

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 97 Your signature below indicates you have:

(a.) I have read Sections 7.1 – 7.4 of Gamelin: _____.

(b.) I have read Cook's Guide to Sections 7.1 – 7.4 of Gamelin: _____.

Problem 98 §VII.4 # 2 (page 208) (integral)

Problem 99 §VII.5 # 1 (page 211) (integral)

Problem 100 §VII.6 # 3 (page ²¹⁵~~211~~) (integral)

Problem 101 §VIII.1 # 1 (page 228) (argument principle)

Problem 102 §VIII.1 # 8 (page 228) (argument principle)

Problem 103 §VIII.2 # 2 (page 230) (Rouché's Theorem)

Problem 104 §VIII.2 # 4 (page 230) (Rouché's Theorem)

Problem 105 §XIII.2 # 3 (page 351) (Mittag-Leffler)

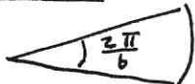
Problem 106 §XIII.2 # 4 (page 351) (Mittag-Leffler)

Problem 107 §XIII.3 # 1b (page 356) (infinite product)

Problem 108 §XIII.3 # 9 (page 356) (infinite product)

MISSION 9

PROBLEM 98 (§ VII.4 #2) (pg. 208)

By integrating around , show $\int_0^{\infty} \frac{dx}{1+x^b} = \frac{\pi}{b \sin(\pi/b)}$ $b > 1$

Consider $f(z) = \frac{1}{1+z^b}$ we have isolated singularities at

the solutions of $1+z^b=0 \Rightarrow z \in (-1)^{1/b}$ in particular

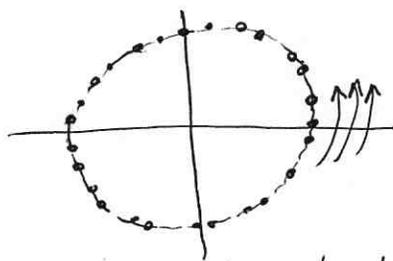
$$z_0 = e^{\pi i/b} \text{ has } (z_0)^b = (e^{\pi i/b})^b = e^{\pi i} = -1.$$

Remark: for $b \in \mathbb{R}$, $b > 1$ we have sequence of solⁿs,

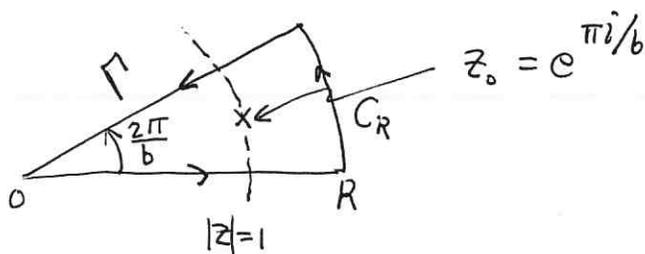
$$\{ z_0, z_0 e^{2\pi i/b}, z_0 \omega^2, z_0 \omega^3, \dots, z_0 \omega^n, \dots \}$$

there is no reason for $\omega^n = 1$ for some n since $b \notin \mathbb{Q}$ in general. $\{ e^{\pi i/b}, e^{3\pi i/b}, e^{5\pi i/b}, \dots \}$ If $b \notin \mathbb{Q}$ then I

believe these singularities densely fall on unit circle.



in view of this, I really doubt what follows.



$$[0, R] : z^b = (|z|e^{i0})^b = x^b$$

$$\Gamma : z^b = t^b (e^{\frac{2\pi i}{b}})^b = +t^b$$

$$dz = \frac{2\pi i}{b} e^{\frac{2\pi i}{b}} dt$$

$$(z = t e^{\frac{2\pi i}{b}} \quad 0 \leq t \leq R)$$

$$(\exp(2\pi i) = 1)$$

$$C_R : |f(z)| = \frac{\pi}{|1+z^b|} \leq \frac{\pi}{R^b - 1} \quad \text{for } b > 1 \text{ and } R \gg 1$$

$$\therefore \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| \leq \lim_{R \rightarrow \infty} \left| \frac{(\frac{2\pi}{b} R) \pi}{R^b - 1} \right| = 0 \quad \text{for } \underline{\underline{b > 1}}$$

