Problem 11 Given that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous mapping from the normed space $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ determine if the following sets of matrices are closed or open in the set of $n \times n$ matrices:

(a.) $GL(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$ (general linear group)

(b.) $SL(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ (special linear group)

Problem 12 Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $F(A) = A^T A - I$ where I is the $n \times n$ identity matrix. Notice that F is clearly continuous since it has a formula which is formed from polynomials in the entries of A. Let $O(n) = F^{-1}\{0\}$.

(a.) Explain why $O(n)$ is closed.

(b.) Show that if $A, B \in O(n)$ then $AB, A^{-1} \in O(n)$ (this makes $O(n)$ a group since it clearly contains the identity matrix and matrix multiplication is associative)

(c.) Is $O(n)$ connected? Hint: consider the determinant function on $O(n)$, recall $\det(AB) = \det(A)\det(B)$.

Problem 13 Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $F(A) = A^2$. Find the Frechet differential $dF_A(H)$ and prove your proposed linear transformation in H satisfies the required limit. My advice is as follows:

(a.) calculate $F(A + H) - F(A)$ and select what appears linear in H and claim it is $dF_A(H)$

(b.) verify the claim by plugging your guess into the definition of differentiable.

Problem 14 Suppose V, W are normed vector spaces. Show that if $F : V \to W$ is differentiable at x_o in a normed vector space V then F is continuous at x_o.

Problem 15 Show that if $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are differentiable at A then the matrix product function FG is likewise differentiable at A.
Hint: intuition suggests $F(A + H) \approx F(A) + dF_A(H)$ for H small, the same for G, multiply and take an educated guess

Problem 16 Find the Jacobian matrix for $F(x, y) = (x^2 - y^2, 2xy)$. Use the Jacobian matrix to help construct a linearization of F centered at $(1, 1)$.

Problem 17 Find the Jacobian matrix for $G(x, y, z) = x + y^2 + z^3$. Use the Jacobian matrix to help construct a linearization of F centered at $(1, \sqrt{2}, \sqrt[3]{3})$.

Problem 18 Find the Jacobian matrix for $H(r, t, s) = (r\cos(t)\sinh(s), r\cos(t)\sinh(s), r\cosh(s))$.

Problem 19 Calculate the differential for the determinant mapping $S : \mathbb{R}^{n\times n} \to \mathbb{R}$ at A where $S(A) = \det(A)$. Note a basis for $n \times n$ matrices is given by the matrix units $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ hence it suffices to calculate $dS_A(E_{ij})$ for arbitrary i, j. Significant partial credit will be awarded for working out the $n = 2$ case.

Problem 20 Calculate the differential for the trace mapping $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ at A where $T(A) = \text{trace}(A) = \sum_{i=1}^{n} A_{ii}$. Once again, it suffices to calculate $dT_A(E_{ij})$ for arbitrary i, j. Significant partial credit will be awarded for working out the $n = 2$ case.