

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

**Problem 1** Your signature below indicates you have:

- (a.) I have read Chapter 1 of Gamelin: \_\_\_\_\_.
- (b.) I have read Cook's Guide to Chapter 1: \_\_\_\_\_.

**Problem 2** The distance between two points  $z, w \in \mathbb{C}$  is given by  $|z - w|$ . In view of this, describe in words the solutions of the complex equations below. Also, find the Cartesian form of the given conditions:

- (a.)  $|z - 2 + 7i| = 3$
- (b.)  $|z - 1| + |z - i| = 1$  O
- (c.)  $|z + \bar{z} - 6 - 2i| \leq 2$

**Problem 3** We discovered the dot-product of  $z$  and  $w$  is given by  $\langle z, w \rangle = \Re(z\bar{w})$  in the first section of the Guide. I claimed that if  $z = x + iy$  and  $w = a + ib$  then  $\Re(z\bar{w}) = xa + yb$ . Show this claim is correct. Also, describe the significance of  $\Im(z\bar{w})$ .

**Problem 4** Let  $A$  be an annulus in the complex plane. In particular, suppose

$$A = \{z \in \mathbb{C} \mid 1 \leq |z - 1 + i| \leq 2\}.$$

Find upper and lower bounds for the modulus of  $p(z) = z^2 + 1$ . This means you should find  $m, M > 0$  such that  $m \leq |p(z)| \leq M$  for all  $z \in A$ .

**Problem 5** §1.2 # 1(a, c, h).

**Problem 6** §1.2 # 5

**Problem 7** §1.4 # 1(b, f) and same for §1.4 # 2(b, f).

**Problem 8** §1.5 # 1(c, d, f).

**Problem 9** §1.6 # 1.

**Problem 10** §1.6 # 4.

**Problem 11** §1.8 # 3.

**Problem 12** §1.7 # 4. And: Let  $p, q \in \mathbb{C}$  and let  $z^p, z^q, z^{p+q}$  denote the principal branches of the given power functions. In particular,  $z^p = \exp(p \operatorname{Log}(z))$ . **Prove or disprove:** for all  $z \in \mathbb{C}^-$ ,  $z^p z^q = z^{p+q}$ .

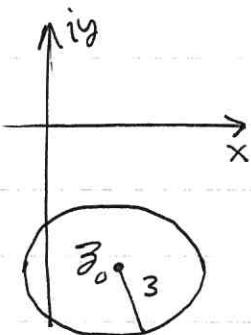
Notice,  $z^a$  is a set of values in #4 but  $z^p$  is a single-value for the problem stated.

Complex Analysis : Mission I solution

PROBLEM 2 (NOTE  $z + \bar{z} = 2x$  and  $z - \bar{z} = 2iy$  for  $z = x + iy$ )

(a.)  $|z - 2 + 7i| = 3 \Rightarrow$  all  $z \in \mathbb{C}$  distance 3 from  $z_0 = 2 - 7i$ . It's a circle centered at  $2 - 7i$  with radius 3.

$$\begin{aligned} |z - 2 + 7i|^2 &= 9 \Rightarrow (z - 2 + 7i)(\bar{z} - 2 - 7i) = 9 \\ &\Rightarrow z\bar{z} + z(-2 - 7i) + \bar{z}(-2 + 7i) + 4 + 49 = 9 \\ &\Rightarrow z\bar{z} + (z + \bar{z})(-2) - 7i(z - \bar{z}) = 9 - 53 \\ &\Rightarrow x^2 + y^2 - 4x - 7i(2iy) = -44 \\ &\Rightarrow x^2 + y^2 - 4x + 14y = -44 \\ &\Rightarrow (x-2)^2 + (y+7)^2 = -44 + 4 + 49 \\ &\Rightarrow (x-2)^2 + (y+7)^2 = 9 \end{aligned}$$

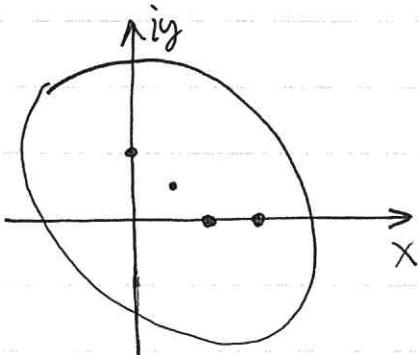


Remark: I didn't require you show all this as it is known once we identify it's a circle.

(b.)  $|z - 1| + |z - i| = 10$

(distance from  $z$  to 1) + (distance from  $z$  to  $i$ ) = 10

This is an ellipse



$$\begin{aligned} (|z-1|)^2 &= (10 - |z-i|)^2 \\ (z-1)(\bar{z}-1) &= 100 - 20|z-i| + |z-i|^2 \\ 3\bar{z} - z - \bar{z} + 1 &\Rightarrow \\ \hookrightarrow &= 100 - 20|z-i| + (z-i)(\bar{z}+i) \\ &= 100 - 20|z-i| + 3\bar{z} + i(z-\bar{z}) - i^2 \\ &= 100 - 20|z-i| + 3\bar{z} + i(2iy) + 1 \end{aligned}$$

$$\therefore -2x + 1 = 100 - 20|z-i| - 2y + 1$$

$$\frac{2(y-x) - 100}{-20} = |z-i| = \sqrt{x^2 + (y-1)^2}$$

$$x - y + 50 = 10\sqrt{x^2 + (y-1)^2}$$

Problem 2b)

$$(x-y+50)^2 = 100(x^2 + (y-1)^2)$$

$$(x-y)^2 + 100(x-y) + 2500 = 100x^2 + 100(y-1)^2$$

$$\cancel{x^2 - 2xy + y^2} + \cancel{100x} - \underline{\underline{100y}} + 2500 = \cancel{100x^2} + \cancel{100y^2} - \underline{\underline{200y}} + 100$$

$$99x^2 + 99y^2 - 100y - 100x + 2xy = 2400$$

(this is equation of rotated ellipse)

