MATH 331: TEST 2 NO NOTECARDS, BOX ANSWERS, SHOW WORK, THANKS !

Problem 1 [15pts] Find the Laurent series centered at $z = 0$ for $f(z) = z^2 \sin(1/z^2)$.

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z^2}\right)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-4n-2} z^2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-4n}$$

Problem 2 [20pts] Find the residue of $f(z)$ at $z = 1 + i$ for $f(z) = \frac{\mathrm{Log}(z)}{(z-1-i)^2}$.

Observe $\mathrm{Log}(z)$ is analytic at $1+i$ hence,

$$\mathrm{Res}_{z=1+i}\left(\frac{\mathrm{Log}(z)}{(z-1-i)^2}\right) = \frac{d}{dz}\left[\mathrm{Log}(z)\right]\Big|_{z=1+i} = \frac{1}{z}\Big|_{z=1+i} = \boxed{\frac{1}{1+i}}$$

$$= \boxed{\frac{1-i}{2}}$$

etc...

Problem 3 [15pts] Exercise on multiplying series:

(**a.**) Multiply $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ and $1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots$

(**b.**) Express your answer from part (a.) in the form $c_o + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + \cdots$

(**c.**) Is this surprising? Explain.

(a.) $\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots\right) =$

$= 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + x + xy + \frac{1}{2}xy^2 + \frac{1}{2}x^2 + \frac{1}{2}x^2y + \frac{1}{6}x^3 + \cdots$

$= 1 + x + y + \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{6}(x^3 + 3x^2y + 3xy^2 + y^3) + \cdots$ (a.)

$= 1 + (x+y) + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 + \cdots$ (b.)

(c.) No. $e^x e^y = e^{x+y}$ and we're just looking at 1st few terms.

Problem 4 [30pts] Derive the Laurent series expansions for $f(z) = \frac{1}{z(1+z^2)}$ on annuli centered about zero (include in each of your two answers the domains on which the series represent $f(z)$)

Suppose $|z| < 1$ so $|-z^2| < 1$ hence

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

Hence,

$$f(z) = \frac{1}{z}\left(\frac{1}{1+z^2}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} = \boxed{\sum_{n=0}^{\infty} (-1)^n z^{2n-1}}$$

for $|z| < 1$

If $|z| > 1$ then $\frac{1}{|z^2|} < 1$ hence,

$$\frac{1}{1+z^2} = \frac{1}{z^2(1+1/z^2)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n$$

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^2}\frac{1}{z^{2n}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^{2n+}$$

Hence,

$$f(z) = \frac{1}{z}\left(\frac{1}{z^2+1}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^{2n+2}$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^{2n+3}}$$

for $|z| > 1$

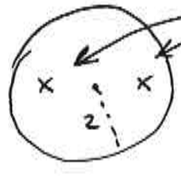
a.k.a.

$1 < |z| < \infty$.

Problem 5 [50pts] Let C denote the positively oriented circle $|z| = 2$. Calculate:

(a.) $\int_C \tan(z)dz$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \qquad \text{now} \quad \cos z = 0 \quad \text{for } z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$



simple poles at $z = \pm \pi/2$

$$\operatorname*{Res}_{z=\pm\pi/2}\left[\frac{\sin(z)}{\cos(z)}\right] = \frac{\sin(\pm\pi/2)}{-\sin(\pm\pi/2)} = -1.$$

$$\int_C \tan(z)\,dz = 2\pi i\left(\operatorname{Res}\left(\tan z; \tfrac{-\pi}{2}\right) + \operatorname{Res}\left(\tan z; \tfrac{+\pi}{2}\right)\right)$$
$$= \boxed{-4\pi i}$$

(b.) $\int_C \underbrace{\frac{\cosh(\pi z)dz}{z(z^2+1)}}$

$\underbrace{z=0,\ z=\pm i}$
$\Rightarrow$ simple poles

$$\operatorname*{Res}_{z=0}\left(\frac{\cosh \pi z}{z(z^2+1)}\right) = \frac{\cosh(0)}{0^2+1} = 1.$$

$$\operatorname*{Res}_{z=\pm i}\left(\frac{\cosh \pi z}{z(z+i)(z-i)}\right) = \begin{cases} \dfrac{\cosh \pi i}{i(i+i)} & : z = i \\ \dfrac{\cosh(-\pi i)}{-i(-i-i)} & : z = -i \end{cases}$$

$$= \frac{1}{2i^2}\cosh(\pi i)$$
$$= \frac{-1}{2}\left(e^{\pi i} + e^{-\pi i}\right) = -\frac{1}{2}\cos\pi$$
$$= \underline{\underline{\frac{1}{2}}}$$

