

Mission 4 SOLUTION

PROBLEM 38 Find FLT mapping $1, i, -1$ to $\infty, 1, 0$

$$f(z) = A \left(\frac{z+1}{z-1} \right) \text{ has } f(1) = \infty \text{ & } f(-1) = 0$$

it remains to set $f(i) = 1$,

$$A \left(\frac{i+1}{i-1} \right) = 1 \Rightarrow A = \frac{i-1}{i+1}$$

Of course, we can clean-up A ,

$$A = \left(\frac{i-1}{i+1} \right) \left(\frac{-i+1}{-i+1} \right) = \frac{-i^2 + i + i - 1}{2} = \frac{2i}{2} = i \text{ funny!}$$

$$f(z) = \frac{(i-1)(z+1)}{(i+1)(z-1)}$$

Remark: cross-ratio method is another way we could have attacked this.

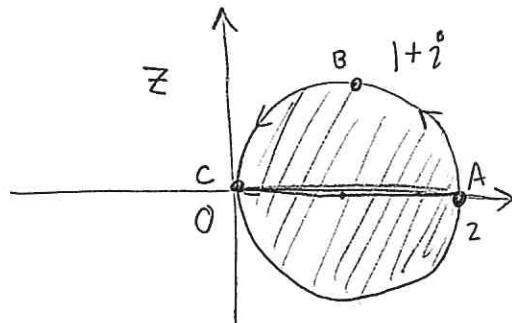
$$f(z) = \frac{i(z+1)}{z-1}$$

PROBLEM 39 ($\S 2.7 \#2$)

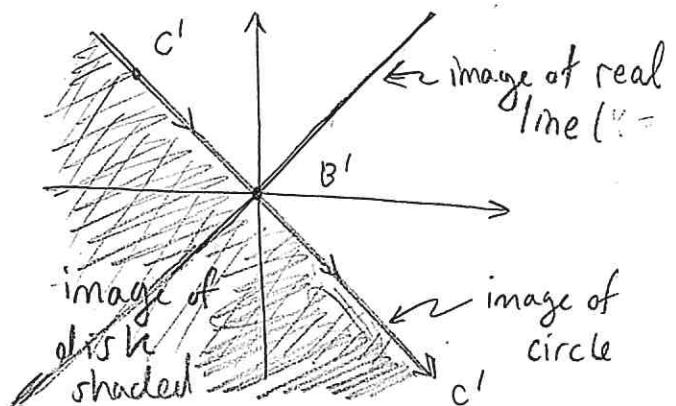
Consider $\begin{array}{ccc} 1+i & \xrightarrow{f} & 0 \\ 2 & \xrightarrow{f} & \infty \\ 0 & \xrightarrow{f} & -1 \end{array}$

w/o using formula for $f(z)$, determine image of circle $|z-1|=1$ and disk $|z-1|<1$ and image of R -axis.

Notice $1+i, 2, 0$ solve $|z-1|=1$ hence fall on the circle



Notice $[0, 2]$ is orthogonal to the half-circle hence the image of $[0, 2]$ must be orthog. to the half-circle.



PROBLEM 40 (§ 3.1 #6, p. 75)

Let P, Q be continuous \mathbb{C} -valued functions on curve γ

Show $F(w) = \int_{\gamma} \frac{Pdx + Qdy}{z-w}$ is analytic for $w \in \mathbb{C} - \gamma$.

Express $F'(w)$ as a line-integral over γ

$$\begin{aligned}\frac{F(w+h) - F(w)}{h} &= \left(\int_{\gamma} \frac{Pdx + Qdy}{z-w-h} - \int_{\gamma} \frac{Pdx + Qdy}{z-w} \right) \frac{1}{h} \\ &= \frac{1}{h} \int_{\gamma} Pdx + Qdy \left(\frac{1}{z-w-h} - \frac{1}{z-w} \right) \\ &= \frac{1}{h} \int_{\gamma} Pdx + Qdy \left(\frac{z-w - (z-w-h)}{(z-w-h)(z-w)} \right) \\ &= \int_{\gamma} Pdx + Qdy \left[\frac{1}{h} \frac{h}{(z-w-h)(z-w)} \right] \\ &= \int_{\gamma} \frac{Pdx + Qdy}{(z-w-h)(z-w)}\end{aligned}$$