— (with 1 instead of 10) —

$$|z-1| + |z-i| = 1$$

$$\sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + (y-1)^2} = 1$$

$$\sqrt{(x-1)^2 + y^2} = 1 - \sqrt{x^2 + (y-1)^2}$$

$$(x-1)^2 + y^2 = 1 - 2\sqrt{x^2 + (y-1)^2} + x^2 + (y-1)^2$$

$$\cancel{x^2 - 2x + 1 + y^2} = \cancel{x^2 - 2\sqrt{x^2 + (y-1)^2} + x^2 + y^2} - 2y + 1$$

$$\frac{2y - 2x - 1}{-2} = \sqrt{x^2 + (y-1)^2}$$

$$\frac{1}{4}(2y - 2x - 1)^2 = x^2 + (y-1)^2$$

$$(y-x - \frac{1}{2})^2 = x^2 + (y-1)^2$$

$$(y-x)^2 - (y-x) + \frac{1}{4} = x^2 + (y-1)^2$$

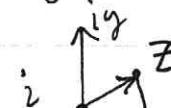
$$\cancel{y^2 - 2xy + x^2} - y + x + \frac{1}{4} = \cancel{x^2 + y^2} - 2y + 1$$

$$2y - y - 2xy + x + \frac{1}{4} - 1 = 0$$

$$y(1 - 2x) = \frac{3}{4} - x$$

$$y = \frac{\frac{3}{4} - x}{1 - 2x}$$

hmm... why is this false?



$$|z-i| + |z-1| \geq |i-1|$$

$|i-1| \geq \sqrt{2}$  bounds! No sol.

PROBLEM 2

$$(c.) |z + \bar{z} - 6 - 2i| \leq 2$$

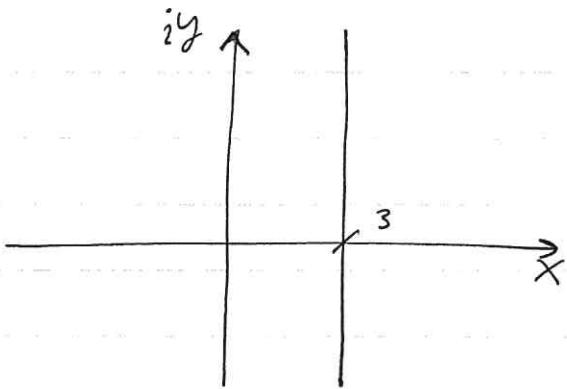
$$|(x+iy) + (x-iy) - 6 - 2i| \leq 2$$

$$|2x - 6 - 2i| \leq 2$$

$$|2(x-3 - i)| \leq 2$$

$$|x-3 - i| \leq 1$$

$$\sqrt{(x-3)^2 + 1^2} \leq 1 \Rightarrow \text{No soln except } \boxed{x=3}$$



PROBLEM 3 Let  $z = x+iy$  and  $w = a+ib$  hence,

$$z\bar{w} = (x+iy)(a-ib)$$

$$= xa - i^2 yb + i(ya - xb)$$

$$= xa + yb + i(ya - xb)$$

$$= \underbrace{\langle x, y \rangle \cdot \langle a, b \rangle}_{\text{Euclidean dot-product}} + i \underbrace{\langle y, -x \rangle \cdot \langle a, b \rangle}_{\text{rotation of } z = x+iy \text{ to } -iz = y - ix}$$

of  $z = x+iy$

and  $w = a+ib$

$$\langle z, w \rangle = \operatorname{Re}(z\bar{w})$$

$$\langle -iz, w \rangle = \operatorname{Im}(z\bar{w})$$

PROBLEM 4  $A = \{z \in \mathbb{C} / 1 \leq |z-1+i| \leq 2\}$

find bounds for  $P(z) = z^2 + 1$

$$\begin{aligned} P(z) &= (z-1+i+1-i)^2 + 1 \quad \leftarrow \text{recentering trick.} \\ &= (z-1+i)^2 + 2(1-i)(z-1+i) + (1-i)^2 + 1 \\ &= (z-1+i)^2 + 2(1-i)(z-1+i) + 1 - 2i - 1 + 1 \end{aligned}$$

Thus,

$$|P(z)| \leq |(z-1+i)^2| + |2(1-i)||z-1+i| + |1-2i|$$

$$\Rightarrow |P(z)| \leq 2^2 + 2\sqrt{2}|z| + \sqrt{5} = \boxed{4(1+\sqrt{2}) + \sqrt{5} = M}$$

for the lower bound we use  $|z+w| \geq ||z|-|w||$

PROBLEM 4  $P(z) = z^2 + 1$  can be expressed as:

$$P(z) = (z - 1 + i)^2 + 2(1-i)(z - 1 + i) + 1 - 2i$$

We know that  $|z-w| \geq |z| - |w|$  hence,

$$\begin{aligned} |P(z)| &\geq |z - 1 + i|^2 - |z(1-i)(z - 1 + i) + 1 - 2i| \\ &\geq |z - 1 + i|^2 - (|z(1-i)(z - 1 + i)| + |1 - 2i|) \\ &\geq |z - 1 + i|^2 - 2\sqrt{2}|z - 1 + i| - \sqrt{5} \quad \text{Given } |z - 1 + i| \geq 1 \\ &\geq 1 - 2\sqrt{2} - \sqrt{5} \end{aligned}$$

Thus, for  $z \in A$

$$1 - 2\sqrt{2} - \sqrt{5} \leq |P(z)| \leq 4(1 + \sqrt{2}) + \sqrt{5}$$

Remark:  $|P(z)| = |x^2 - y^2 + 1 + 2xyi|$  for  $z = x + iy$ .

$$|P(z)| = \sqrt{(x^2 + 1 - y^2)^2 + 4x^2y^2}$$

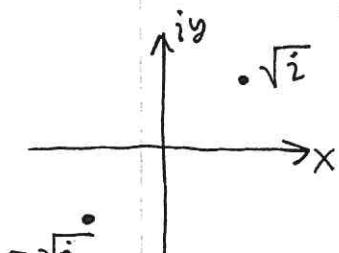
we could find extreme values for  $|P(z)|$  subject to  $1 \leq \sqrt{(x-1)^2 + (y+1)^2} \leq 2$ . I think I prefer complex notation.

Remark: the sol<sup>n</sup> above is what I intended for you to do for future skill-set-sake. HOWEVER, one of you (at least, likely several) noticed that

$P(z) = z^2 + 1$  has  $P(-i) = i^2 + 1 = 0$  and in fact,  $i \in A$  since  $|-i - 1 + i| = ||i|| = 1$  thus  $|P(-i)| = 0$  and clearly  $0 \leq |P(z)| \leq 4(1 + \sqrt{2}) + \sqrt{5}$  is a sharper estimate for  $|P(z)|$ .

