MATH 331: TEST 3	MAY WORK TOGETHER, RESEARCH ON INTERNET, BOOKS ETC...

Please write your solutions on separate paper. While you may work together, you must write the solution you turn in by yourself in your own words (no copying). Thanks! Warning: this test appears more difficult than it actually is... notice your assignments are mostly just the bold part.

Problem 1 [10pts] Suppose $r, R \in \mathbb{R}$ with $0 \leq r < R$. **Prove that** $S = \{z \in \mathbb{C} \mid r < |z - z_o| < R\}$ **is an open set.**

Problem 2 [10pts] Let $f(z)$ be analytic in a domain D and suppose $f(z) - f(z_o)$ has a zero of order n at $z_o \in D$. Prove that for $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that for all w with $|w - f(z_o)| < \delta$ the equation $f(z) - w = 0$ has exactly n roots in $|z - z_o| < \epsilon$. You may (probably should) use Rouche's Theorem in your proof.

Problem 3 [10pts] Suppose $\sum_{n=0}^{\infty}(1 + 3^{-n-1})z^n$ represents $f(z)$ for $0 < |z| < 1$. Calculate $\int_C f(z)dz$ for C positively oriented $|z - 1| < 1/2$. Analytic continuation is key concept here, not direct calculation, although some calculation is needed.

Problem 4 [20pts] The solution of $\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0$ on a closed domain D subject to boundary conditions for ϕ on ∂D can often be seen from conformally mapping the solution from one of the basic cases below. Here we merely seek to familarize ourselves with these basic cases:

(a.) $\phi(z) = ARe(z) + B$ solves $\nabla^2\phi = 0$ on vertical strips $x_1 \leq Re(z) \leq x_2$ with $\phi(x_1) = c_1$ and $\phi(x_2) = c_2$ for appropriate choices of A, B as to impose the boundary conditions. **Choose** A, B **such that** $\phi(-1) = 0$ **and** $\phi(1) = 2$ **and explain why** $\phi(z) = ARe(z) + B$ **is indeed a solution to Laplace's equation**

(b.) $\phi(z) = AIm(z) + B$ solves $\nabla^2\phi = 0$ on horizontal strips $y_1 \leq Im(z) \leq y_2$ with $\phi(y_1) = c_1$ and $\phi(y_2) = c_2$ for appropriate choices of A, B as to impose the boundary conditions. **Choose** A, B **such that** $\phi(i) = 10$ **and** $\phi(4i) = 20$ **and explain why** $\phi(z) = AIm(z) + B$ **is indeed a solution to Laplace's equation**

(c.) $\phi(z) = ALog(|z|) + B$ solves $\nabla^2\phi = 0$ for $z \in \mathbb{C}$ with $R_1 \leq |z| < R_2$ and for appropriate choices of the constants A, B we may impose $\phi|_{|z|=R_1} = c_1$ and $\phi|_{|z|=R_2} = c_2$ for any given set of boundary values c_1, c_2. **Explain why** $ALog(|z|) + B$ **solves Laplace's equation and find** A, B **for which** $\phi(z) = 0$ **if** $|z| = 2$ **and** $\phi(z) = 10$ **if** $|z| = 7$**.**

(d.) Let us denote $Arg_\alpha(z) \in arg(z) \cap (\alpha, 2\pi + \alpha]$. For example, $\alpha = -\pi$ is the principle argument function. Generally this gives us branch of the argument for which the discontinuity occurs at angle α. Claim: $\phi(z) = BArg_\alpha(z) + A$ solves $\nabla^2\phi = 0$ for $z \in \mathbb{C}$ with $\theta_1 \leq Arg_\alpha(z) \leq \theta_2$ and for appropriate choices of the constants A, B we may impose $\phi|_{Arg_\alpha(z)=\theta_1} = c_1$ and $\phi|_{Arg_\alpha(z)=\theta_2} = c_2$ for any given set of boundary values c_1, c_2. **Explain why** $BArg_\alpha(z) + A$ **solves Laplace's equation in the upper half-plane and find** α, B, A **such that** $\phi(z) = 1$ **for** $Re(z) > 0$ **and** $\phi(z) = 3$ **for** $Re(z) < 0$**.**

(e.) Suppose $x_1 < x_2 < \cdots < x_n$ and $A_1, A_2, \ldots A_n$ are constants which are used to formulate the function $\phi(z) = \sum_{j=1}^{n} A_j Arg(z - x_j)$. **Show that** $\phi(z) = \sum_{j=1}^{n} A_j Arg(z - x_j)$ **solves** $\nabla^2\phi = 0$ **in the upper half-plane. Also, find the boundary values of the given solution on the real axis.**

Problem 5 [50pts] If $\phi(z) \in \mathbb{R}$, $z = x + iy$, solves $\phi_{xx} + \phi_{yy} = 0$ on some domain D of $\mathbb{C}$ and if f is an injective analytic function where we denote $w = f(z)$, $w = u + iv$, then $\Psi(u, v) = \phi(f^{-1}(w)) = \phi(x(u, v), y(u, v))$ solves $\Psi_{uu} + \Psi_{vv} = 0$. The proof is simple: ϕ is harmonic hence we can find an analytic function g such that $Re(g) = \phi$. Then, since the composition of analytic functions is once more analytic, $\Psi = Re(g \circ f^{-1})$ is the real component of an analytic function and is hence harmonic. Therefore, we can replace the Dirichlet problem in the $z = x + iy$ plane for a possibly simpler Dirichet problem in the $w = u + iv$ plane. Once we find Ψ in u, v then we simply invert $\Psi = \phi \circ f^{-1}$ to find $\phi = \Psi \circ f$. This mapping equation simply means to take the u, v in the Ψ solution and replace them with the appropriate expressions in x, y as defined by $f(x, y) = u(x, y) + iv(x, y)$. We have many injective analytic functions to utilize in this technique: rotations, magnifications, translations, inversion (these give us the Mobius transformations collectively), exponential mapping,... there are many more techniques I have not shared in lecture, the literature here is vast.

(a.) Solve the Dirichlet problem on the slanted strip $x \leq y \leq x + 1$ for $z = x + iy$ where $\phi(x + ix) = 10$ and $\phi(x + i(x + 1)) = 20$. *Hint: use a rotation and the result of either (a.) or (b.) of the previous problem*

(b.) Solve the Dirichlet problem on the annulus $2 \leq |z - 1 + 2i| \leq 7$ where the inner-circle has constant boundary value of $\phi = 0$ whereas the outer-circle takes constant boundary value 10. *Hint: use a translation and the result of (c.) of the previous problem*

(c.) Solve the Dirichlet problem on the disk $|z| \leq 1$ such that $\phi(z) = 1$ for $Im(z) > 0$ and $\phi(z) = 3$ for $Im(z) < 0$ *Hint: use some Mobius transformation and the result of (d.) of the previous problem*

(d.) Solve the Dirichlet problem on the disk around infinity $|z| \geq 1$ such that $\phi(z) = 1$ for $|z| = 1$ with $Re(z) > 0$ and $\phi(z) = 3$ for $|z| = 1$ and $Re(z) < 0$ *Hint: use some Mobius transformation and the result of (d.) of the previous problem*

Remark: the other half of this test will concern infinite products and the Mittag Leffler expansion which you have already begun work on in Test 2...