + \left[(i-1)^2 + (1-i)^2\right] e^{-ia}\right)$$

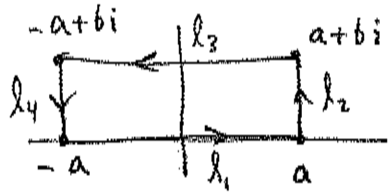
$$= \frac{\pi i e^{-a}}{8}\left(-4i e^{ia} - 4i e^{-ia}\right)$$

$$= \pi e^{-a}\,\frac{1}{2}\left(e^{ia} + e^{-ia}\right)$$

$$= \boxed{\pi e^{-a} \cos(a)}$$

Remark: this is a problem.

PROBLEM 69 | #13 of §61, $\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ for $b>0$



a.) show horizontal legs give $\int e^{-z^2}dz$ of

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)\,dx$$

and vertical legs,

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{i2ay}e^{y^2}dy$$

thus,

$$\int_0^a e^{-x^2}\cos(2bx)\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy$$

(b.) use $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$ to obtain claimed formula.

(a.) l_1 : $z = x$ for $-a \le x \le a$ and $dz = dx$.

l_2 : $z = a + yi$ for $0 \le y \le b$ and $dz = i\,dy$

$-l_3$: $z = x + ib$ for $-a \le x \le a$ and $dz = dx$

$-l_4$: $z = -a + yi$ for $0 \le y \le b$ and $dz = i\,dy$

Let $f(z) = e^{-z^2}$ hence,

$$\text{Horizontal} = \int_{l_1} f(z)dz + \int_{l_3} f(z)dz = \int_{l_1} f(z)dz - \int_{-l_3} f(z)dz = \int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{-(x+ib)^2}dx$$

$-(x^2+2ixb-b^2)$

$$\text{Horizontal Legs} = \int_{-a}^{a} e^{-x^2}dx - e^{b^2}\int_{-a}^{a} e^{-x^2}\big(\cos(2bx) + \underline{i\sin(2bx)}\big)dx$$

gives odd fnct which integrates to zero.

$$= \int_{-a}^{a} e^{-x^2}dx - e^{b^2}\int_{-a}^{a} e^{-x^2}\cos(2bx)\,dx$$

$$= \underline{2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)\,dx}\ . \quad \text{(I)}$$

$$\text{Vertical Legs} = \int_{l_2} f(z)dz - \int_{-l_4} f(z)dz$$

$$= \int_0^b \exp(-(a+yi)^2)\,i\,dy - \int_0^b \exp(-(-a+yi)^2)\,i\,dy$$

$$= i\int_0^b \exp(-a^2 - 2ayi + y^2)dy - i\int_0^b \exp(-a^2 + 2ayi + y^2)\,dy$$

$$= ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy$$

$$= \underline{2e^{-a^2}\int_0^b e^{y^2}\sin(2ay)\,dy} \quad \text{(II)}$$

$\left(\text{as } \sin(2ay) = \frac{1}{2i}\left(e^{2iay} - e^{-2iay}\right) = -2i\left(e^{2iay} - e^{-2iay}\right) \right)$

PROBLEM 69 Continued:

Observe $f(z) = e^{-z^2}$ is entire, hence analytic in and on $l_1 \cup l_2 \cup l_3 \cup l_4$

$$\underbrace{\int_{l_1 \cup l_2 \cup l_3 \cup l_4} f(z)\,dz = 0}_{\text{Cauchy Goursat}} = \underbrace{2\int_0^a e^{-x^2}dx}_{\sqrt{\pi}} - 2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)\,dx + 2e^{-a^2}\int_0^b e^{y^2}\sin(2ay)\,dy$$

$$\lim_{a\to\infty}\int_0^a e^{-x^2}\cos(2bx)\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-b^2} + \underbrace{\lim_{a\to\infty}\left(e^{-a^2-b^2}\int_0^b e^{y^2}\sin(2ay)\,dy\right)}_{\text{vanishes due to } e^{-a^2}}$$

$$\boxed{\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}}$$

PROBLEM 70 #4 of §62: $\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ for $-1<a<1$ (show it)