P98 continued

Let $\gamma = \Gamma \cup [0, R] \cup C_R$ and suppose $R \rightarrow \infty$
and assume $b \in \mathbb{R}$ for which $z_0 = e^{\pi i/b}$ is the
only isolated singularity within γ ,

$$\begin{aligned}\int_{\gamma} f(z) dz &= 2\pi i \operatorname{Res} \left[\frac{1}{1+z^b}, e^{\pi i/b} \right] = 2\pi i \left[\frac{1}{b z^{b-1}} \Big|_{z=e^{\pi i/b}} \right] \\ &= 2\pi i \left(\frac{1}{b e^{\pi i} e^{-\pi i/b}} \right) \\ &= \frac{2\pi i}{b(-1)} e^{\pi i/b} \\ &= \frac{-2\pi i e^{\pi i/b}}{b}\end{aligned}$$

Thus, (using $x=t$ for Γ) \Downarrow

$$\int_0^R \frac{dx}{1+x^b} - \frac{2\pi i e^{\frac{2\pi i}{b}}}{b} \int_0^R \frac{dx}{1+x^b} = \frac{-2\pi i e^{\pi i/b}}{b} \quad \text{as } R \rightarrow \infty$$

$$\left(1 - \frac{2\pi i e^{\frac{2\pi i}{b}}}{b} \right) \int_0^{\infty} \frac{dx}{1+x^b} = \frac{-2\pi i e^{\pi i/b}}{b}$$

$$\int_0^{\infty} \frac{dx}{1+x^b} = \frac{2\pi i \exp(\pi i/b)}{2\pi i \exp(\frac{2\pi i}{b}) - b}$$

=



~~in progress~~

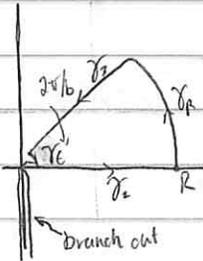
Stolen 😊 \Downarrow

MATH 331 - Mission 9

Problem 98 - By integrating around the boundary of a pie-slice domain of aperture $2\pi/b$,

show that

$$\int_0^{\infty} \frac{dx}{1+x^b} = \frac{\pi}{b \sin(\pi/b)}, \quad b > 1$$



$$\text{let } f(z) = \frac{1}{1+z^b} = \frac{1}{1+|z|^b e^{ib \arg z}}$$

$$\int_{\gamma_1} f(z) dz = \text{Res} \left[\frac{1}{1+z^b}, e^{\pi i/b} \right] = \frac{2\pi i}{b z^{b-1}} \Big|_{z=e^{i\pi/b}} = \frac{2\pi i}{b} e^{\pi i/b}$$

$$\int_{\gamma_2} f(z) dz = \int_{\epsilon}^R \frac{1}{1+|z|^b e^{ib \arg z}} dz = \int_{\epsilon}^R \frac{1}{1+x^b} dx \rightarrow \int_0^{\infty} \frac{1}{1+x^b} dx$$

$$\left| \int_{\gamma_R} f(z) dz \right| \leq \frac{1}{R^{b-1}} \cdot \frac{2\pi R}{b} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\int_{\gamma_3} f(z) dz = \int_{R\epsilon}^{\epsilon} \frac{1}{1+|z|^b e^{ib \arg z}} dz = \int_R^{\epsilon} \frac{1}{1+x^b} e^{2\pi i/b} dx \rightarrow \int_{\infty}^0 e^{2\pi i/b} \frac{1}{1+x^b} dx = -e^{2\pi i/b} \int_0^{\infty} \frac{1}{1+x^b} dx$$

$$\left| \int_{\gamma_{\epsilon}} f(z) dz \right| \leq \frac{1}{1-\epsilon^b} \cdot \frac{2\pi \epsilon}{b} \rightarrow 0 \text{ as } \epsilon \rightarrow 0^+$$

$$\text{So } \int_{\gamma_0} f(z) dz = \frac{2\pi i}{b} e^{\pi i/b} = \int_0^{\infty} \frac{1}{1+x^b} dx - e^{2\pi i/b} \int_0^{\infty} \frac{1}{1+x^b} dx$$

$$\rightarrow \frac{-2\pi i}{b} e^{\pi i/b} = \int_0^{\infty} \frac{1}{1+x^b} dx \left[1 - e^{2\pi i/b} \right]$$

$$\rightarrow \frac{-2\pi i}{b} = \int_0^{\infty} \frac{1}{1+x^b} dx \left[e^{-\pi i/b} - e^{\pi i/b} \right]$$

