Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps.

Problem 1 Calculate the cartesian forms of

- **(a.)** $\frac{3}{2-i}$
- **(b.)** $(\sqrt{3}+i)^7$
- **(c.)** $Log(-2i)$

Problem 2 Let $z = 3+4i$. Find the polar and Cartesian form of z^{42}.

Problem 3 Find all solutions of $z^5 = 32i$. Please give your solutions in polar form.

Problem 4 Let S be formed by the union of the solution sets of $|z-2i| \leq 3$ and $|z+3i| < 2$. Sketch S in the complex plane and prove or disprove that S is connected. Proof by picture will suffice here.

Problem 5 Factor $f(z) = z^4 + 4z^2 + 4i$ completely over $\mathbb{C}$.

Problem 6 Let $f(z) = \sin(z^2)$ verify that $f(z)$ satisfies the Cauchy Riemann equations by explicitly calculating the component functions u, v for which $f = u + iv$. Derive $f'(z) = 2z\cos(z^2)$ from the Cauchy Riemann approach.

Problem 7 Let $u(z) = e^{ax-by}\sin(bx+ay)$ where $a, b \in \mathbb{R}$. Find the harmonic conjugate of u on $\mathbb{C}$.

Problem 8 Find real-valued functions u, v such that $f = u + iv$ given that:

- **a.** $f(z) = \cos(2z - 3i)$
- **b.** $f(z) = Log(iz + 2)$

Problem 9 If the function below is holomorphic on some domain then explain why and calculate its complex derivative. Otherwise, explain why the function is not holomorphic on any domain. (I do not expect you to describe the domain of holmorphicity in the cases it exists)

- **(a.)** $f(x+iy) = x^3 - y + ixy^2$,
- **(b.)** $f(x+iy) = \cos(x^2+y^2)$,
- **(c.)** $f(z) = \frac{z}{z^2-1}$,
- **(d.)** $f(z) = z^2 Log(2z+1)$,
- **(e.)** $f(z) = \tanh^2(\sinh(\sqrt{z}))$

Problem 10 Does $\lim_{z\to 0} \frac{\sin z}{z}$ exist? If so, what is its value? Support your claim.

Problem 11 Consider $f(z) = z^4$. Let $S_\alpha = \{\ re^{i\theta} \mid \alpha \leq \theta < \alpha + \pi/2,\ r \in [0, \infty)\}$.

- **(a.)** Show that $f(S_\alpha) = \{f(z) \mid z \in S_\alpha\}$ is the complex plane.

(b.) Show the restriction of f to S_α is injective.
Restriction of a function means to keep the same formula and replace the domain; $f|_{S_\alpha} : S_\alpha \to \mathbb{C}$ where $f|_{S_\alpha}(z) = f(z)$ for each $z \in S_\alpha$ defines the restriction of f to S_α which I denote as $f|_{S_\alpha}$. The term "injective" for a function g means that if $z_1, z_2 \in dom(g)$ and $g(z_1) = g(z_2)$ then $z_1 = z_2$.

(c.) Let $w = u + iv$ and $z = x + iy$. If $w = z^4$ for $z \in S_\alpha$ then solve for x, y as functions of u, v. *I highly recommend you use constructions we made in the first week of class and not try to do this in terms of direct cartesian calculation !*

(d.) Let $g(u+iv) = x(u+iv) + iy(u+iv)$ where $x(u+iv), y(u+iv)$ are the formulas you derived in the previous part. Determine the largest set on which g is holomorphic.

Problem 12 Let $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$. If possible, find a branch of the following multiply valued functions which is holomorphic on $\mathbb{H}$

(a.) $\log(3z - 1)$

(b.) $\log(z^2 - i)$

(c.) $\log(z^4 + 1)$

The best solution here would also answer the question of what is the maximal set on which a branch of the expressions above could be holomorphic, but I put $\mathbb{H}$ here so we can make the question easier to understand. You can think about Log_α replacing log *and try to solve for the bad place to find the curves which forbid holomorphicity given the α-branch*

Problem 13 Saff and Snider §7.2#1. (find order of zero and show function not one-to-one)

Problem 14 Saff and Snider §7.3#4. (Mobius transformation which maps lower half plane to disk)

Problem 15 Saff and Snider §7.3#7. (Mobius transformations involving ∞)