Hence,

$$F'(w) = \lim_{h \rightarrow 0} \int_{\gamma} \frac{Pdx + Qdy}{(z-w-h)(z-w)} = \int_{\gamma} \frac{Pdx + Qdy}{(z-w)^2}$$

(we exchanged order of $\lim_{h \rightarrow 0}$ and \int_{γ} which is allowed as $\frac{1}{(z-w-h)(z-w)} \xrightarrow{h \rightarrow 0} \frac{1}{(z-w)^2}$ uniformly)

as $h \rightarrow 0.$)

(There is another way, sketch below:-)

Let $w = x_1 + ix_2$ we need $\partial_{x_1} F = -i \partial_{x_2} F$ and continuity.

$$\begin{aligned}\frac{\partial}{\partial x_1} \int_{\gamma} \frac{Pdx + Qdy}{z-w} &= \int_{\gamma} \frac{\partial}{\partial x_1} \left[\frac{P}{z-w} \right] dx + \frac{\partial}{\partial x_1} \left[\frac{Q}{z-w} \right] dy \\ &= \int_{\gamma} \frac{-Pdx + Qdy}{(z-w)^2} \frac{\partial w}{\partial x_1} = \int_{\gamma} \frac{Pdx + Qdy}{(z-w)^2}\end{aligned}$$

Likewise $\frac{\partial}{\partial x_2} \int_{\gamma} \frac{Pdx + Qdy}{z-w} = \int_{\gamma} \frac{Pdx + Qdy}{(z-w)^2} \frac{\partial w}{\partial x_2} = i \int \frac{Pdx + Qdy}{(z-w)^2}$

PROBLEM 41 (S3.2 #1 pg. 82)

(a.) $x dx + y dy = d \left(\underbrace{\frac{1}{2}x^2 + \frac{1}{2}y^2}_h \right).$

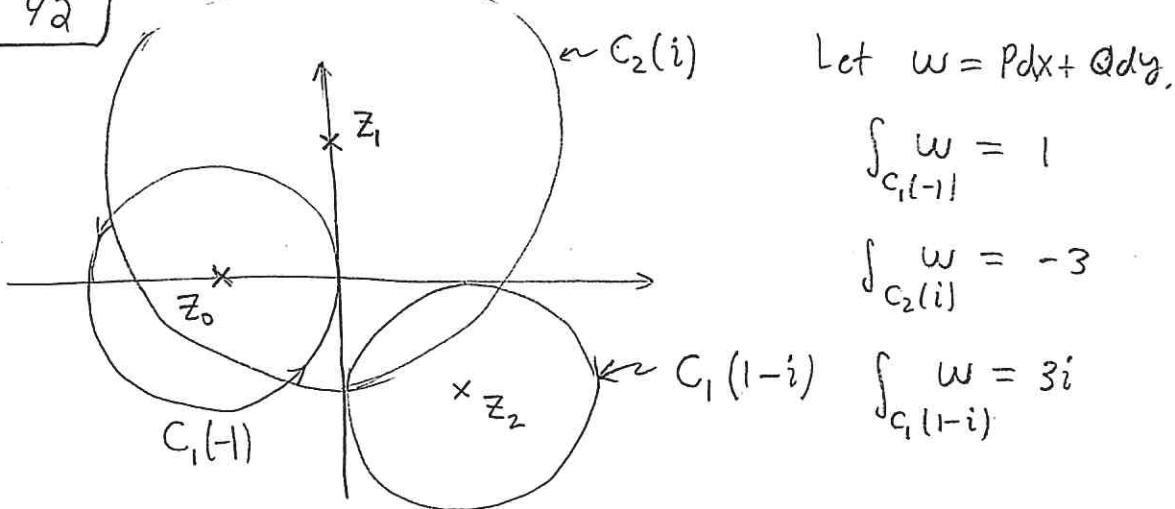
(b.) $x^2 dx + y^5 dy \Rightarrow h = \underbrace{\frac{1}{3}x^3 + \frac{1}{6}y^6}_.$