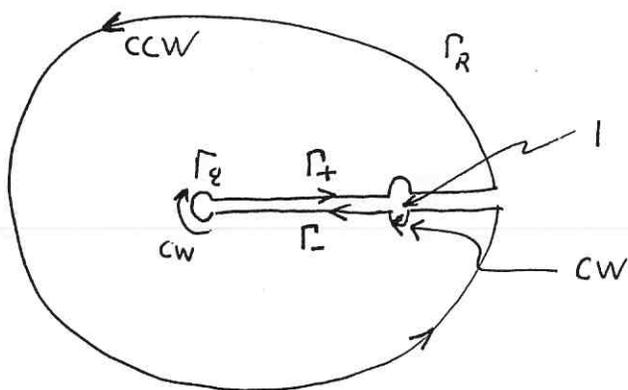
$$\rightarrow \frac{i\pi}{b} = \int_0^{\infty} \frac{1}{1+x^b} dx \left[\frac{e^{\pi i/b} - e^{-\pi i/b}}{2} \right]$$

$$\rightarrow \frac{i\pi}{b} = \int_0^{\infty} \frac{1}{1+x^b} dx \left(\sin(\pi/b) \right)$$

$$\frac{\pi}{b \sin(\pi/b)} = \int_0^{\infty} \frac{1}{1+x^b} dx, \quad b > 1. //$$

P99 § VII.5#1

show $\int_0^{\infty} \frac{\log(x) dx}{x^a(x-1)} = \frac{2\pi^2}{1 - \cos(2\pi a)} \quad 0 < a < 1$



$$\Gamma_+ : z = re^{i0} = r \rightarrow f(z) = \frac{\log(z)}{z^a(z-1)} = \frac{\ln(r)}{r^a(r-1)}$$

$$\Gamma_- : z = re^{i0} = re^{2\pi i} \rightarrow f(z) = \frac{\log(re^{2\pi i})}{r^a e^{2\pi i a}(r-1)} = \frac{\ln(r) + 2\pi i}{r^a e^{2\pi i a}(r-1)}$$

$$\text{Res}[f(z), 1^+] = \text{Res}\left[\frac{1}{r^a(r-1)}(r-1 + \dots), 1^+\right] = 0 \quad (\text{removable on } \Gamma_+)$$

$$\text{Res}[f(z), 1^-] = \text{Res}\left[\frac{\ln(r) + 2\pi i}{r^a e^{2\pi i a}(r-1)}, 1\right] = \frac{\ln(1) + 2\pi i}{1^a e^{2\pi i a}} = \frac{2\pi i}{\exp(2\pi i a)}$$

Hence, supposing $\int_{\Gamma_\epsilon} f(z) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$ we have,

$$\int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} - \frac{1}{e^{2\pi i a}} \int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} + \frac{2\pi i}{e^{2\pi i a}} \int_0^{\infty} \frac{dx}{x^a(x-1)} = -\pi i \left(\frac{2\pi i}{e^{2\pi i a}} \right)$$

$$e^{2\pi i a} \int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} - \int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} + 2\pi i \int_0^{\infty} \frac{dx}{x^a(x-1)} = 2\pi^2$$

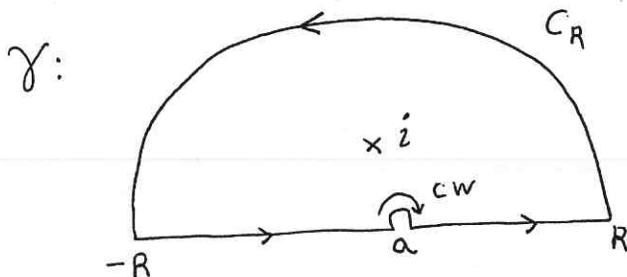
$$[\cos(2\pi a) - 1] \int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} + i[2\pi + \sin(2\pi a)] \int_0^{\infty} \frac{dx}{x^a(x-1)} = 2\pi^2$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\ln(x) dx}{x^a(x-1)} = \frac{2\pi^2}{\cos(2\pi a) - 1}}$$

P100 §VII. 6 #3 p. 211

$$\text{Show P.V. } \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)} dx = \frac{-\pi a}{a^2+1} \text{ for } -\infty < a < \infty$$

By integrating around indented half-disk in upper half-plane.