Let $z = e^{i\theta}$ thus $dz = ie^{i\theta}d\theta = iz\,d\theta \hookrightarrow d\theta = dz/iz$

and $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right)$ thus identity for unit circle C

Note: $|a|<1 \Rightarrow a^2<1$
$\Rightarrow \frac{1}{a^2}>1 \Rightarrow \frac{1}{a^2}-1>0$

$$\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta} = \int_C \frac{dz}{iz\Big[\underbrace{\frac{a}{2}\left(z+\frac{1}{z}\right)}_{a\cos\theta}+1\Big]}$$

$$= \int_C \frac{dz}{i\left(z+\frac{a}{2}z^2+\frac{a}{2}\right)}$$

$$= \int_C \frac{2\,dz}{ia\left(z^2+\frac{2z}{a}+1\right)}$$

$$= \frac{2}{ia}\int_C \frac{dz}{z^2+\frac{2z}{a}+1}$$

$$= \frac{2}{ia}\int_C \frac{dz}{\left(z+\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}\right)\left(z+\frac{1}{a}+\sqrt{\frac{1}{a^2}-1}\right)}$$

$$= \frac{4\pi i}{ia}\left(\frac{1}{z+\frac{1}{a}+\sqrt{\frac{1}{a^2}-1}}\right)\Bigg|_{z=-\frac{1}{a}+\sqrt{\frac{1}{a^2}-1}}$$

$$= \frac{4\pi}{a}\left(\frac{1}{\cancel{\frac{-1}{a}}+\sqrt{\frac{1}{a^2}-1}+\cancel{\frac{1}{a}}+\sqrt{\frac{1}{a^2}-1}}\right)$$

$$= \frac{2\pi}{a\sqrt{\frac{1}{a^2}-1}} = \boxed{\frac{2\pi}{\sqrt{1-a^2}}}\ (a>0) \quad \text{OR} \quad \boxed{\frac{2\pi}{-\sqrt{1-a^2}}}\ (a<0)$$

Note:
$$\cancel{z^2+\frac{2z}{a}+1=0}$$
$$\Rightarrow \cancel{z = \frac{-2/a \pm\sqrt{4/a^2-4}}{2}}$$
$$\Rightarrow \cancel{z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2}-1}}$$
principal root.

Observe $\left|\sqrt{\frac{1}{a^2}-1}\right| < \left|\frac{1}{a}\right|$

$$z = \frac{-1\pm\sqrt{1-a^2}}{a}$$

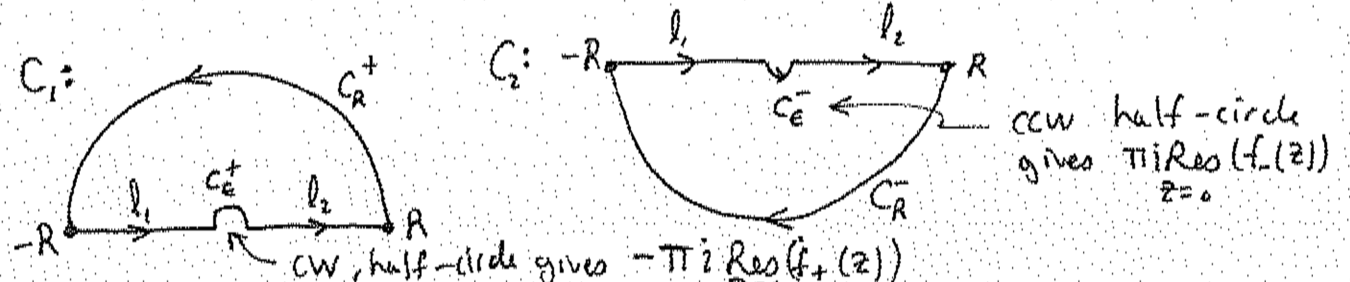
PROBLEM 71 | #1 of §64 of Churchill: for $a \geq 0$, $b \geq 0$,

Derive $\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \pi\left(\frac{b-a}{2}\right)$ then argue $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

