Remember, I can only give you hints if you work on this before the day before it's due...

Problem 51 Given that $\mathbb{R}$ is complete, show that $\mathbb{R}^{n\times n}$ is complete. (I proved $\mathbb{R}^n$ is complete in lecture, think a bit, you can easily modify my argument)

Problem 52 Let A be a square matrix and P and invertible matrix. Show that $P^{-1}e^AP = e^{P^{-1}AP}$.

Problem 53 Show that if $AB = BA$ then $e^Ae^B = e^{A+B}$.

Problem 54 Show that $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ where $[A, B] = AB - BA$ and the omit terms involve three or more terms in A and B. This is the Baker-Campbell-Hausdorff formula, the higher terms I have not written are all expressed as nestings of multiple commutators. For example, the next higher terms involve $[A, [A, B]$ and $[B, [B, A]]$.

Problem 55 Show that $\det(e^A) = e^{\text{tr}(A)}$. You can assume A is diagonalizable or, if you know what it means, assume A is in Jordan form. (this identity is general)

Problem 56 Suppose $SL(n) = \{A \in \mathbb{R}^{n\times n} \mid det(A) = 1\}$ defines the group of **special linear** matrices. Find a condition on B such that $\gamma(t) = e^{tB}$ forms a smooth curve in $SL(n)$ near the identity matrix. For future reference, the set of all matrices such as B forms $sl(n)$.

Problem 57 Suppose $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^TA = I\}$ defines the group of **orthogonal** matrices. Find a condition on B such that $\gamma(t) = e^{tB}$ forms a smooth curve in $O(n)$. For future reference, the set of all matrices such as B forms $o(n)$.

Problem 58 Assume f_n, f are real-valued functions of a real variable and $n \in \mathbb{N}$. Suppose $f_n(x) \to f(x)$ for all $x \in [a, b]$ as $n \to \infty$. In such a case, we say that $f_n \to f$ pointwise on $[a, b]$. We also say, $\{f_n\}$ converges pointwise to f on $[a, b]$. We know continuous functions are Riemann integrable. If each f_n is continuous on $[a, b]$ then we can calculate $\int_a^b f_n(x)\,dx$ for $n = 1, 2, \ldots$. On the other hand, we may be able to calculate $\int_a^b f(x)\,dx$. Is it true that:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx \quad ?$$

The answer is, in general, **no.** Show how the following provides a **counter-example**

$$f_n(x) = \begin{cases} 4n^2x & 0 \le x \le 1/2n \\ 4n - 4n^2x & 1/n \le x \le 1/n \\ 0 & 1/n \le x \le 1 \end{cases}.$$

this exercise shows that it may not be possible to interchange the order of a limit and an integral. Clearly pointwise convergence does not allow the interchange in general. If for all $\epsilon > 0$ there exists N (independent of x) such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ then we say $\{f_n\}$ converges to f uniformly on $[a, b]$. Uniform convergence often allows for the interchange of limits. See Rosenlicht for a nice introduction to these ideas, I got this example from page 138 of his text Introduction to Analysis.

Problem 59 It can be shown that $\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$. Show that:

(a.) for $t > 0$ the change of variables $x = ty$ changes the integral to $\int_0^\infty \frac{\sin(ty)}{y} dx = \frac{\pi}{2}$

(b.) differentiating under the integral sign with respect to t yields $\int_0^\infty \cos(ty) dy = 0$.

this exercise shows that differentiating under the integral can lead to nonsensical results. I will share conditions under which this cannot happen a little after this homework is due. Please remind me if I forget!

Problem 60 Derive the formulas for the indefinite integrals

$$\int x^n \sin(x)\, dx \qquad \int x^n \cos(x)\, dx$$

via differentiating under an appropriate integral. Hint: begin your derivations by observing that $\int \cos(tx) dx = \frac{\sin(tx)}{t}$ and $\int \sin(tx) dx = -\frac{\cos(tx)}{t}$.
this exercise shows tha differentiating under the integral yields surprising new derivations of formulas which are otherwise much more tedious to derive. Of course, to justify this calculation certain analytical details must be checked, again I will let you in on that soon after this is done.