Please solve these problems on other paper and clearly label each problem in the order which they are given here. Staple your work in the upper left corner with a metal staple and do not fold. Please show work for full credit. I am interested in the steps. This instruction applies for the remainder of the problem sets and it is assumed you remember this. **Thanks!**

Please do your best with the time you have, I would rather get two solved problems then none. Also, notice some of these problems are very simple. I marked the ones that I suspect take deeper thinking with $*$. In this particular Problem Set, many of the problems are little more than understanding notation and doing a simple, short calculation. Also, remember your calculus III, to find vector from P to Q use $v = Q - P$ this is especially relevant to Problems 1 and 2.

Problem 1 Suppose a line $\mathcal{L}$ in $\mathbb{R}^3$ contains the points $(1,1,1)$ and $(2,4,6)$. Describe $\mathcal{L}$ as follows:

(a.) parametrically with parameter λ.

(b.) as the solution set of one or more equations in x, y, z.

Problem 2 Suppose a plane $\mathcal{P}$ contains the points $(1,1,0\dots,0)$, $(1,2,0\dots,0)$ and $(2,1,0\dots,0)$ in $\mathbb{R}^n$. Describe $\mathcal{P}$ as follows:

(a.) parametrically with parameters α, β

(b.) as the solution set of one of more equations in $x_1, x_2, \dots, x_n$.

Problem 3 Suppose $\beta = \{1, x-1, (x-1)^2\}$ is a basis for polynomials P_2. Find the coordinate vector of $f(x) = ax^2 + bx + c$ with respect to β. That is, find $\Phi_\beta(f(x))$. Furthermore, suppose $T : P_2 \to P_2$ is defined by $T(f(x)) = f''(x)$. Find the matrix of T w.r.t. the basis β; that is, calculate $[T]_{\beta,\beta}$.

Problem 4 Suppose X, Y are sets and $V \subseteq Y$ and $U \subseteq X$. Let $f : X \to Y$ be a function, we define the **inverse image of V under f** by:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\},$$

likewise, define the **image of U under f** by:

$$f(U) = \{y \in Y \mid \text{there exists } x \in U \text{ with } f(x) = y\} = \{f(x) \mid x \in U\}.$$

Given the definitions above, calculate the images and inverse images given below (find the set which describes the image or inverse image and if possible identify it geometrically) :

(a.) if $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by $F(x,y) = y - x^2 - 1$ then describe $F^{-1}\{0\}$ as a point-set.

(b.) if $G : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $G(x,y,z) = (x^2 + y^2 + z^2, z)$ describe $G^{-1}(R^2, k)$ geometrically. What condition do we need for $G^{-1}(R^2, k) \neq \emptyset$? (assume $R, k \in \mathbb{R}$)

(c.) let $X : \mathbb{R}^2 \to \mathbb{R}^n$ be defined by $X(s,t) = p + sv + tw$
where v, w are linearly independent n-vectors and $p \in \mathbb{R}^n$. Describe $X([0,1]^2)$.

Problem 5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ the **standard matrix** of T is denoted $[T] \in \mathbb{R}^{m\times n}$ and is defined by $[T]_{ij} = [T(e_j)]_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. In matrix notation, this definition is nicely written as $[T] = [T(e_1)|T(e_2)|\cdots|T(e_n)]$. Use the given definition to find the standard matrix for the linear transformations given below:

(a.) $T_1(x, y) = (x + 2y, 3x + 4y)$

(b.) $T_2(x, y) = (x + y, 2x + 2y, 3x + 3y)$

(c.) $T_3(x, y, z) = x + 2y + 3z$

(d.) $T_4(x) = (x, 2x, 3x)$

Problem 6 Calculate the composition $T_3 \circ T_2 \circ T_1$ in two ways:

(a.) from the definition of function composition $(T_3 \circ T_2 \circ T_1)(x, y) = T_3(T_2(T_1(x, y)))$,

(b.) via matrix multiplication $(T_3 \circ T_2 \circ T_1)(x, y) = [T_3 \circ T_2 \circ T_1](x, y) = [T_3][T_2][T_1][x, y]^T$.

remark: this is why the product of matrices is defined as it is.

Problem 7 Let $A_3 = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. Find a basis $\{f_1, f_2, f_3\}$ for A_3 by writing $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and studying the condition $A^T = -A$. You should find $\dim(A_3) = 3$.

Bonus: *study the isomorphism* $\Phi : A_3 \to \mathbb{R}^3$ *defined by linearly extending* $\Phi(f_i) = e_i$, *if we think of antisymmetric* 3×3 *matrices as vectors then what is the geometric meaning of matrix multiplication for* A_3*?*

Problem 8* Suppose P is a paralellogram in $\mathbb{R}^3$ in the octant with positive coordinates. Furthermore, define $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2\}$.

(a.) find $Area(P)$.

(b.) define $L_{ij}(\vec{v}) = (\vec{v} \bullet \widehat{x}_i)\,\widehat{x}_i + (\vec{v} \bullet \widehat{x}_j)\,\widehat{x}_j$ and prove using properties of dot-products that L_{ij} is a linear transformation.

(c.) assume that linear transformations map parallelograms to lines, parallelograms or points and use this presupposition to establish the following equation:

$$Area(P)^2 = Area(L_{12}(P))^2 + Area(L_{31}(P))^2 + Area(L_{23}(P))^2$$

Problem 9 Let $c \in \mathbb{R}$, $A, X \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ we define $AB \in \mathbb{R}^{m\times p}$, $A + X, cA \in \mathbb{R}^{m\times n}$ by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \qquad (A + X)_{ij} = A_{ij} + X_{ij}, \qquad (cA)_{ij} = cA_{ij}.$$

Using the index notation given above, show (for appropriately sized matrices)

(a.) $A(B + C) = AB + AC$

(b.) if $(A^T)_{ij} = A_{ji}$ for all i, j then $(AB)^T = B^T A^T$ (socks-shoes identity)

(c.) If $I \in \mathbb{R}^{n\times n}$ such that $I_{ij} = \delta_{ij}$ and $X \in \mathbb{R}^{n\times p}$ then $IX = X$.

Problem 10 It is convenient to introduce some notation: the *Kronecker delta* is defined by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Note the standard basis for $\mathbb{R}^n$ is nicely described by $(e_i)_j = \delta_{ij}$.

It is also often convenient to introduce the completely antisymmetric symbol in n-dimensions (the Levi-Civita symbol);

$$\epsilon_{i_1 i_2 \dots i_n} = \det[e_{i_1}|e_{i_2}|\cdots|e_{i_n}].$$

The formula above means that $\epsilon_{12\dots n} = det[I] = 1$ and all other nontrivial values can be obtained by swapping indices. Each swap changes the sign. For example, $\epsilon_{123\dots n} = -\epsilon_{213\dots n} = -1$. If any index is repeated then the antisymmetric symbol is zero.

A popular convention in physics and some math is that when an index is repeated a summation is implied. For example, $A_i B_i = \sum_i A_i B_i$ where the $\sum_i$ ranges over whatever is the accepted range of the index i. Let A_i, B_i denote three-dimensional vectors. Verify that

(a.) $A \bullet B = A_i B_i$

(b.) $A \times B = A_i B_j \epsilon_{ijk} e_k$