PROBLEM 5 Express in polar & cartesian and plot,

$$\text{§1.2 #1a)} \quad \sqrt{i} = \exp\left(i \frac{\operatorname{Arg}(i)}{2}\right) \quad \begin{cases} \text{see defn 1.4.4} \\ \text{so } |w|=1 \text{ and } \sqrt[2]{1}=1 \end{cases}$$



$$\begin{aligned} &= \exp\left(\frac{i\pi}{2}\right) : \text{polar} \\ &= \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \\ &= \boxed{\frac{1+i}{\sqrt{2}}} : \text{cartesian} \end{aligned}$$

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

Note:  $(i)^{1/2} = \{\sqrt{i}, -\sqrt{i}\}$

PROBLEM 5/

$$\underline{\text{§1.2 #1d}} \quad \sqrt[4]{-1} = \sqrt[4]{1 \cdot \exp(i\pi)} = \boxed{e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}}$$

Sol 1 has error, this is principal 4<sup>th</sup> root of -1.

the  $e^{-i\pi/4}$  is in  $(-1)^{1/4} = \{e^{i\pi/4}, ie^{i\pi/4}, -e^{i\pi/4}, -ie^{i\pi/4}\}$   
 $(-1)^{1/4} = \{\frac{1+i}{\sqrt{2}}, \frac{i-1}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{-i+1}{\sqrt{2}}\}$

$ie^{i\pi/4}$ .  $\uparrow iy \cdot e^{i\pi/4} \leftarrow$  principal 4<sup>th</sup> root



$-e^{i\pi/4} \cdot \bullet -ie^{i\pi/4}$

§1.2 #1h

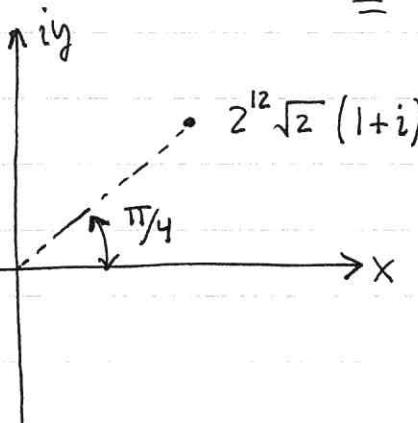
$$\left(\frac{1+i}{\sqrt{2}}\right)^{25} = \left(\sqrt{2} e^{i\pi/4}\right)^{25}$$

$$= (\sqrt{2})^{25} e^{\frac{25\pi i}{4}}$$

$$= [(\sqrt{2})^2]^{12} \sqrt{2} \exp\left(i\left(6\pi + \frac{\pi}{4}\right)\right)$$

$$= \underbrace{2^{12} \sqrt{2}}_{\text{polar}} e^{i\pi/4} = \underbrace{2^{12} \sqrt{2}(1+i)}_{\text{nearly Cartesian}}$$

(\*)  $\star$   
 $\sqrt{2} e^{i\pi/4} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$   
 $= \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)$   
 $= 1+i$



PROBLEM 6] (§1.2 #5)

(a.) For  $n \geq 1$  show  $1 + z + z^2 + \dots + z^n = \frac{1-z^{n+1}}{1-z}$  for  $z \neq 1$ .

If  $n=1$  then  $\frac{1-z^2}{1-z} = \frac{(1+z)(1-z)}{1-z} = 1+z$  for  $z \neq 1$ .

Inductively assume  $1 + z + z^2 + \dots + z^n = \frac{1-z^{n+1}}{1-z}$  and consider, we have  $z^{n+1} = 1 - (1-z)[1 + z + z^2 + \dots + z^n]$  (\*)

$$\begin{aligned}
 \frac{1-z^{n+2}}{1-z} &= \frac{1}{1-z}(1-z(z^{n+1})) \quad \rightarrow (*) \\
 &= \frac{1}{1-z}\left(1 - (1-z)[1 + z + z^2 + \dots + z^n]\right) \\
 &= \frac{1}{1-z} - z\left[\frac{1}{1-z} - [1 + z + z^2 + \dots + z^n]\right] \\
 &= \frac{1}{1-z} - \frac{z}{1-z} + z + z^2 + z^3 + \dots + z^{n+1} \\
 &= \frac{1-z}{1-z} + z + z^2 + \dots + z^{n+1} \\
 &= 1 + z + z^2 + \dots + z^{n+1} \quad (\text{for } z \neq 1)
 \end{aligned}$$

Hence the claim is true for  $n+1$  and we find by induction  $1 + z + \dots + z^n = \frac{1-z^{n+1}}{1-z}$  for  $z \neq 1$  is true  $\forall n \in \mathbb{N}_0$ .

$$\begin{aligned}
 (b.) \quad z = e^{i\theta} \Rightarrow 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \quad (\text{used deMoivre,}) \\
 z = e^{-i\theta} \Rightarrow 1 + e^{-i\theta} + e^{-2i\theta} + \dots + e^{-ni\theta} &= \frac{1 - e^{-i(n+1)\theta}}{1 - e^{-i\theta}}
 \end{aligned}$$

Thus, adding the eq's above

$$2 + e^{i\theta} + e^{-i\theta} + e^{2i\theta} + e^{-2i\theta} + \dots + e^{ni\theta} + e^{-ni\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-i(n+1)\theta}}{1 - e^{-i\theta}}$$

divide by 2 and recall  $\cos \beta = \frac{1}{2}(e^{i\beta} + e^{-i\beta})$ ,

$$1 + \cos \theta + \cos 2\theta + \dots + \cos(n\theta) = \frac{1}{2} \left[ \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-i(n+1)\theta}}{1 - e^{-i\theta}} \right]$$



continued



PROBLEM 6 continued

$$\begin{aligned}
 \star &= \frac{1}{2} \left[ \frac{e^{-i\theta/2} (1 - e^{i(n+1)\theta})}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{i\theta/2} (1 - e^{-i(n+1)\theta})}{e^{i\theta/2} - e^{-i\theta/2}} \right] \\
 &= \frac{1}{4i} \left[ \frac{e^{-i\theta/2} - e^{i(n-\frac{1}{2})\theta}}{-\frac{1}{2}i(e^{i\theta/2} - e^{-i\theta/2})} + \frac{e^{i\theta/2} - e^{-i(n-\frac{1}{2})\theta}}{\frac{1}{2}i(e^{i\theta/2} - e^{-i\theta/2})} \right] \\
 &= \frac{1}{4i} \left[ \frac{e^{i(n-\frac{1}{2})\theta} - e^{-i\theta/2} + e^{i\theta/2} - e^{-i(n-\frac{1}{2})\theta}}{\sin(\theta/2)} \right] \\
 &= \frac{1}{2} \left\{ \frac{1}{\sin \theta/2} \left[ \frac{1}{2i}(e^{i\theta/2} - e^{-i\theta/2}) + \frac{1}{2i}(e^{i(n-\frac{1}{2})\theta} - e^{-i(n-\frac{1}{2})\theta}) \right] \right\} \\
 &= \frac{1}{2} \left[ \frac{1}{\sin \theta/2} \left[ \sin \theta/2 + \sin(n - \frac{1}{2})\theta \right] \right] \\
 &= \frac{1}{2} \left( 1 + \frac{\sin(n - \frac{1}{2})\theta}{\sin \theta/2} \right)
 \end{aligned}$$

