Problem 61 Let $T[f] = \int_0^1 f(t)\,dt$ for all $f \in V = P_2 = \text{span}\{1, x, x^2\}$. Show that $T \in V^*$.

Problem 62 Suppose V has basis $\beta = \{f_i\}_{i=1}^n$ and V^* has dual basis $\beta^* = \{f^j\}_{j=1}^n$ for which we assume $f^j(f_i) = \delta_{ij}$. We learn how to select components via the appropriate basis-evaluation:

(a.) let $v \in V$. Show that if $v = \sum_{i=1}^n c^i f_i$ then $c^i = f^i(v)$.

(b.) let $\alpha \in V^*$. Show that if $\alpha = \sum_{i=1}^n c_i f^i$ then $c_i = \alpha(f_i)$.

Problem 63 Suppose we have two bases for V: $\beta = \{f_i\}_{i=1}^n$ and $\bar{\beta} = \{\bar{f}_i\}_{i=1}^n$. A nice notation when dealing with two coordinate systems is to use bars to denote quantities attached to the bar-basis. In particular, we write:

$$v = \sum_{i=1}^n v^i f_i \qquad \text{verses} \qquad v = \sum_{i=1}^n \bar{v}^j \bar{f}_j$$

Since $f_i \in \text{span}\bar{\beta}$ there exists a matrix $P \in \mathbb{R}^{n\times n}$ for which $f_i = \sum_{j=1}^n P_i^{\ j} \bar{f}_j$. Moreover, because the inverse expansion must also exist as β is also a basis we can expect $P \in \text{Gl}(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$. In particular, $Q \in \mathbb{R}^{n\times n}$ exists for which $\bar{f}_j = \sum_{k=1}^n Q_j^{\ k} f_k$. Observe that substituting the Q-expansion into the P-expansion yields:

$$f_i = \sum_{k,j=1}^n P_i^{\ j} Q_j^{\ k} f_k \qquad \Rightarrow \qquad \sum_{j=1}^n P_i^{\ j} Q_j^{\ k} = \delta_{ik}.$$

some people write $\delta_i^{\ k}$ in place of δ_{ik} to emphasize the pattern. Do note that $P^{-1} = Q$, we introduce Q to avoid writing P^{-1}. I think I've given enough background at this point to ask you the following:

(a.) let $x \in V$. Show that $\bar{x}^j = \sum_{i=1}^n P_i^{\ j} x^i$

(b.) likewise, let $\alpha \in V^*$ and suppose $\alpha = \sum_{i=1}^n \alpha_i f^i$ and $\alpha = \sum_{i=1}^n \bar{\alpha}_i \bar{f}^i$ show $\bar{\alpha}_j = \sum_{i=1}^n Q^i_{\ j} \alpha_i$.

(c.) differentiate part $(a.)$ with respect to x^i to obtain $P_i^{\ j} = \frac{\partial \bar{x}^j}{\partial x^i} = \partial_i \bar{x}^j$.

Problem 64 Suppose V, W are vector spaces and V^*, W^* are the corresponding dual spaces. Moreover, suppose V has basis $\beta = \{f_i\}_{i=1}^n$ and V^* has dual basis $\beta^* = \{f^j\}_{j=1}^n$ for which we assume $f^j(f_i) = \delta_{ij}$. Likewise, suppose W has basis $\gamma = \{g_i\}_{i=1}^m$ and W^* has dual basis $\gamma^* = \{g^j\}_{j=1}^m$ for which we assume $g^j(g_i) = \delta_{ij}$. Furthermore, we suppose there exist barred-versions of all the bases given above; $\bar{\beta}, \bar{\beta}^*, \bar{\gamma}, \bar{\gamma}^*$. Using partial-derivatives to capture the coordinate change:

$$f_i = \sum_{j=1}^n \frac{\partial \bar{x}^j}{\partial x^i} \bar{f}_j, \qquad f^i = \sum_{j=1}^n \frac{\partial x^i}{\partial \bar{x}^j} \bar{f}^j, \qquad g_i = \sum_{j=1}^m \frac{\partial \bar{y}^j}{\partial y^i} \bar{g}_j, \qquad g^i = \sum_{j=1}^m \frac{\partial y^i}{\partial \bar{y}^j} \bar{g}^j.$$

Here I introduce coordinate systems y and $\bar{y}$ for W which correspond the the γ and $\bar{\gamma}$ bases in the natural manner ($y = \sum_{i=1}^m y^i g_i = \sum_{i=1}^m \bar{y}^i \bar{g}_i$ etc...). Let $T : V \to W$ be a linear transformation. Define, for the sake of tradition, (later, it is the custom to use $T^i_{\ j}$ instead of $A^i_{\ j}$)

$$A^i_{\ j} = g^i(T(f_j)) \qquad \text{and, in the same way,} \qquad \bar{A}^i_{\ j} = \bar{g}^i(T(\bar{f}_j))$$

(a.) show that:

$$T(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} A^i{}_j x^j g_i.$$

(b.) find the relation between $A^i{}_j$ and $\bar{A}^i{}_j$.

Problem 65 Let $b : V \times V \to \mathbb{R}$ be a bilinear mapping on a vector space V with bases $\beta, \bar{\beta}$ (continuing to use the notation of the previous problem) then **show** $b(x, y) = \sum_{i,j} b_{ij} x^i y^j$ where $b_{ij} = b(f_i, f_j)$. Moreover, if $\bar{b}_{ij}$ are the components of b with respect to $\bar{\beta}$-basis then **show**

$$\bar{b}_{ij} = \sum_{k,l=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} b_{kl}.$$

Problem 66 Consider the tensor $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g = \sum_{i=1}^{n} e^i \otimes e^i$ on $\mathbb{R}^n$ where $e^i(e_j) = \delta_{ij}$ and e_j denotes the usual j-th element of the standard basis for $\mathbb{R}^n$. Identify this tensor.

Problem 67 Define $\omega_{e_i} = e^i$ and show that linearly extending ω to $\mathbb{R}^3$ provides an isomorphism to $\Lambda^1(\mathbb{R}^3) = \mathbb{R}^{3*}$. Also Define $\Phi_{e_i} = \sum_{j,k=1}^{3} \epsilon_{ijk} e^j \otimes e^k$. Show that extending Φ linearly gives an isomorphism $\Phi : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3)$. Verify that $\omega_{\vec{F}} \wedge \omega_{\vec{G}} = \Phi_{\vec{F} \times \vec{G}}$.

Problem 68 Let $V = \mathbb{R}^4$ and denote $V^* = \text{span}\{dt, dx, dy, dz\}$. Let $\alpha = 3dt + 6dx$ and $\beta = dx + dy$. Calculate $\alpha \wedge \alpha$, $\beta \wedge \beta$ and $\alpha \wedge \beta$.

Problem 69 Suppose $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is given by $g(x, y) = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3$. Suppose we change coordinates by a matrix Λ where $\Lambda^T \eta \Lambda = \eta$ where we define $\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the formula for g in the $\bar{x}$-coordinate system. In particular, $\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$.

Problem 70 Finally, a physical application, the four-momentum is P^μ for $\mu = 0, 1, 2, 3$ where $P^0 = E = m_o \gamma(v)$ (total energy) and $P^i = m_o \gamma(v) v^i$ (relativistic momentum) and $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$. Using the notation of the previous problem, the quantity $g(P, P)$ is the invariant interval of the 4-momentum. Calculate its value. Here the constant m_o is called the **rest-mass**.