Let $f(z) = \frac{1}{(z^2+1)(z-a)}$ has isolated poles at $\pm i$ and a

$$\text{Res}[f(z), i] = \frac{1}{(z+i)(z-a)} \Big|_{z=i} = \frac{1}{2i(i-a)} = \frac{1}{-2-2ia} = -\frac{1}{2}(1+ia)^{-1}$$

$$\text{Res}[f(z), a] = \frac{1}{(z^2+1)} \Big|_{z=a} = \frac{1}{a^2+1}$$

Observe for $|z|=R$ with $R > 1, a$,

$$|f(z)| = \frac{1}{|z^2+1||z-a|} \leq \frac{1}{(R^2-1)(R-a)}$$

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi R}{(R^2-1)(R-a)} \rightarrow 0 \text{ as } R \rightarrow \infty$$

Hence, as $R \rightarrow \infty$, applying Cauchy's Th^m and the fractional residue theorem,

$$\begin{aligned} \text{P.V. } \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x-a)} &= \frac{2\pi i}{2i(i-a)} + \frac{\pi i}{a^2+1} = \frac{-\pi(a+i)}{(a-i)(a+i)} + \frac{\pi i}{a^2+1} \\ &= \boxed{\frac{-\pi a}{a^2+1}} \end{aligned}$$

Show $z^4 + 2z^2 - z + 1$ has exactly one root in each quadrant.

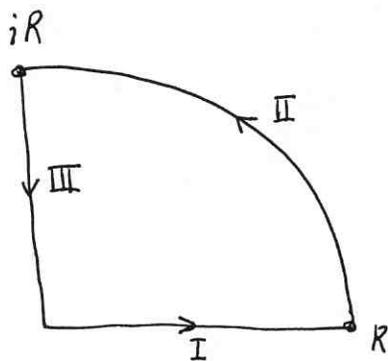
Observe $z^4 + 2z^2 - z + 1 = (z^2 + 1)^2 - z$ hence

$P(z) = 0 \Rightarrow (z^2 + 1)^2 = z$ hence real sol^s must be positive, however, $(z^2 + 1)^2 = z$ for $z > 0 \Rightarrow (1 + \frac{1}{z})^2 = 1$

which is impossible. Thus, $P(z) \neq 0$ for all $z \in \mathbb{R}$.

As $P(z)$ has real coeff. we find $\exists z_1, z_2 \in \mathbb{C}$ for which $P(z) = (z - z_1)(z - z_2)(z - \bar{z}_1)(z - \bar{z}_2)$. Thus,

if we show there is just one solⁿ in quadrant I then that suffices to prove there is just one root \star in each quadrant ($P(iy) \neq 0 \forall y \in \mathbb{R}$ can be shown)



Need to calculate $\Delta \arg(P)$ along each of $\textcircled{I}, \textcircled{II}, \textcircled{III}$. Then $\frac{\Delta \arg(P)}{2\pi}$ gives # of z zeros.

\textcircled{I} $z = x$, $P(x) = x^4 + 2x^2 - x + 1$, $P(0) = 1 \Rightarrow P(x) > 0$
Hence $\Delta \arg(P) = 0$ along \textcircled{I} as $\Theta(P(x)) = 0$ along whole $[0, R]$

\textcircled{II} $z = Re^{i\theta}$, $0 \leq \theta \leq \pi/2$

$P(R) \in [0, \infty) \therefore \text{Arg}(P(R)) = 0$

$$P(Re^{i\pi/2}) = R^4(e^{i\pi/2})^4 + 2(Re^{i\pi/2})^2 - Re^{i\pi/2} + 1 \sim R^4 e^{2\pi i}$$

~~$\Delta \arg(P) = \arg(P(R)) - \arg(P(Re^{i\pi/2})) = 0 - 2\pi = -2\pi$~~ $\Delta \arg(P) = 2\pi$