(c.) $y dx + x dy \Rightarrow h = xy.$

(d.) $y dx - x dy$ is not closed as $\frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial x}(-x).$

$$\begin{aligned} \oint_{|z|=1} y dx - x dy &= \int_0^{2\pi} (\sin t)(-\sin t dt) - (\cos t)(\cos t dt) && ; z = \cos t + i \sin t \\ &= - \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt && dx = -\sin t dt \\ &= - \int_0^{2\pi} dt && dy = \cos t dt \\ &= -2\pi. && \text{(other loops will also work)} \end{aligned}$$

PROBLEM 42



Any $C_R(\alpha)$ which encloses z_0 picks up 1. Notice for $\epsilon < 1$ we deduce $\int_{\gamma_\epsilon(z_1)} w = -4$ from $\int_{C_{1(-1)}} w = 1$ and the deformation theorem.

$$\int_{C_R(\alpha)} Pdx + Qdy = \begin{cases} 0 & \text{if } 0 < R < 1 \\ -3 & \text{if } 1 < R < \sqrt{2} \\ -3 + 3i & \text{if } R > \sqrt{2} \end{cases}$$

PROBLEM 43 (§ 3.3 #2)

Show that complex-valued fnct. $h(z)$ on star-shaped domain D is harmonic $\Leftrightarrow h(z) = f(z) + \overline{g(z)}$ where $f(z), g(z) \in \mathcal{O}(D)$

\Leftarrow Suppose $f(z), g(z) \in \mathcal{O}(D)$ and let $h(z) = f(z) + \overline{g(z)}$ where we know D is star-shaped. Let $f = u+iv$ and $g = a+ib$ hence $h = u+iv+a-ib = (u+a)+i(v-b)$

$$h_{xx} = u_{xx} + a_{xx} + i(v_{xx} - b_{xx})$$

$$h_{yy} = u_{yy} + a_{yy} + i(v_{yy} - b_{yy})$$

Recall $u+iv, a+ib \in \mathcal{O}(D) \Rightarrow u, v, a, b$ harmonic, therefore:

$$h_{xx} + h_{yy} = u_{xx} + u_{yy} + a_{xx} + a_{yy} + i(v_{xx} + v_{yy} - (b_{xx} + b_{yy})) = 0.$$

\Rightarrow Suppose D is star-shaped and $h(z)$ is complex-valued fnct. such that $h_{xx} + h_{yy} = 0$. Let $h = u+iv$ and observe $(u_{xx} + iv_{xx}) + (u_{yy} + iv_{yy}) = 0 \Rightarrow u_{xx} + u_{yy} = 0 \text{ & } v_{xx} + v_{yy} = 0$.

Hence u is harmonic on a star-shaped domain.

Therefore, $\exists b: D \rightarrow \mathbb{R}$ such that $u+ib \in \mathcal{O}(D)$.

Likewise v harmonic on $D \Rightarrow \exists$ harmonic conjugate a for which $v+ia \in \mathcal{O}(D)$. Consider,

$$h = u+iv \quad \dots \quad (\text{unfinished})$$

PROBLEM 44 ($\S 3.5 \# 2, 3$)

a.) Fix $n \geq 1$, $r > 0$, and $\lambda = \rho e^{i\phi}$. What is the max. mod. of $z^n + \lambda$ over the disk $D = \{z \mid |z| \leq r\}$? Where does $z^n + \lambda$ attain its maximum modulus on D ?