Therefore, we've shown:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos(n\theta) = \frac{1}{2} \left( 1 + \frac{\sin(n - \frac{1}{2})\theta}{\sin \theta/2} \right)$$

PROBLEM 7

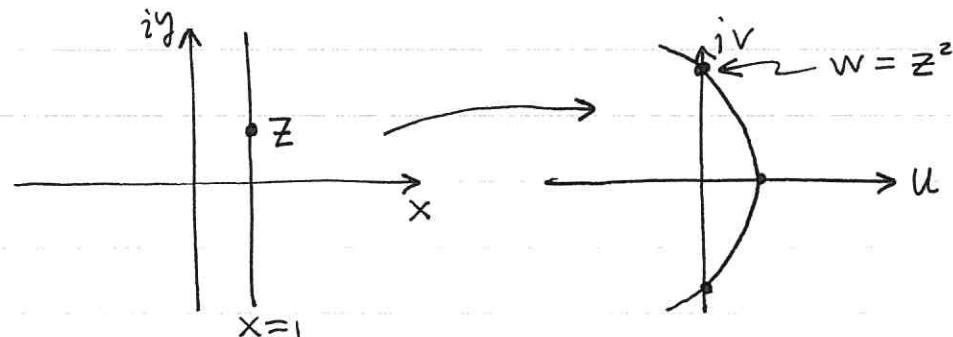
[§1.4 #1b] Sketch curve  $x=1$  in  $z$ -plane and its image under  $z \mapsto w=z^2$

$$x=1 \Rightarrow z = 1+iy \Rightarrow z^2 = (1+iy)^2 = 1-y^2+2iy$$

Let  $w=u+iv$  where  $u,v$  serve as Cartesian coord. of  $w$ -plane.

We have  $u=1-y^2$  and  $v=2y$  where  $y$  parametrizes image curve.

$$\text{Eliminate } y = \frac{v}{2} \rightarrow u = 1 - \frac{1}{4}v^2$$



(PROBLEM 7)

§1.4 # 1f]  $y = \frac{1}{x}, x \neq 0 \Rightarrow z = x + \frac{i}{x}$

thus  $z^2 = x^2 - \frac{1}{x^2} + 2ix = x^2 - \frac{1}{x^2} + 2i$

Set  $w = u + iv = x^2 - \frac{1}{x^2} + 2i \Rightarrow u = x^2 - \frac{1}{x^2}, v=2$

Simplify  $u$  to  $u = \frac{x^4 - 1}{x^2} = \frac{(x^2 - 1)(x^2 + 1)}{x^2} = \frac{(x+1)(x-1)(x^2+1)}{x^2}$

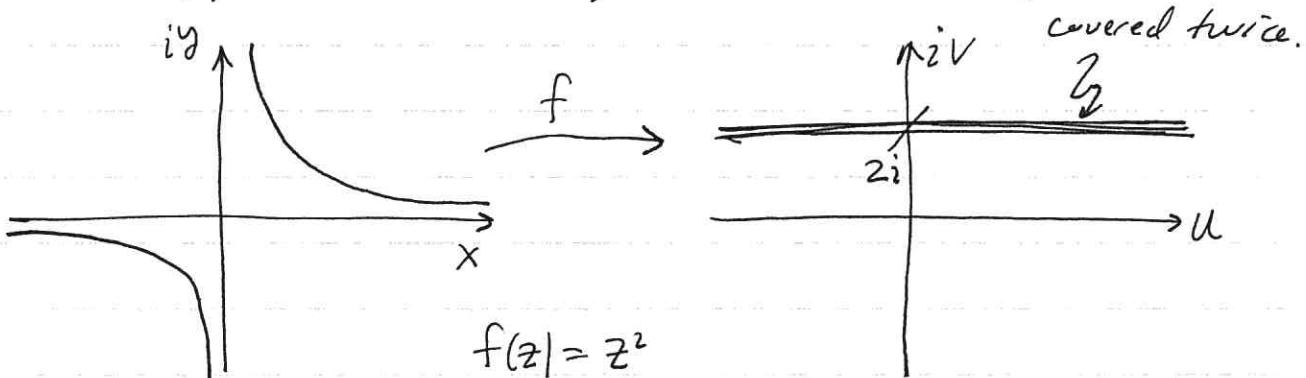
I'm just working to see what values are attained by  $u$  as  $x$  varies over  $\mathbb{R}^x = (-\infty, 0) \cup (0, \infty)$ .

Let's see as  $x \rightarrow 0^+$  we have  $u = x^2 - \frac{1}{x^2} \rightarrow -\infty$

as  $x \rightarrow \infty$  we have  $u = x^2 - \frac{1}{x^2} \rightarrow \infty$

Likewise as  $x \rightarrow 0^-$ ,  $u \rightarrow -\infty$  &  $x \rightarrow -\infty$ ,  $u \rightarrow \infty$