\textcircled{III} $z = iy$, $P(iy) = y^4 - 2y^2 - iy + 1$

for $y \rightarrow \infty$ has $\text{Arg}(P(iy)) \rightarrow 0$

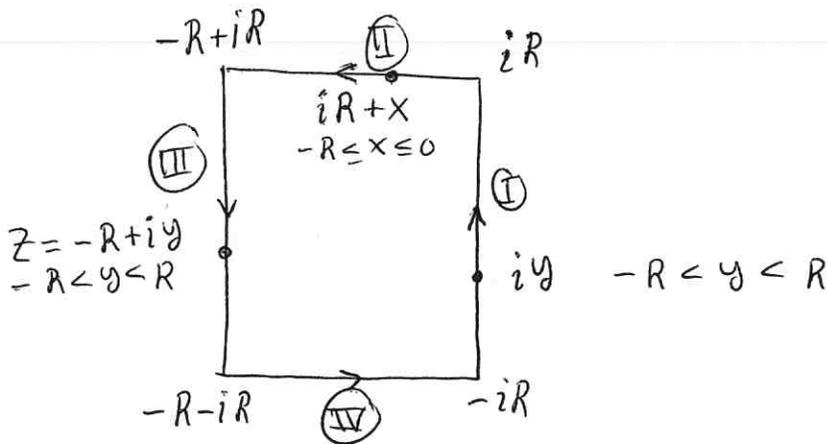
as $y \rightarrow 0$ have $P(iy) \approx 1 \rightarrow \text{Arg}(P(iy)) \rightarrow 0$

~~But we also need +~~ thus net change \Rightarrow $\frac{\Delta \arg(P)}{2\pi} = 1$ zero \Rightarrow claim \star true.

P102 § VIII, #8

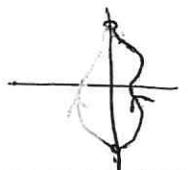
Show that if $\operatorname{Re}(\lambda) > 1$ then $e^z = z + \lambda$ has one solⁿ in left-half-plane

Consider the rectangle below, and $f(z) = z + \lambda - e^z$



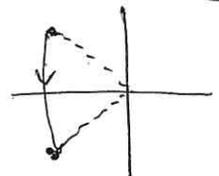
I should be more careful. It matters how the values of $f(z)$ change as we go along each leg. Study composition of $f(z)$ and parametrizations.

(I.) $z = iy$, $f(iy) = iy + \lambda - e^{iy}$ for $-R < y < R$
 As $R \rightarrow \infty$ the iy term dominates, starts at $\theta = 3\pi/2$, then as $\operatorname{Re}(\lambda) - \cos(y) \gg 0$ travel back to $\pi/2 \Rightarrow \Delta_I = +\pi$.



(II.) $z = iR + x$, $f(iR + x) = iR + x + \lambda - e^{iR} e^x$
 begin at $x = 0$, $R \rightarrow \infty$ $f(z) \sim iR - e^{iR} \sim iR \hookrightarrow \theta = \frac{\pi}{2}$
 As $x \rightarrow -R$ get $f(z) \sim iR - R + \lambda$
 $\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \hookrightarrow \Delta_{II} = \frac{\pi}{4}$
 (of course, see last step, duh.)

(III.) $z = -R + iy$, $f(-R + iy) = -R + iy + \lambda + e^{-R} e^{iy}$
 start at $f(z) \sim -R + iR$
 end at $f(z) \sim -R - iR$ always
 $\Delta_{III} = \frac{\pi}{2}$



④ $f(x-iR) = x - iR + \lambda - e^x e^{-iR}$
 begin at $\theta \approx \frac{5\pi}{4}$ as $x \rightarrow 0$

$f(x-iR) \rightarrow \underline{-iR + \lambda} - e^{-iR} \rightarrow \theta = \frac{3\pi}{2}$
 dominates

$\Delta_{IV} = \frac{\pi}{4}$

Hence the net change in argument

is $\pi + \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = 2\pi \iff \underline{f(z)}$ has

one zero on the rectangle, but $R \rightarrow \infty$ gives us the half-plane.

P103 § VIII.2 #2 (pg. 230)

How many roots does $z^9 + z^5 - 8z^3 + 2z + 1$ have between circles $\{|z|=1\}$ and $\{|z|=2\}$

Let $P(z) = z^9 + z^5 - 8z^3 + 2z + 1 = \underbrace{-8z^3}_{f(z)} + \underbrace{z^9 + z^5 + 2z + 1}_{h(z)}$

- If $|z|=1$ then $|f(z)|=8$ and $|h(z)| \leq 1+1+2+1=5$
 Hence apply Rouché's Th^m, $f(z) \neq 0$ thrice hence $P(z)$ has 3 zeros inside $|z|=1$.