Consider $f(z) = \frac{\cos(az) - \cos bz}{z^2} = \underbrace{\frac{1}{2z^2}\left(e^{iaz} - e^{ibz}\right)}_{f_+(z)} + \underbrace{\frac{1}{2z^2}\left(e^{-iaz} - e^{-ibz}\right)}_{f_-(z)}$

$$\operatorname*{Res}_{z=0}\left(f_+(z)\right) = \frac{d}{dz}\left(e^{iaz} - e^{ibz}\right)\Big|_{z=0} = ia - ib = i(a-b).$$

$$\operatorname*{Res}_{z=0}\left(f_-(z)\right) = \frac{d}{dz}\left(e^{-iaz} - e^{-ibz}\right)\Big|_{z=0} = -ia + ib = i(b-a).$$



$$\int_{C_1} f_+(z)\,dz = \int_{\ell_1 \cup \ell_2} \frac{\cancel{\quad}}{x^2}\,dx + \int_{C_\epsilon^+} f_+(z) + \int_{C_R^+} f_+(z)\,dz$$

$\epsilon \to 0^+$, $R \to \infty$

$$0 = P.V.\int_{-\infty}^{\infty} \frac{e^{iax} - e^{ibx}}{x^2}\,dx - \pi i\left(i(a-b)\right) + 0$$ ← Jordan's Lemma.

Likewise, using the half-residue Th^{m} & Jordan's Lemma once again: $\epsilon \to 0^+$, $R \to \infty$

$$0 = \int_{C_2} f_-(z)\,dz = \lim_{R\to\infty}\int_{\ell_1 \cup \ell_2} \frac{e^{-iax} - e^{-ibx}}{x^2}\,dx + \pi i\left(i(b-a)\right) + 0$$

$$0 = P.V.\int_{-\infty}^{\infty} \frac{e^{-iax} - e^{-ibx}}{x^2}\,dx + \pi i\left(i(b-a)\right)$$

$$\Rightarrow P.V.\int_{-\infty}^{\infty} \frac{2\cos ax - 2\cos bx}{x^2}\,dx = \pi i\left(i(a-b)\right) - \pi i\left(i(b-a)\right) = 2\pi i^2(a-b)$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \pi\left(\frac{b-a}{2}\right)}$$

$1 - \cos(2x) = 2\sin^2 x$ consider $a = 0$, $b = 2$

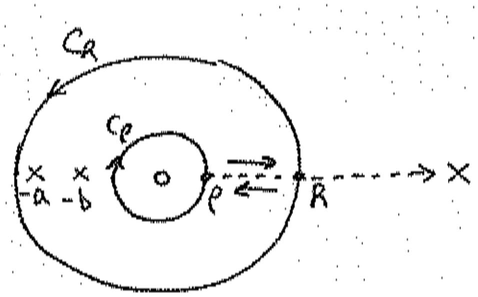
$$\int_0^\infty \frac{1 - \cos 2x}{x^2}\,dx = \pi\left(\frac{2}{2}\right) \Rightarrow \int_0^\infty \frac{2\sin^2 x\,dx}{x^2} = \pi \quad \therefore \boxed{\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}}$$

PROBLEM 72 #5 of §64 of Churchill

Use $f(z) = \dfrac{z^{1/3}}{(z+a)(z+b)} = \dfrac{e^{\frac{1}{3}\log(z)}}{(z+a)(z+b)}$ for $|z| > 0$, $0 < \arg z < 2\pi$

and the closed contour pictured below to show <u>formally</u> that

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\,\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0)$$



We'll examine $\rho \to 0^+$ and $R \to \infty$

<u>upper path</u>: $z = x$, $\arg(z) = 0$

<u>lower path</u>: $z = x$, $\arg(z) = 2\pi$

up: $\log(z) = \ln(x)$, down: $\log(z) = \ln(x) + 2\pi i$

$$\int_{C_R \cup C_\rho \cup \rightleftarrows} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=-a}(f(z)) + \operatorname*{Res}_{z=-b}(f(z))\right)$$

$\arg(z) = \pi$ for $z = -a, -b$.