Consider $f(z) = |z^n + \lambda| \leq |z^n| + |\lambda| = |z|^n + |\lambda| \leq r^n + \rho$

Let us attempt to solve $|z^n + \lambda| = r^n + \rho$ for $z \in \partial D$.

$$\text{Let } z = re^{i\theta} \Rightarrow |r^n e^{ino} + \rho e^{i\phi}| = r^n + \rho$$

need $e^{ino} = e^{i\phi}$ for this.

$$\Rightarrow n\theta = \phi + 2\pi k$$

$$\Rightarrow \theta = \frac{\phi}{n} + \frac{2\pi k}{n}$$

$$\Rightarrow z = r e^{i\phi/n} e^{\frac{2\pi ki}{n}}$$

Consider $|z| = R$ in what follows, (here is the ugliness of direct attack)

$$f(z) = |z^n + \lambda|^2 \Rightarrow f(r, \theta) = |r^n e^{ino} + \lambda|^2 \stackrel{\text{via calc. III}}{=} \dots$$

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r} [(r^n e^{ino} + \lambda)(r^n e^{-ino} + \bar{\lambda})]$$

$$= \frac{\partial}{\partial r} [r^{2n} + \lambda r^n e^{-ino} + \bar{\lambda} r^n e^{ino} + \lambda \bar{\lambda}]$$

$$= 2nr^{2n-1} + nr^{n-1}(\lambda e^{-ino} + \bar{\lambda} e^{ino})$$

$$\frac{\partial f}{\partial \theta} = -in\lambda r^n e^{-ino} + in\bar{\lambda} r^n e^{ino} = inr^n(\bar{\lambda} e^{ino} - \lambda e^{-ino})$$

Method of Lagrange Multipliers for circle $x^2 + y^2 = R^2$

$$\nabla f = \beta \nabla g \Rightarrow \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} = 2r\beta \hat{r} \quad \text{or } r^2 = R^2$$

$$\text{Need } \frac{\partial f}{\partial \theta} = 0 \Rightarrow \bar{\lambda} e^{ino} = \lambda e^{-ino} \Rightarrow e^{-i\phi} e^{ino} = e^{i\phi} e^{-ino}$$

$$\text{Hence, } e^{2i\phi} = e^{2ino} \Rightarrow n\theta = \phi + 2\pi k \Rightarrow \boxed{\theta = \frac{\phi + 2\pi k}{n}}$$

PROBLEM 44 Continued

[§ 3.5 #3] Use max-principle to prove Fundamental Th^m of Algebra.

Suppose $n \geq 1$ and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$.

Suppose $P(z) \neq 0 \quad \forall z \in \mathbb{C}$ towards a contradiction.

Notice that we may factor out z^n for $z \neq 0$,

$$P(z) = \left(a_n + \underbrace{\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}}_{\text{call this } w} \right) z^n$$

call this w
notice it has n -terms

Choose R large enough that $|z| \geq R$ implies

$$\left| \frac{a_0}{z^n} \right|, \left| \frac{a_1}{z^{n-1}} \right|, \dots, \left| \frac{a_{n-1}}{z} \right| < \frac{|a_n|}{z^n}$$

Remark: See
Churchill Ed. 6
pg. 131

It follows we may bound w ,

$$|w| \leq \left| \frac{a_{n-1}}{z} \right| + \left| \frac{a_{n-2}}{z^2} \right| + \dots + \left| \frac{a_1}{z^{n-1}} \right| + \left| \frac{a_0}{z^n} \right| < n \left(\frac{|a_n|}{z^n} \right) = \frac{|a_n|}{z}$$

Observe,

$$|a_n + w| \geq | |a_n| - |w| | > \frac{|a_n|}{2}$$

Thus, for $|z| \geq R$ yet again,

$$|P(z)| = |a_n + w| |z^n| > \frac{|a_n|}{2} |z|^n \geq \frac{|a_n|}{2} R^n$$

Let $f(z) = \frac{1}{P(z)}$. Note $f(z)$ is holomorphic on \mathbb{C} as we assume $P(z) \neq 0 \quad \forall z \in \mathbb{C}$. Consider the max-principle applied to $|z|=R$ for R as described above, note,

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n| R^n} = L_R$$

Thus $|f(z)| \leq L_R$ for all $z \in \mathbb{C}$ with $|z| \leq R$. Furthermore, as $R \rightarrow \infty$ we see $L_R \rightarrow 0$ and hence $|f(z)| \leq 0$

$$\Rightarrow f(z) = \frac{1}{P(z)} = 0 \quad \text{which is impossible}$$

thus our assumption $P(z) \neq 0 \quad \forall z \in \mathbb{C}$ is false and we find \exists at least one zero for $P(z)$.