In summary,  $v=2$  fixed,  $u$  free to vary over  $\mathbb{R}$



§1.4 # 2b] Sketch  $x=1$  image under  $f(z) = \sqrt{z}$

For  $x=1$ , we're in quadrants I & IV hence inverse tangent works directly  $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(y)$  for  $z=1+iy$

$$f(z) = \sqrt{1+iy} = \sqrt{\sqrt{1+y^2} \exp(i\tan^{-1}(y))}$$

$$w = (1+y^2)^{1/4} \exp(i\tan^{-1}(y)/2)$$

Analyze in

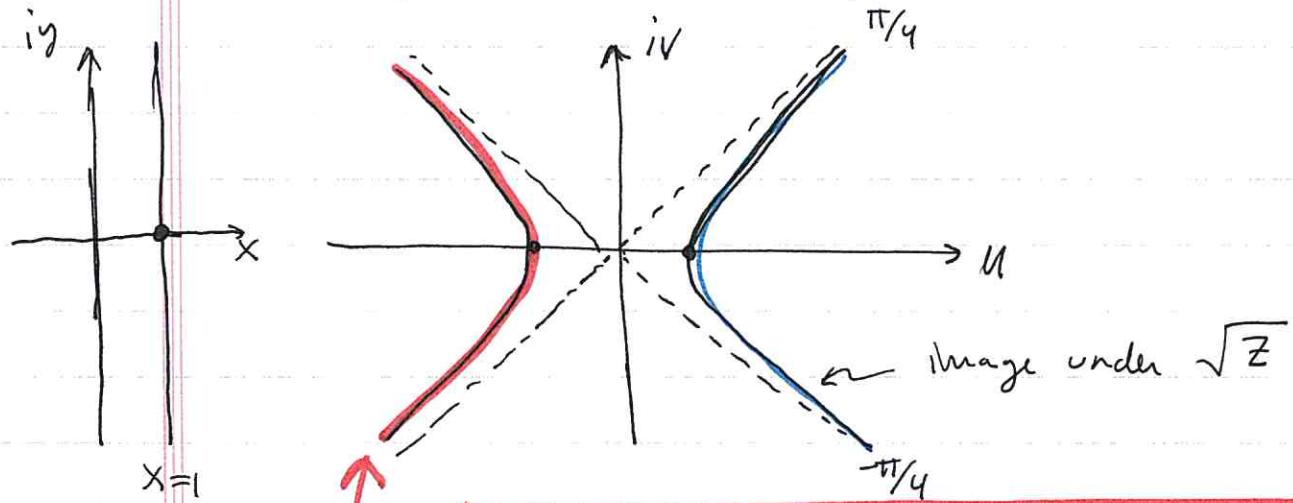
polar coordinates as elimination method is no longer reasonable!

$w = pe^{i\phi}$  we see  $p = (1+y^2)^{1/4} \geq 1$  then

$\frac{\tan^{-1}(y)}{2} \rightarrow \pm \frac{\pi}{4}$  as  $y \rightarrow \pm \infty$ .

§1.4 #26 continued

$\rho \rightarrow \infty$  as  $y \rightarrow \pm\infty$   
 $\phi$  ranges from  $-\frac{\pi}{4} \rightarrow \frac{\pi}{4}$  as  $-\infty \xrightarrow{y} \infty$



$$f_2(z) = \sqrt{z} = -\sqrt{z} \quad (\text{other branch})$$

$$\exp\left(\frac{2\pi i}{2}\right) = \exp(\pi i) = -1$$

§1.4 # 2f  $y = \frac{1}{x} \Rightarrow z = x + \frac{i}{x} = \sqrt{x^2 + \frac{1}{x^2}} \exp(i \operatorname{Arg}(x + \frac{i}{x}))$   
 for  $x > 0$ , note  $\operatorname{Arg}(x + \frac{i}{x}) = \tan^{-1}(\frac{1/x}{x}) = \tan^{-1}(\frac{1}{x^2})$

thus  $z = \sqrt{x^2 + \frac{1}{x^2}} \exp(i \tan^{-1}(\frac{1}{x^2})) \quad (\text{for } x > 0)$

so  $\sqrt{z} = (x^2 + \frac{1}{x^2})^{1/4} \exp\left[i \frac{\tan^{-1}(\frac{1}{x^2})}{2}\right]$

as  $x \rightarrow 0$  we have  $(x^2 + \frac{1}{x^2})^{1/4} \rightarrow \infty$  and  $\frac{\tan^{-1}(\frac{1}{x^2})}{2} \rightarrow \frac{\pi}{4}$

as  $x \rightarrow \infty$  we have  $\rho \rightarrow \infty$  and  $\phi \rightarrow 0$

For  $x < 0$ ,  $z = x + \frac{i}{x}$  is in quad III hence  $-\pi < \operatorname{Arg}(z) < -\frac{\pi}{2}$

thus  $\operatorname{Arg}(x + \frac{i}{x}) = \tan^{-1}(\frac{1}{x^2}) - \pi$  thus,

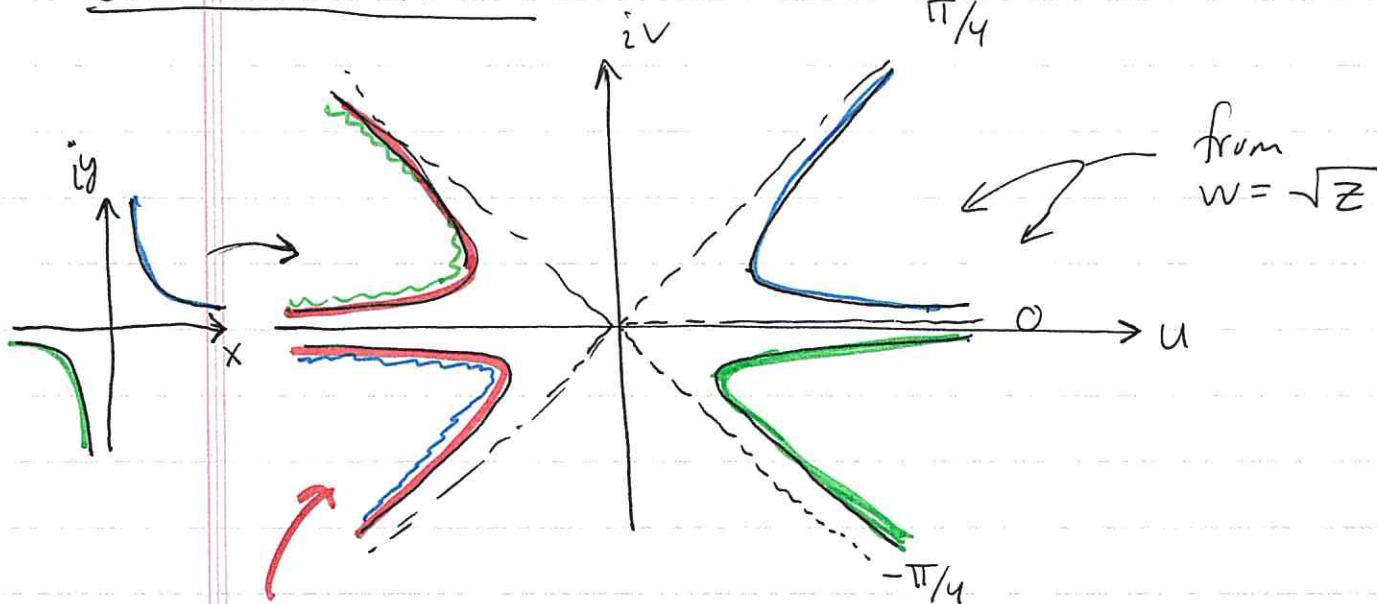
$$\sqrt{z} = (x^2 + \frac{1}{x^2})^{1/4} \exp\left(i \frac{\tan^{-1}(\frac{1}{x^2}) - \pi}{2}\right), e^{-\frac{i\pi}{2}} = -i$$