$$\underbrace{\int_{C_R} f(z)\,dz}_{0} + \underbrace{\int_{C_\rho} f(z)\,dz}_{0} + \int_{\rightarrow} f(z)\,dz + \int_{\leftarrow} f(z)\,dz = 2\pi i\left(\frac{a^{1/3}}{b-a} + \frac{b^{1/3}}{a-b}\right)e^{\pi i/3}$$

As $R \to \infty$ and $\rho \to 0^+$

$$\int_0^\infty \frac{e^{\frac{1}{3}\ln(x)}}{(x+a)(x+b)}\,dx - \int_0^\infty \frac{e^{\frac{1}{3}[\ln(x)+2\pi i]}}{(x+a)(x+b)}\,dx = 2\pi i\left(\frac{a^{1/3}}{b-a} - \frac{b^{1/3}}{b-a}\right)e^{\pi i/3}$$

$$\underbrace{\left(e^{-\pi i/3} - e^{\pi i/3}\right)}\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi i}{b-a}\left(a^{1/3} - b^{1/3}\right)$$

$$\cancel{1 - \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = 1 + \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{3-i\sqrt{3}}{2}}$$

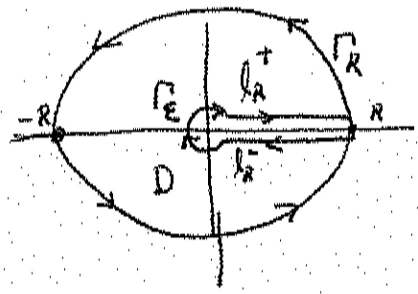
$$\cancel{1 - \frac{1}{2} - i\frac{\sqrt{3}}{2}} \qquad \cancel{\left(\frac{3-i\sqrt{3}}{2}\right)\left(\frac{3+i\sqrt{3}}{2}\right) = \frac{9+3}{4} = \frac{12}{4} = 3}$$

$$\frac{1}{2i}\left(e^{\pi i/3} - e^{-\pi i/3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \text{thus,}$$

$$\boxed{\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\left(\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}\right)}$$

$\boxed{\text{PROBLEM 73}}$ #1 of pg. 207 of Gamelin (keyhole contour problem)

Show that $\int_0^\infty \frac{x^{-a}\,dx}{1+x} = \frac{\pi}{\sin \pi a}$ for $0<a<1$ by integrating around the keyhole contour pictured below



the total curve is ∂D which includes Γ_R (ccw) and Γ_ε (cw) and ℓ_R^+, ℓ_R^-

consider $f(z) = \frac{z^{-a}}{1+z}$ (multiply-valued

use $0 < \arg(z) < 2\pi$

Recall $z^{-a} = \exp(-a\log(z))$ thus for ℓ_R^+ where $z=x$ and $\arg(z)=0$ we have $z^{-a} = \exp(-a\ln|x|) = x^{-a}$ however on $-\ell_R^-$ where $z=x$ but $\arg(z)=2\pi$ hence

$$z^{-a} = \exp(-a[\ln|x| + 2\pi i]) = x^{-a}e^{-2\pi a i} = x^{-a}(\cos 2\pi a - i\sin 2\pi a)$$

On Γ_R: $z = Re^{i\theta} \Rightarrow z^{-a} = \exp(-a\ln R - ia\theta) = R^{-a}e^{-ia\theta}$

thus $|f(z)| = \left|\frac{R^{-a}e^{-ia\theta}}{1+Re^{i\theta}}\right|$ .. ok need Jordan for Γ_R.

$$2\pi i \operatorname*{Res}_{z=-1}\left(\frac{z^{-a}}{1+z}\right) = 2\pi i(-1)^{-a} = 2\pi i\exp(\ln|-1| - a(i\pi)) = \underline{2\pi i e^{-ai\pi}}$$

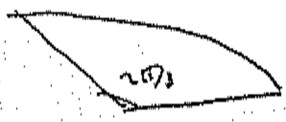
$$\int_0^R \frac{x^{-a}}{1+x}dx - \int_0^R \frac{x^{-a}e^{-2\pi a i}}{1+x}dx = 2\pi i e^{-ai\pi}$$

$$\int_0^R \frac{x^{-a}}{1+x}dx = \frac{2\pi i e^{-ai\pi}}{1-e^{-2\pi a i}} = \frac{2\pi i}{e^{ai\pi}-e^{-ai\pi}} = \underline{\frac{\pi}{\sin(a\pi)}}.$$

- Jordan's Lemma for Γ_R.
- Bounding Th^m for Γ_ε.