Remark: Liouville's Th^m: Every bounded entire function is constant is another nice way to end the argument above.

PROBLEM 45 § 3.6 #4 AND work through Example on pg. 95

in case $\alpha = \pi/3$. Find the flow, potential and stream lines explicitly and sketch them in the appropriate sector. Let

$g(z) = \log(z+1)$, find the flow created by mapping the $\alpha = \pi/3$ flow through the injective holomorphic mapping g . See pg. 94 for the idea.

$$\underbrace{h(z) = z^{\frac{\pi}{\alpha}} = z^{\frac{\pi}{(\pi/3)}}}_{\text{complex velocity potential}} = \boxed{z^3 = (r e^{i\theta})^3 = r^3 (\cos 3\theta + i \sin 3\theta)}$$

Potential

$$\text{stream function} = \operatorname{clm}(h(z)) = \boxed{r^3 \sin(3\theta) = \psi} \quad \frac{\text{stream}}{\text{function}}$$

As Gamelin comments, $\psi = 0$ along $\theta = \pi/3$, and $\theta = 0$.

Setting $\psi = C$

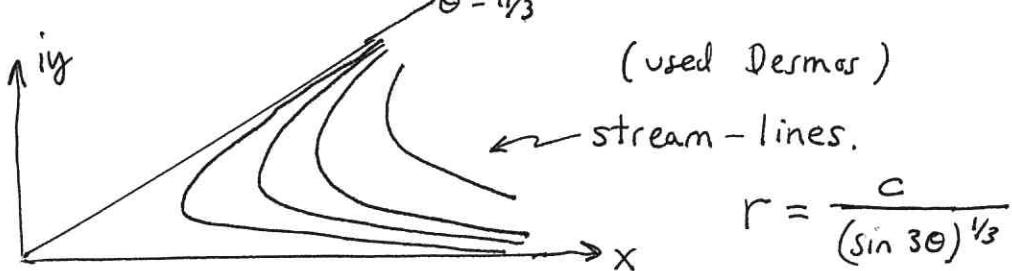
$$r^3 \sin(3\theta) = C \rightarrow r = \sqrt[3]{\frac{C}{\sin 3\theta}} = C (\sin 3\theta)^{-1/3}$$

(I think $r = c(\sin(\frac{\pi\theta}{3}))^{-\frac{1}{3}}$
has typo on p. 95)

The flow $V(z) = h'(z) = 3z^2$

$$|V(z)| = |3z^2| = 3r^2 \text{ slow down as } r \rightarrow 0$$

Q. The flow is $\boxed{V(z) = 3(x^2 - y^2) + 6ixy} = \boxed{3r^2(\cos 2\theta + i \sin 2\theta)}$.



PROBLEM 4.5 continued

If $h: D \rightarrow U$ is one-to-one holomorphic map
and if $f_0(w) = \phi_0(w) + i\psi_0(w)$ is complex velocity potential on U
then $f(z) = f_0(h(z))$ is complex velocity potential on D .

We have $f_0(w) = w^3$ and $\underline{h(z)} = \text{Log}(z+1)$
~~in pg. 94 terms.~~

$$\Rightarrow f(z) = [\text{Log}(z+1)]^3$$

$$\Rightarrow V_{new}(z) = f'(z) = \boxed{\frac{3 \text{Log}^2(z+1)}{z+1}}$$

§3.6#4 — (unfinished) —

PROBLEM 46 (§ 3.7 #4)

Find steady-state heat distribution in laminar plate corresponding to vertical half-strip $\{z \mid |\operatorname{Re}(z)| < \pi/2, y = \operatorname{dm}(z) > 0\} = \text{VHS}$ where $x = \pm \pi/2$ are held at temperature $T_0 = 0$.

and the bottom edge $(-\pi/2, \pi/2)$ held at $T_b = 100$

Make rough sketch of isothermal curves and lines of heat flow.