- If $|z|=2$ then $|z^9|=2^9$ but $|z^5 - 8z^3 + 2z + 1| \leq 2^5 - 8 \cdot 2^3 + 4 + 1$

Let $f(z) = z^9$ and $h(z) = z^5 - 8z^3 + 2z + 1$

it is clearly the case for $|z|=2$ that $|f(z)| < |h(z)|$

Moreover, $f(z) = 0$ nine times over, we count that

there are 9 zeros for $P(z)$ inside $|z|=2$

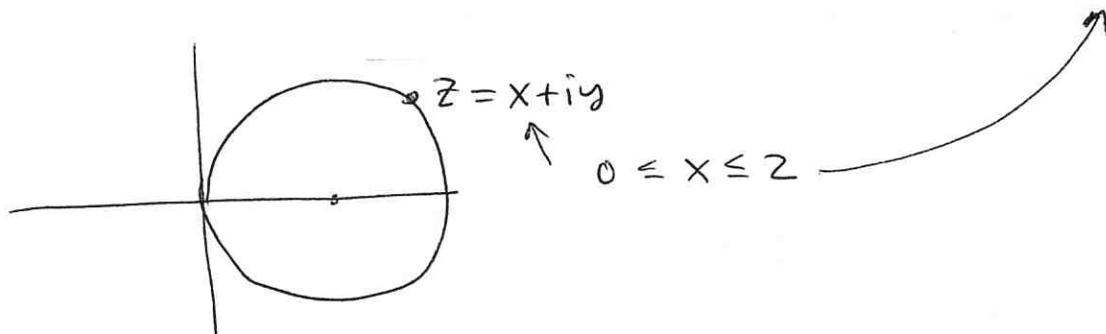
- It follows there must be $9 - 3 = \boxed{6}$ in the annulus 

P104 § VIII.2 #4 (p. 230)

Fix complex λ s.t. $|\lambda| < 1$. For $n \geq 1$, show that $(z-1)^n e^z - \lambda$ has n -zeros for $|z-1| < 1$ and no other zeros in the right half-plane.

Let $f(z) = (z-1)^n e^z$ and $h(z) = -\lambda$

Observe for $|z-1|=1 \Rightarrow |f(z)| \leq |e^z| = e^x \geq 1$



Thus, $|\lambda| < 1 \Rightarrow |h(z)| < e^x = |f(z)|$ for $|z-1|=1$.

Clearly $f(1) = 0$ n -fold times as $e^1 \neq 0$ thus,

by Rouché's Th^m $f(z)$ and $g(z) = (z-1)^n e^z - \lambda$

have same # of zeros inside $|z-1|=1$, namely n -zeros.

//

The argument above is repeated w/o much modification on $|z-a|=a$ for $a > 1$ since

$$|(z-a)^n e^z| = a^n e^x \text{ for } 0 \leq x \leq 2a$$

and so $|\lambda| < 1 < a^n e^x = |f(z)| \Rightarrow (z-a)^n e^z - \lambda = 0$ for n -times inside $|z-a|=a$. But, we already know \exists n -zeros inside $|z-1|=1$ hence \nexists other zeros outside the disk in the right half-plane.

P105 § ~~X~~ III.2 #3 /

Show that
$$f(z) = \frac{\pi}{\sin \pi z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2} = g(z)$$

I leave the proof that the series above is uniformly convergent to the reader. That said, it is easy to see these $f(z)$, $g(z)$ share the same poles.

$f(z)$: ~~$\sin(\pi z) = 0$~~ $\sin(\pi z) = 0$ for $z \in \mathbb{Z}$.