$$= (x^2 + \frac{1}{x^2})^{1/4} \exp\left(i \frac{\tan^{-1}(\frac{1}{x^2})}{2}\right) (-i)$$

behaves same as  $x \rightarrow \infty$  when  $x \rightarrow -\infty$

However, multiply curve by  $-i$  rotates CW  $90^\circ$

§ 1.4 #2f continued



from  $f_2(z) = -\sqrt{z}$

(I tried to show where the  $x > 0$  vs.  $x < 0$  branch of the  $y = \sqrt{x}$  graph maps to)

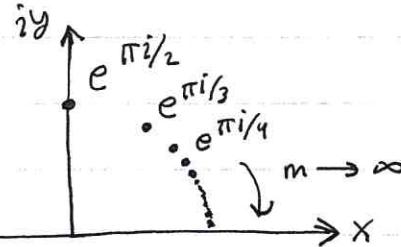
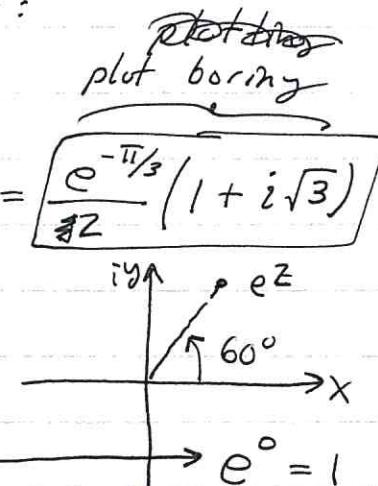
[PROBLEM 8] § 1.5 # 1(c, d, f) Calculate & plot  $e^z$  for:

§ 1.5 #1c)  $z = \frac{\pi(i-1)}{3} = -\frac{\pi}{3} + \frac{\pi i}{3}$

$$e^z = e^{-\frac{\pi}{3}} e^{\frac{\pi i}{3}} = e^{-\pi/3} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = \boxed{\frac{e^{-\pi/3}}{2}(1 + i\sqrt{3})}$$

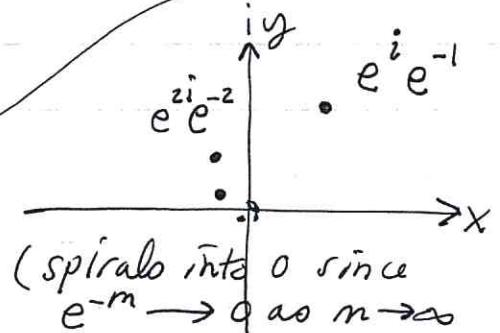
§ 1.5 #1d)  $z = \frac{\pi i}{m}$ ,  $m = 1, 2, 3, \dots$  *plot not so boring*  $\circlearrowleft$

$$e^z = e^{\frac{\pi i}{m}} = e^{\pi i}, e^{\frac{\pi i}{2}}, e^{\frac{\pi i}{3}}, e^{\frac{\pi i}{4}}, \dots$$



§ 1.5 #1f)  $z = m(i-1)$ ,  $m = 1, 2, 3, \dots$

$$e^z = e^{-m} e^{mi}$$

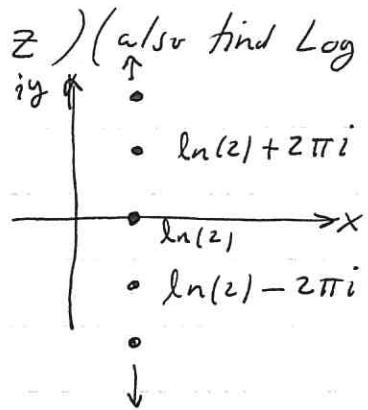


PROBLEM 9 (S1.6 #1) (find & plot  $\log(z)$  for  $z$ ) (also find  $\text{Log}(z)$ )

$$(a.) \log(z) = \text{Log}(z) + 2\pi i \mathbb{Z}$$

$$= [\ln(z) + 2\pi i \mathbb{Z}]$$

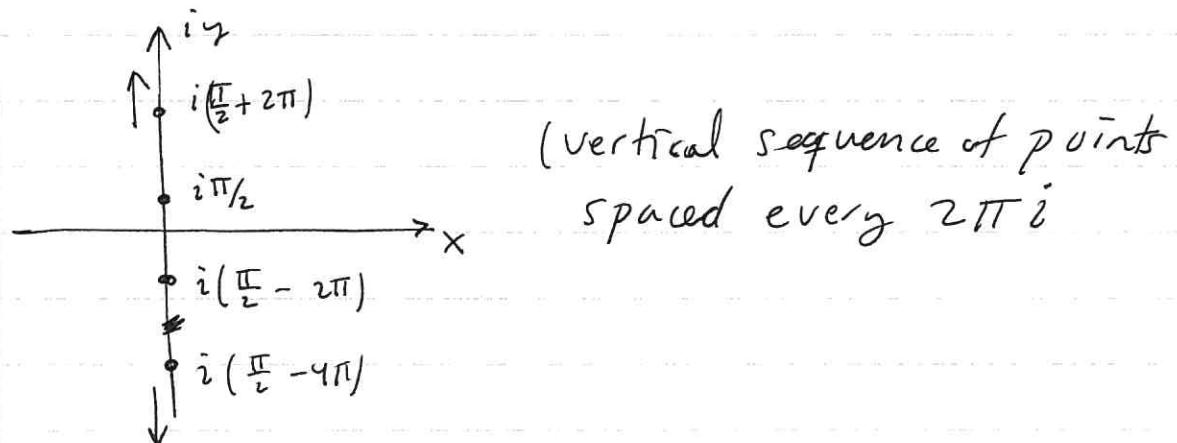
$$[\text{Log}(z) = \ln(z) : \text{principal log}]$$



$$(b.) \log(i) = \text{Log}(i) + 2\pi i \mathbb{Z}$$

$$= \text{Log}(e^{i\pi/2}) + 2\pi i \mathbb{Z}$$

$$= [i\pi/2 + 2\pi i \mathbb{Z}] \quad \notin [\text{Log}(i) = i\pi/2]$$



$$(c.) \log(1+i) = \text{Log}(\sqrt{2}e^{i\pi/4}) + 2\pi i \mathbb{Z}$$

$$= [\ln\sqrt{2} + \frac{i\pi}{4} + 2\pi i \mathbb{Z}] \quad \notin \text{Log}(1+i) =$$