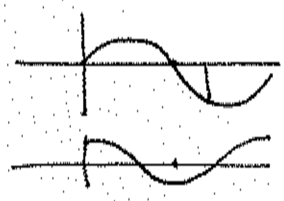
#8 of pg. 208

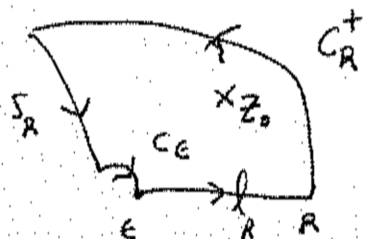
integrate $\frac{\log z}{z^3+1}$ around sector of $2\pi/3$ to show $\int_0^\infty \frac{\log(x)\,dx}{x^3+1} = \frac{-2\pi^2}{27}$

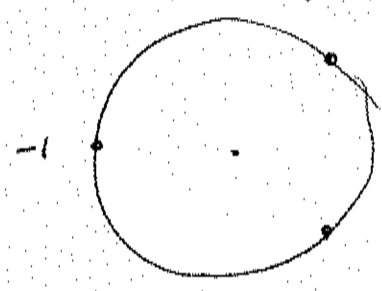


$z^3+1=0$

$-1,\ -e^{2\pi i/3},\ -e^{4\pi i/3}$







$-e^{4\pi i/3} = -\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}$

$z_0 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

$C_R^+ : z = Re^{i\theta},\ 0 \le \theta \le \frac{2\pi}{3}$

$-S_R : z = te^{\frac{2\pi i}{3}},\ \epsilon \le t \le R$

$-C_\epsilon : z = \epsilon e^{i\theta},\ 0 \le \theta \le \frac{2\pi}{3}$

$l_R : z = x,\ \epsilon \le x \le R$

$C_R^+ : |f(z)| = \left|\frac{\log(Re^{i\theta})}{(Re^{i\theta})^3+1}\right| \le \frac{|\ln R + i\theta|}{R^3-1} \le \frac{\sqrt{(\ln R)^2 + 2\pi}}{R^3-1} \longrightarrow 0.$

$\frac{\infty \cdot 0}{1}$

$-S_R$: non trivial.

$C_\epsilon : |f(z)| = \left|\frac{\ln\epsilon + i\theta}{\epsilon^3-1}\right|$ $\qquad \frac{\sqrt{(\ln\epsilon)^2 + 2\pi}}{1-\epsilon^3}\,\frac{2\pi}{3}\epsilon \longrightarrow$

l_R : non trivial

$< \left(\frac{\ln\epsilon + 2\pi}{1-\epsilon^3}\right)\frac{2\pi\epsilon}{3}$

$\frac{2\pi}{3}\left[\frac{\ln\epsilon + 2\pi}{\frac{1}{\epsilon} - \epsilon^2}\right] \longrightarrow \frac{2\pi}{3}\left[\frac{\frac{1}{\epsilon}}{\frac{-1}{\epsilon^2} - 2\epsilon}\right] \longrightarrow \frac{0}{-1-0} =$