$g(z)$: $\frac{1}{z} \leftrightarrow z=0$ pole

$$\frac{(-1)^n(zz)}{z^2 - n^2} = \frac{(-1)^n}{z-n} + \frac{(-1)^n}{z+n} \quad \hookrightarrow \quad z = \pm n \quad \text{for } n \in \mathbb{N}.$$

We see $\text{Res}[g(z), n] = (-1)^n$

We calculate, $\text{Res}\left[\frac{\pi}{\sin \pi z}, n\right] = \frac{\pi}{\pi \cos(\pi n)} = (-1)^n$ (Rule 4)

Thus $f(z)$ & $g(z)$ share same principal part as all the poles were simple. Moreover,

$$h(z) = f(z) - g(z) \in \mathcal{O}(\mathbb{C})$$

Hence, if we can show $\lim_{z \rightarrow \infty} h(z) = 0$ then

by Liouville's Th^m it follows $h(z) = 0 \quad \forall z \in \mathbb{C}$

hence the identity $f(z) = g(z) \quad \forall z \in \mathbb{C}$ holds. We

Now show just that:

Observe, for $z = iy$ we have

$$f(iy) = \frac{\pi}{\sin(\pi iy)} = \frac{\pi}{i \sinh(\pi y)} \rightarrow \frac{\pi}{\infty} = 0 \text{ as } |y| \rightarrow \infty$$

Thus, by periodicity $f(z+2n) = f(z)$ we deduce $f(z) \rightarrow 0$.
(note $g(z) \rightarrow 0 \Rightarrow h(z) \rightarrow 0$ as $z \rightarrow \infty$.)

P106 (§ ~~XIII~~ 2 #4) (p. 351)

$$\text{Show } \frac{\pi}{\cos \pi z} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)}{z^2 - (n - \frac{1}{2})^2} = g(z)$$

Once again, I leave proof of uniform convergence of the series forming $g(z)$ to the reader. Observe

$$f(z): \cos \pi z = 0 \Rightarrow \pi z = \pi(n + \frac{1}{2}) \text{ for } n \in \mathbb{Z} \\ \underline{z = n + \frac{1}{2} \text{ for } n \in \mathbb{Z}}$$

$$g(z) = \frac{(-1)^n (2n-1)}{z^2 - (n - \frac{1}{2})^2} = \frac{(-1)^n}{z - (n - \frac{1}{2})} - \frac{(-1)^n}{z + (n - \frac{1}{2})}$$

Hence, same simple poles. Moreover,

$$\text{Res} \left[\frac{\pi}{\cos \pi z}, n + \frac{1}{2} \right] = \frac{\pi}{-\pi \sin(\pi(n + \frac{1}{2}))} = \frac{-1}{-\cos n\pi \sin \frac{\pi}{2}} = (-1)^n$$

Likewise, $\text{Res} [g(z), n - \frac{1}{2}] = (-1)^n$ hence

$f(z) - g(z) = h(z) \in \mathcal{O}(\mathbb{C})$. We need to show $h(z) \rightarrow 0$ as $z \rightarrow \infty$. Clearly, $g(z) \rightarrow 0$

and if we study $z = iy$ then

$$f(iy) = \frac{\pi}{\cos(\pi iy)} = \frac{\pi}{\cosh(\pi y)} \rightarrow 0 \text{ as } |y| \rightarrow \infty$$

thus by $2\mathbb{Z}$ -periodicity of $f(z)$ we find $f(z) \rightarrow 0$ as $z \rightarrow \infty$. Therefore, $h(z) \rightarrow 0$ as $z \rightarrow \infty$ and, by Liouville's Th^m we find $h(z) = 0 \forall z \in \mathbb{C}$ and thus $f(z) = g(z) \forall z \in \mathbb{C}$ which proves the desired assertion.

P107 (§ XIII.3 #16) (pg. 356)

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{25}\right)\dots$$
$$= \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)\left(\frac{24}{25}\right)\dots = \boxed{\frac{1}{2}}$$

See pg. 170 of Guide. It's solved there 😊

Example 13.3.3

P108 § XIII.3 #9 (p. 356)

Show
$$\prod_{n=1}^{\infty} \underbrace{\left(1 + \frac{1}{n^2}\right)}_{P_n} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

~~Sorry folks, out of time for now.~~

Recall,
$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Set $z = i$ to obtain $1 - \frac{z^2}{n^2} = 1 + \frac{1}{n^2}$ hence,

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) = \frac{\sin(\pi i)}{\pi i}$$

$$= \frac{1}{\pi i} \left[\frac{1}{2i} (e^{-\pi} - e^{\pi}) \right]$$

$$= \frac{-1}{2\pi} (e^{-\pi} - e^{\pi})$$

$$= \frac{e^{\pi} - e^{-\pi}}{2\pi} = \boxed{\frac{\sinh(\pi)}{\pi}}$$

oh, but → 😊