Same instruction as Mission 1. Enjoy !

Problem 46 Sketch the proof of Cauchy's Theorem.

Problem 47 Let $C = \{z_o + e^{it} \mid 0 \leq t \leq 2\pi\}$. Calculate $\int_C (z - z_o)^m dz$ where $m \in \mathbb{Z}$. You will need to treat the $m = -1$ case and $m \neq -1$ case separately.

Problem 48 Jeff's Problem: let $c \in \mathbb{C}$ where $c \neq -1$ and define $z^c = exp(cLog(z))$ for $z \in \mathbb{C}^-$. Let $C_\varepsilon = \{Re^{it} \mid -\pi + \varepsilon \leq t \leq \pi - \varepsilon\}$ and $R > 0$. Calculate

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} z^c \, dz.$$

Jeff's idea to calculate this is really interesting and a great add-on to the previous problem. Of course, we could investigate other branch cuts to see if the result depends on the choice of branch, but, Jeff said I should save that for the test.

Problem 49 Let $C = [z_1, z_2]$ and calculate $\int_C z \, dz$ in two distinct ways. First, from the definition of the integral. Second, using FTC II for complex integrals.

Problem 50 Let $C = [z_1, z_2] \subset \mathbb{C}^-$ and calculate $\int_C \sqrt{z} \, dz$.

Problem 51 Calculate $\int_C z^3 dz$ where C is a path from P to Q. Given your result, what pair of real vector fields are path-independent ?
We can always think of a given complex integral as calculating the work done by a pair of associated real vector fields. So, every complex integral can be viewed as saying possibly interesting things about corresponding real line integrals

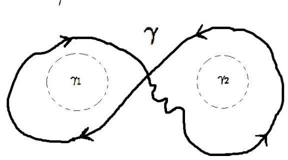
Problem 52 Explain why $\int_{|z|=1} ze^z \, dz = 0$. Set-up the integral $\int_{|z|=1} ze^z \, dz$ explicitly in terms of the parametrization $z = e^{it}$ for $0 \leq t \leq 2\pi$. Expand the formulas into cartesian form. What real integrals do we find must be zero ?

Problem 53 Let $R = \{z = x + iy \in \mathbb{C} \mid |z - 2i| \leq 2, \ x \geq 0\}$. Calculate

$$\int_{\partial R} z^{42} dz \qquad \& \qquad \int_{\partial R} \frac{z}{z - 1 - i} dz.$$

I strongly recommend you don't calculate directly.

Problem 54 Let γ_1 and γ_2 be CCW-oriented circles as illustrated by the dotted circles in the diagram. Let γ be as indicated. Given $\int_{\gamma_1} f(z)dz = -21$ and $\int_{\gamma_2} f(z)dz = 21$, calculate the value of $\int_\gamma f(z)dz$. You are given that f is holomorphic outside dotted circles.



Problem 55 Suppose $R > e^{\pi}$. Show $\left| \oint_{|z|=R} \frac{Log(z)}{z^2}\, dz \right| \leq 2\sqrt{2}\pi \frac{\ln R}{R}$.

Problem 56 Show $\int_{\partial D} \bar{z} dz = 2i\text{Area}(D)$.

Problem 57 Show that $\left| \oint_{|z-1|=1} \frac{e^z}{z+3}\, dz \right| \leq 2\pi e^2$.

Problem 58 Define $f(z) = \int_i^z \frac{d\zeta}{\zeta}$ for all $z \in \mathbb{C}$ except z along the negative imaginary axis. Explain why the given formula is well-defined and calculate the formula for $f(z)$ by explicit integration given that $z = x + iy$ has $x > 0$ and $|z| > 1$. Can you identify what function $f(z)$ is in terms of notation we introduced earlier in the course ?

Problem 59 Integrate $exp(-z^2/2)$ around the rectangle with vertices $\pm R$ and $it \pm R$ in the limit $R \to \infty$ to show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\, dx = e^{-t^2/2}$$

for $-\infty < t < \infty$. You will need to recall from Calculus III that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$.

Problem 60 **Complexified Green's Theorem:** We define the integral of a complex component vector field $\vec{F} + i\vec{G}$ (here $\vec{F}$ and $\vec{G}$ are real vector fields) along a curve by the natural rule below:

$$\int_C (\vec{F} + i\vec{G}) \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + i \int_C \vec{G} \bullet d\vec{r}$$

In the differential notation,

$$\int_C (P_1 + iP_2)\, dx + (Q_1 + iQ_2)\, dy = \int_C P_1\, dx + Q_1\, dy + i \int_C P_2\, dx + Q_2\, dy.$$

Likewise, we define the integral of a complex-valued function over some region R by the obvious rule $\int_R (f + ig) dA = \int_R f\, dA + i \int_R g\, dA$ where f, g are real-valued functions. Given this backdrop. Prove that Green's Theorem for $\vec{F} = \langle P, Q \rangle$ where P, Q are complex valued and continuously differentiable on R and near the positively oriented boundary ∂R,

$$\int_{\partial R} P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Your proof may assume the usual real Green's Theorem and we suppose R is a bounded region with piecewise smooth positively oriented boundary ∂R.