Naturally  $\text{Log}(\sqrt{2}e^{i\pi/4}) = [\ln\sqrt{2} + i\frac{\pi}{4} = \text{Log}(z)]$

plot is similar.

$$(d.) \log\left(\frac{1+i\sqrt{3}}{2}\right) = \text{Log}\left(e^{i\pi/3}\right) + 2\pi i \mathbb{Z}$$

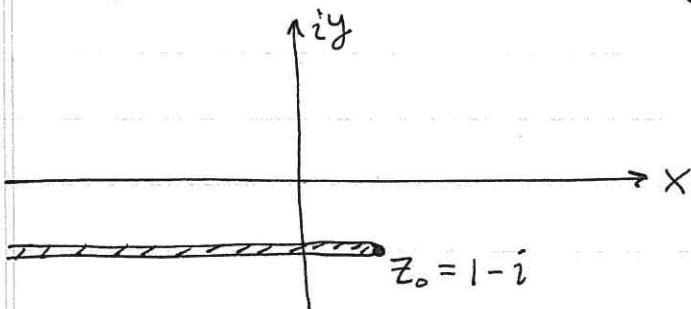
$$\text{Log}\left(\frac{1+i\sqrt{3}}{2}\right) = \frac{i\pi}{3} + 2\pi i \mathbb{Z}$$

$$\text{Log}\left(\frac{1+i\sqrt{3}}{2}\right) = \frac{i\pi}{3}$$

Sorry, this  
got a little  
boring towards  
the end.

PROBLEM 10 ( $\S 1.6 \# 4$ ) How would you make a branch cut to define single-valued branch of function  $\log(z+i-1)$ ? What about  $\log(z-z_0)$ ?

Notice,  $z = 1-i$  gives  $\log(z+i-1)$  trouble. It is analogous to  $z=0$  for  $\log(z)$  so...



Let  $f(z) = \text{Log}(z+i-1) \Leftrightarrow$  requires  $z+i-1 \in \mathbb{C}$   
 This means  $z \in \mathbb{C}$  such that  $\text{Im}(z+i-1) = 0$  and  
 $\text{Re}(z+i-1) < 0$  is not allowed (these are the deleted pts.)  
 $\text{Im}(x+iy+i-1) = y+1 = 0 \Rightarrow y = -1$ .  
 $\text{Re}(x+iy+i-1) = x-1 < 0 \Rightarrow x < 1$ .

Therefore,  $\boxed{\text{dom}(f) = \mathbb{C} - (\infty - i, 1 - i)}$ . For arbitrary  $z_0$ , we could use  $f(z) = \text{Log}(z-z_0)$  or as developed in notes,  $f(z) = \text{Log}_\alpha(z-z_0)$ .

*Remark:*

PROBLEM 11 ( $\S 1.8 \# 3$ ) Find all zeros and periods of  $\cosh z$  &  $\sinh z$ .

Notice first,

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}) \quad \& \quad \sinh(z) = \frac{1}{2}(e^z - e^{-z})$$

are built from  $\exp(z)$  for which  $\exp(z+2\pi ik) = \exp(z)$

$\forall k \in \mathbb{Z}$ . It follows  $\cosh(z+2\pi ik) = \cosh z$  and  $\sinh(z+2\pi ik) = \sinh(z) \quad \forall k \in \mathbb{Z}$ .

To find zeros, it helps to set  $z = x+iy$  and find explicit formulas for  $\cosh(z)$  &  $\sinh(z)$



PROBLEM 11 continued

$$\begin{aligned}
 \cosh(x+iy) &= \frac{1}{2} [e^{x+iy} + e^{-(x+iy)}] \\
 &= \frac{1}{2} [e^x e^{iy} + e^{-x} e^{-iy}] \\
 &= \frac{1}{2} [(\cosh x + \sinh x)e^{iy} + (\cosh x - \sinh x)e^{-iy}] \\
 &= \frac{1}{2} \cosh x (e^{iy} + e^{-iy}) + \frac{i}{2} \sinh x (e^{iy} - e^{-iy}) \\
 &= \frac{1}{2} \cosh x \cos(y) + i \sinh x \sin(y)
 \end{aligned}$$

For  $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y = 0$

We must solve simul/tan easily

$$\cosh x \cos(y) = 0 \Rightarrow \cancel{\cos x \neq 0} \quad y = \frac{\pi}{2}(2k+1).$$

$$\sinh x \sin(y) = 0 \Rightarrow \pm \sinh(x) = 0 \Rightarrow x = 0.$$

We find ~~not infinitely many zeros,~~

$$\boxed{\cosh^{-1}\{0\} = \left\{ \frac{\pi i}{2}(2k+1) \mid k \in \mathbb{Z} \right\}}$$

We can derive similar identity for  $\sinh(z)$

one nice trick ;  $\sinh(x+iy) = \frac{\partial}{\partial x} [\cosh(x+iy)]$

$$\begin{aligned}
 \sinh(x+iy) &= \frac{\partial}{\partial x} [\cosh x \cos y + i \sinh x \sin y] \\
 &= \underline{\sinh x \cos y + i \cosh x \sin y}.
 \end{aligned}$$

For  $\sinh(z) = 0$  we need :

$$\sinh x \cos y = 0 \quad \& \quad \cosh x \sin y = 0$$

Notice,  $\cosh x \neq 0 \quad \forall x \in \mathbb{R}$  hence  $\sin y = 0$  hence

$y = h\pi$  for some  $h \in \mathbb{Z}$ . Then  $\cos(y) = \cos(h\pi) = (-1)^h$   
 thus  $\sinh(x) = 0 \Rightarrow x = 0$ . We find,

$$\boxed{\sinh^{-1}\{0\} = \{h\pi i \mid h \in \mathbb{Z}\}}$$

Remark: might be fun to think about  $\cosh(iz) = \cos(z) \dots$

PROBLEM 12 I'll begin with a Lemma. Ultimately the subtle points here are all due to  $\text{Arg}(z)$  vs.  $\arg(z)$ .