By Residue Th^{m}

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}dz = 2\pi i\left[1 + \frac{1}{2} + \frac{1}{2}\right]$$
$$= \boxed{4\pi i}$$

Problem 6 [15pts] Express the integral

$$\int_0^{2\pi} \frac{2}{6 + 42\cos(\theta)} d\theta$$

as an contour integral on the positively oriented unit circle $|z| = 1$. **DO NOT TRY TO CALCULATE THE CONTOUR $\int$.**

$C: \; z = e^{i\theta}$ for $0 \le \theta \le 2\pi$

$dz = i e^{i\theta} d\theta$

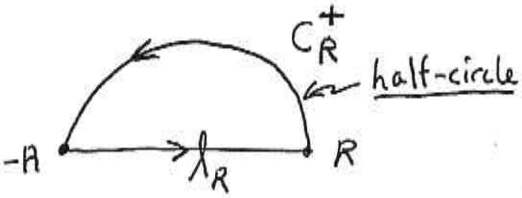
$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Hence,

$$\int_0^{2\pi} \frac{2\, d\theta}{6 + 42\cos\theta} = \boxed{\int_C \frac{2\, dz/iz}{6 + 21\left(z + \frac{1}{z}\right)}}$$

Problem 7 [25pts] Suppose C_R is as pictured below. Calculate



$$\lim_{R\to\infty}\int_{C_R}\frac{dz}{(z-i)(z+2i)} \quad : \quad C_R = C_R^+ \cup \ell_R, \quad R > 3.$$

and derive from the above limit the value of $p.v. \int_{-\infty}^{\infty} f(x)dx$ for appropriate $f(x)$.

$f(z) = \dfrac{1}{(z-i)(z+2i)}$ has simple poles at $z = i \in \mathrm{Int}(C_R)$ and $z = -2i \notin \mathrm{Int}(C_R)$ consequently for $R > 3$ (well 2 will do)

$$\int_{C_R}\frac{dz}{(z-i)(z+2i)} = 2\pi i \operatorname*{Res}_{z=i}[f(z)] = 2\pi i\left(\frac{1}{i+2i}\right) = \frac{2\pi}{3}.$$

$\underline{\text{However for } |z|=R:}$ $\quad |f(z)| = \dfrac{1}{|z-i||z+2i|} \leq \dfrac{1}{||z|-1|\,||z|-2|} = \dfrac{1}{(R-1)(R-2)}$

Hence $\left|\int_{C_R^+} f(z)dz\right| \leq \dfrac{\pi R}{(R-1)(R-2)} \longrightarrow 0$ as $R \to \infty$.

Therefore, as $\int_{C_R} = \int_{C_R^+} + \int_{\ell_R}$ we find,

$$\lim_{R\to\infty}\int_{\ell_R} f(z)dz = \frac{2\pi}{3} \quad : \ell_R : z = x \text{ for } -R \leq x \leq R$$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x-i)(x+2i)} = \frac{2\pi}{3} \qquad \star = x - 2x = -x$$

Note that

$$\frac{1}{(x-i)(x+2i)}\left[\frac{(x+i)(x-2i)}{(x+i)(x-2i)}\right] = \frac{(x+i)(x-2i)}{(x^2+1)(x^2+4)} = \frac{x^2+2}{(x^2+1)(x^2+4)} + i[\star]$$

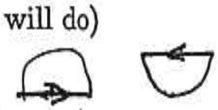
Thus,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2+2+i(-x)}{(x^2+1)(x^2+4)}dx = \frac{2\pi}{3}$$

Consequently from the real part : $\boxed{\int_{-\infty}^{\infty}\frac{x^2+2}{(x^2+1)(x^2+4)}dx = \frac{2\pi}{3}}$

(I know the integral $\int_{-\infty}^{\infty}$ exists by comparison)

Problem 8 [10pts] Indicate two other types of contour integration which we considered in homework, but which have not appeared on this test. (an example with a brief sentence will do)

- $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ had to break into  pieces to apply Jordan Lemma...
- $\dfrac{1}{z\sqrt{1-z}}$ had to use dog-bone contour
- also keyhole contour or •  pacman. aka branch cut.