Hint : use $w = \sin z$ to map strip to $H = \{x+iy \mid y \geq 0\}$ and make use of harmonic functions $\operatorname{Arg}(w-a)$

$$\text{Answer : } \phi = 50 [\operatorname{Arg}(\sin z + i) - \operatorname{Arg}(\sin z - i)]$$

$$\text{Consider, } \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$\Rightarrow \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\text{For } x+iy \in \text{VHS} \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } y > 0$$

$$\Rightarrow -1 \leq \frac{\sin x}{\cos(x)} \leq 1 \text{ and } \sinh(y) > 0$$

$$\Rightarrow \sin(x+iy) = \underbrace{\sin(x)\cosh(y)}_{\text{ranges over IR}} + i\underbrace{\cos(x)\sinh(y)}_{\text{ranges over } (0, \infty)}$$

Now $z \mapsto \sin z$ is conformal, so, as $\frac{d}{dz}(\sin z) = \cos z \neq 0$

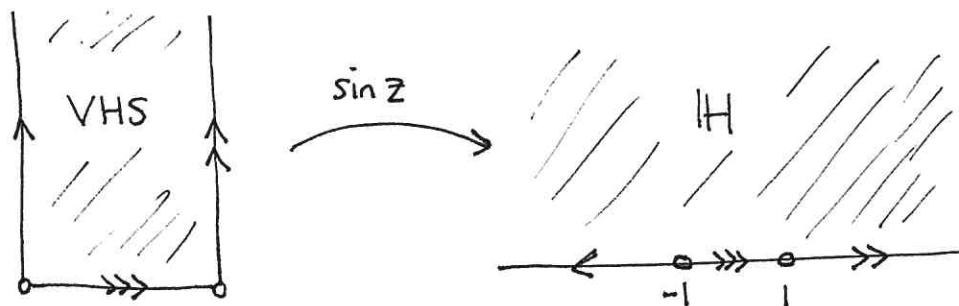
for $z \in \text{int(VHS)}$. Actually for $\partial(\text{VHS})$ we have $\cos(\pm \frac{\pi}{2}) = 0$

But, away from those corner points, we have conformal mapping.

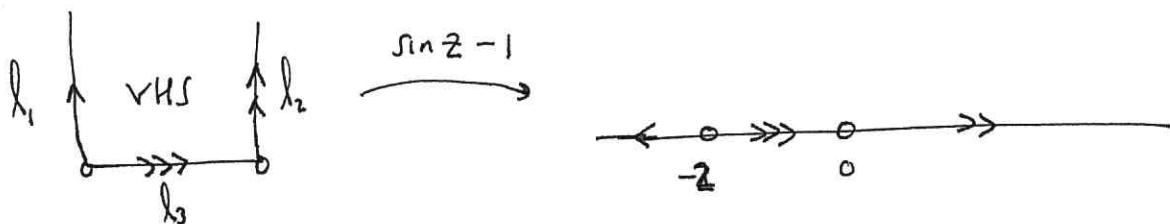
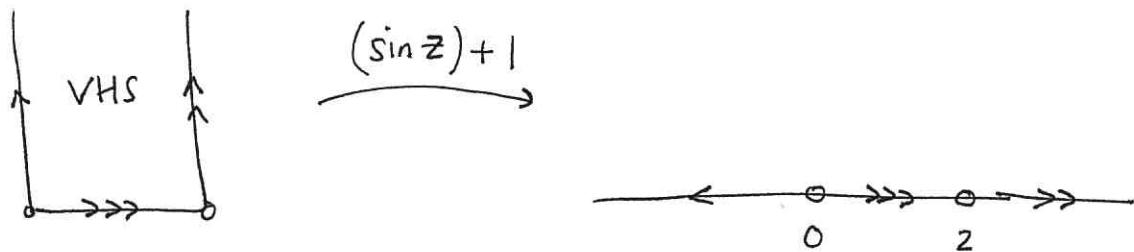
$$\sin(-\frac{\pi}{2}, \frac{\pi}{2}+i\infty) = (-1, -\infty)$$