Lemma: Let  $z \in \mathbb{C}^*$  then

- ①  $\text{Arg}(z) + \text{Arg}(w) = \text{Arg}(zw) + 2\pi k$  for some  $k \in \mathbb{Z}$ .
- ②  $\text{Log}(z) + \text{Log}(w) = \text{Log}(zw) + 2\pi ki$  for some  $k \in \mathbb{Z}$ .

Proof: we know  $z, w \in \mathbb{C}^*$  have  $z = |z| \exp(i \text{Arg}(z))$

and  $w = |w| \exp(i \text{Arg}(w))$  thus,

$$\begin{aligned} zw &= |z| \exp(i \text{Arg}(z)) |w| \exp(i \text{Arg}(w)) \\ &= |z||w| \exp(i \text{Arg}(z) + i \text{Arg}(w)) \\ &= |zw| \exp(i(\text{Arg}(z) + \text{Arg}(w))) \end{aligned}$$

However, we also know  $zw = |zw| \exp(i \text{Arg}(zw))$  hence we find  $\exp(i(\text{Arg}(z) + \text{Arg}(w))) = \exp(i \text{Arg}(zw))$  and by  $2\pi i \mathbb{Z}$  periodicity of exponential we obtain  $k \in \mathbb{Z}$  s.t.

$$\underline{\text{Arg}(z) + \text{Arg}(w) = \text{Arg}(zw) + 2\pi k}.$$

This proves ①. To prove ②, consider

$$\text{Log}(z) = \ln|z| + i \text{Arg}(z)$$

$$\text{Log}(w) = \ln|w| + i \text{Arg}(w)$$

add the relations above and use  $\ln|z| + \ln|w| = \ln|z||w| = \ln|zw|$ ,

$$\text{Log}(z) + \text{Log}(w) = \ln|zw| + i(\text{Arg}(z) + \text{Arg}(w))$$

Now from ① we know  $\exists k \in \mathbb{Z}$  such that  $\text{Arg}(z) + \text{Arg}(w) = \text{Arg}(zw) + 2\pi k$ . Consequently,

$$\text{Log}(z) + \text{Log}(w) = \ln|zw| + i \text{Arg}(zw) + 2\pi ik$$

$$\therefore \underline{\text{Log}(z) + \text{Log}(w) = \text{Log}(zw) + 2\pi ik. //}$$

We will use these facts to complete #4 of §1.7.

$$\log(z) = \text{Log}(z) + 2\pi i \mathbb{Z}.$$

PROBLEM 12 continued :  $z^a = \exp(a \log(z))$  gives ②

§1.7 #4  $z^a = \{ \exp(a \log(z) + 2\pi i m) / m \in \mathbb{Z} \}$

$$w^a = \{ \exp(a \text{Log}(w) + 2\pi i n) / n \in \mathbb{Z} \}$$

Let  $x \in z^a$  and  $y \in w^a$  then  $\exists m, n \in \mathbb{Z}$  such that

$$x = \exp(a \log(z) + 2\pi i m a)$$

$$y = \exp(a \log(w) + 2\pi i n a)$$

Thus,

$$xy = \exp(a \log(z) + 2\pi i m a) \exp(a \log(w) + 2\pi i n a)$$

$$= \exp(a \log(z) + a \log(w) + 2\pi i m a + 2\pi i n a)$$

$$= \exp(a(\log(z) + \log(w)) + 2\pi i(m+n)a)$$

$$= \exp(a(\text{Log}(zw) + 2\pi i k) + 2\pi i(m+n)a) \text{ by Lemma ②}$$

$$= \exp(a \text{Log}(zw) + 2\pi i(k+m+n)a)$$

Hence  $xy \in (zw)^a$  and we have shown

$$z^a w^a \subseteq (zw)^a. \text{ Conversely suppose } \tilde{x} \in (zw)^a \text{ then}$$

$$\exists k \in \mathbb{Z} \text{ s.t. } \tilde{x} = \exp(a \log(zw) + 2\pi i a k)$$

and by Lemma ②  $\exists m \in \mathbb{Z}$  s.t.

$$\tilde{x} = \exp(a(\log(z) + \log(w) + 2\pi i m) + 2\pi i a k)$$

$$= \underbrace{\exp(a \log(z))}_{\in z^a} \underbrace{\exp(a \log(w) + 2\pi i(m+k)a)}_{\in w^a}$$

Thus  $\tilde{x} \in z^a w^a$  as  $\tilde{x}$  is a product of elements in  $z^a$  and  $w^a$ . Thus  $(zw)^a \subseteq z^a w^a$ . We conclude  $\underline{z^a w^a = (zw)^a}$ .

Remark: Notice, this is false if we fail to think of

$z^a$  &  $w^a$  as sets. If we used principal branch

$$z^a = \exp(a \text{Log}(z)) \text{ and } w^a = \exp(a \text{Log}(w)) \text{ then}$$

$$z^a w^a = \exp(a(\log(z) + \log(w))) \neq \exp(a \log(zw)) \text{ (in some cases)}$$

try  $a = \frac{1}{k}$   
and  $z = -1, w = -1$

(PROBLEM 12) Let  $z \in \mathbb{C}^- = \mathbb{C} - (-\infty, 0]$ .

Define  $z^p = \exp(p \operatorname{Log}(z))$  for what remains below,  
Likewise for  $q \in \mathbb{C}$  let  $z^q = \exp(q \operatorname{Log}(z))$ . We  
calculate,

$$\begin{aligned} z^p z^q &= \exp(p \operatorname{Log}(z)) \exp(q \operatorname{Log}(z)) \\ &= \exp(p \operatorname{Log}(z) + q \operatorname{Log}(z)) \quad \textcircled{I} \end{aligned}$$

On the other hand, we also have  $z^{p+q}$  defined as:

$$\begin{aligned} z^{p+q} &= \exp((p+q) \operatorname{Log}(z)) \\ &= \exp(p \operatorname{Log}(z) + q \operatorname{Log}(z)) \rightarrow \text{by } \textcircled{I} \\ &= z^p z^q \end{aligned}$$

Hence  $\underline{z^p z^q = z^{p+q}}$   $\forall p, q \in \mathbb{C}$  given  $z^p$  denotes  
the principal power function given by  $\operatorname{Log}$ . It's  
clear this works the same for  $z^p = \exp(p \operatorname{Log}_\alpha(z))$ .