$\frac{\epsilon}{-1-2\epsilon^3}$

$$\operatorname{Res}\left(\frac{\log(z)}{z^3+1};\ \frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = \left.\frac{\log\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{3z^2}\right|_{z=\frac{1}{2}+i\frac{\sqrt{3}}{2}}$$

$$= \frac{\ln\frac{1}{2} + \frac{i\sqrt{3}}{2}}{3\left(\frac{1}{4}\left(1+i\sqrt{3}\right)^2\right)}$$

$$= \frac{-\ln(2) + \frac{1}{2}i\sqrt{3}}{\frac{3}{4}\left(1+2i\sqrt{3}-3\right)}$$

$$= \cancel{2}\ \frac{4}{3}\left(\frac{-\ln(2)+\frac{i}{2}\sqrt{3}}{-2+2i\sqrt{3}}\right)$$

$$= \frac{2}{3}\left[\frac{-\ln(2)+i\sqrt{3}/2}{-1+i\sqrt{3}}\left[\frac{-1-i\sqrt{3}}{-1-i\sqrt{3}}\right]\right]$$

$$= \frac{2}{3}\left[\frac{\ln(2) - i\sqrt{3}/2 + 3/2 + i\ln(2)\sqrt{3}}{1+3}\right]$$

$$= \frac{1}{6}\left[\frac{3}{2} + \ln(2) + i\left(\sqrt{3}\ln(2) - \frac{\sqrt{3}}{2}\right)\right] = R_0$$

$$\int_{-S_R}\frac{\log(z)}{z^3+1}\,dz = \int_\epsilon^R \frac{\ln(t)+\frac{2\pi i}{3}}{(te^{2\pi i/3})^3+1}e^{\frac{2\pi i}{3}}\,dt \qquad \left(e^{\frac{2\pi i}{3}}\right)^3 = e^{2\pi i}$$

$$= \cos 2\pi + i\sin 2\pi$$

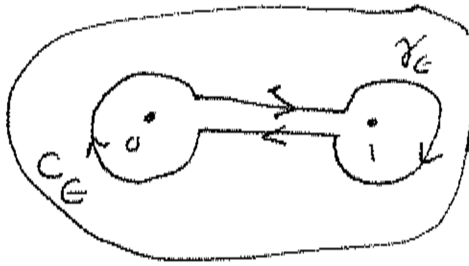
$$= 1$$

$$= \int_\epsilon^R \frac{\ln(t)}{t^3+1}e^{\frac{2\pi i}{3}}\,dt + \int_\epsilon^R \frac{\frac{2\pi i}{3}e^{\frac{2\pi i}{3}}\,dt}{t^3+1}$$

$$\left(1+e^{\frac{2\pi i}{3}}\right)\int_\epsilon^R \frac{\ln x}{x^3+1}\,dx + \int_\epsilon^R \frac{\exp\left(\frac{2\pi i}{3}\right)\frac{2\pi i}{3}}{t^3+1}\,dt = 2\pi i R_0$$

#2 of p. 222 Gamelin

$$\int_0^1 \frac{x^4\,dx}{\sqrt{x(1-x)}} = \frac{35\pi}{128}$$



all together Γ_ϵ

$$f(z) = \frac{z^4}{\sqrt{z(1-z)}}$$

by argument similar to that on 220, as $\epsilon \to 0$
only $\rightleftarrows$ nontrivial and

$$\underbrace{\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} f(z)\,dz} = \underbrace{2\int_0^1 \frac{x^4\,dx}{\sqrt{x(1-x}} = 2\pi i \left[\frac{-35 i}{128}\right]}$$

$$\hookrightarrow \int_0^1 \frac{x^4\,dx}{\sqrt{x(1-x)}} = \frac{35\pi}{128}.$$

$$2\pi i \, Res(f(z);\infty) =$$

$$\frac{z^4}{\sqrt{z(1-z)}} = \pm \frac{iz^4}{z}\left[1 - \frac{1}{2}\left(\frac{-1}{z}\right) + \frac{(-1/2)(-3/2)}{2!}\left(\frac{-1}{z}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(\frac{-1}{z}\right)^3\right.$$

$$= \pm i\left[z^3 + \frac{1}{2}z^2 - \frac{3}{6}z + \frac{15}{8\cdot 6} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\left(\frac{-1}{z}\right)^4 z^3 + \dots\right) + \dots\Big]$$

$$= \pm i\left[\dots \cancel{+} - \frac{\cancel{3}\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2\cdot 4\cdot \cancel{3}\cdot 2}\frac{1}{z} + \dots\right]$$

$$= \pm i\left[\dots - \frac{35}{128}\frac{1}{z}\right]$$