$$\sin(\frac{\pi}{2}, \frac{\pi}{2}+i\infty) = (1, \infty)$$

$$\sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1, 1)$$



PROBLEM 46 continued



Ok,

$$\begin{cases} \sin\left(-\frac{\pi}{2}, -\frac{\pi}{2}+i\infty\right) + 1 = (0, -\infty) \rightarrow \text{Arg gives } +\pi \\ \sin\left(\frac{\pi}{2}, \frac{\pi}{2}+i\infty\right) + 1 = (2, \infty) \rightarrow \text{Arg gives } 0 \\ \sin\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) + 1 = (0, 2) \end{cases}$$

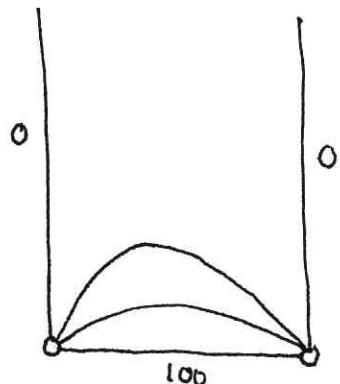
$$\begin{cases} \sin\left(-\frac{\pi}{2}, -\frac{\pi}{2}+i\infty\right) - 1 = (-2, -\infty) \\ \sin\left(\frac{\pi}{2}, \frac{\pi}{2}+i\infty\right) - 1 = (0, \infty) \quad \text{Arg of } \pi \\ \sin\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - 1 = (-2, 0) \quad \text{Arg of } 0 \end{cases}$$

Thus,

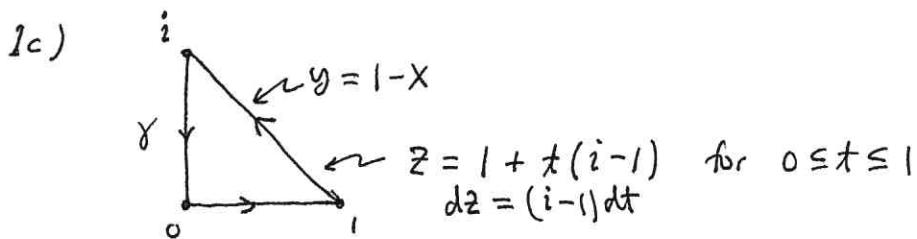
$$\text{Arg}[\sin(z)+1] - \text{Arg}[\sin(z)-1] = \begin{cases} 0 & : z \in l_1 \\ 0 & : z \in l_2 \\ \pi & : z \in l_3 \end{cases}$$

Hence, the solⁿ is a bit off, we should set

$$\boxed{T(z) = \frac{100}{\pi} (\text{Arg}(\sin(z)+1) - \text{Arg}(\sin(z)-1))}$$



PROBLEM 47 (§ 4.1 # 1c, 2b, 3c)



$$\begin{aligned}
 \int_{\gamma} z dz &= \int_0^i x dx + \int_0^1 ((1+t(i-1))(i-1)) dt - \int_0^1 (iy)(idy) \\
 &= \frac{1}{2} + i-1 + (i-1)^2 \int_0^1 t dt + \int_0^1 y dy \\
 &= \cancel{\frac{1}{2}} + i-1 + [(i^2 - i - i + 1)] \frac{1}{2} + \cancel{\frac{1}{2}} \\
 &= i - i \\
 &= \boxed{0}.
 \end{aligned}$$

(well, that's a relief, we know $F(z) = \frac{1}{2}z^2$ is a primitive for $f(z) = z$ on the triangle hence it ought to be zero)
 (by now we know several reasons...)

Let $m \in \mathbb{Z}$,

$$\begin{aligned}
 2b.) \quad \int_{\gamma} \bar{z}^m dz &= \int_{|z|=1} \bar{z}^m dz = \int_0^{2\pi} (\overline{e^{i\theta}})^m i e^{i\theta} d\theta \\
 &= \int_0^{2\pi} (e^{-i\theta})^m i e^{i\theta} d\theta \\
 &= \int_0^{2\pi} e^{-mi\theta} e^{i\theta} id\theta \\
 &= i \int_0^{2\pi} \exp(i(1-m)\theta) d\theta \quad \curvearrowright m \neq 1 \\
 &= i \left. \frac{1}{i(1-m)} e^{i(1-m)\theta} \right|_0^{2\pi} \\
 &= \frac{1}{1-m} (e^{i(1-m)2\pi} - 1) \\
 &= \frac{1}{1-m} (0) \\
 &= 0.
 \end{aligned}$$

If $m=1$ then $\int_{\gamma} \bar{z} dz = i \int_0^{2\pi} d\theta = \boxed{2\pi i}$

PROBLEM 47 continued

3c) Let $\gamma: |z|=R$ with CCW-orientation, and $m \in \mathbb{Z}$,

$$\begin{aligned} \int_{\gamma} \bar{z}^m dz &= \int_0^{2\pi} (\overline{Re^{i\theta}})^m iRe^{i\theta} d\theta && : z = Re^{i\theta} \text{ parametrizes} \\ &= \int_0^{2\pi} R^{m+1} e^{i(1-m)\theta} d\theta && |z|=R \text{ with CCW direction} \\ &= \begin{cases} 2\pi i R^{m+1} & \text{for } m=1 \\ 0 & \text{for } m \neq 1 \end{cases} && \text{for } 0 \leq \theta \leq 2\pi \\ &= \begin{cases} 2\pi i R^2 & \text{for } m=1 \\ 0 & \text{for } m \neq 1 \end{cases} && \text{Same as 2b.} \\ &&& \text{Sorry so boring.} \end{aligned}$$

unless I've made
a silly error, it
seems the text
has error here.
(it says $2\pi i R$)

PROBLEM 48 (S 4.1 #6): $\left| \oint_{|z|=R} \frac{\operatorname{Log}(z)}{z^2} dz \right| \leq 2\sqrt{2}\pi \frac{\ln(R)}{R}$ for $R > e^\pi$

Let $\bar{z} = Re^{i\theta}$ then $\operatorname{Log}(\bar{z}) = \operatorname{Log}(Re^{i\theta}) = \ln(R) + i\theta$

$$\text{then } \left| \frac{\operatorname{Log}(z)}{z^2} \right| = \left| \frac{\ln(R) + i\theta}{z^2} \right| = \frac{|\ln R + i\theta|}{R^2} = \frac{|\ln(R) + i\ln(e^\theta)|}{R^2}$$

$$\text{Hence } \left| \frac{\operatorname{Log}(z)}{z^2} \right| = \frac{1}{R^2} \sqrt{\ln(R)^2 + \ln(e^\theta)^{2\pi}} \leq \frac{1}{R^2} \sqrt{\ln^2(R) + \ln^2(R)} \quad \xrightarrow{\text{provided } e^\theta \leq e^\pi < R.}$$

Therefore, for $R > e^\pi$ we find for $|z|=R$,

$$\left| \frac{\operatorname{Log}(z)}{z^2} \right| \leq \frac{\sqrt{2}\ln(R)}{R^2}$$

Apply the ML-estimate for $R > e^\pi$ to obtain ($L = 2\pi R$)

$$\left| \oint_{|z|=R} \frac{\operatorname{Log}(z)}{z^2} dz \right| \leq \frac{2\sqrt{2}\ln(R)\pi R}{R^2} = \frac{2\sqrt{2}\pi\ln(R)}{R}.$$

(finally an interesting reasonable problem, sorry folks,
I made some bad picks on this one 