Problem 71 Suppose $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a linearly **dependent** set of dual-vectors over a finite dimensional vector space V. **Show that** $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = 0$. For your convenience, you may assume that $\alpha_1 = \sum_{j=2}^{k} c_j \alpha_j$ for some constants $c_2, \dots, c_k \in \mathbb{R}$.

Problem 72 Suppose γ is a p-form in $\Omega(V)$ where V is a n-dimensional vector space over $\mathbb{R}$. Assume $p \leq n$. If $\gamma \wedge e^j = 0$ for $j = 1, 2, \dots, n$ then does it follow $\gamma = 0$? Discuss.

Problem 73 Finish the proof of Theorem 10.2.6 part 3. In particular, show: for smooth functions f, g on a manifold $\mathcal{M}$ with coordinates (x^i):

$$\frac{\partial}{\partial x^i}(fg) = \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}$$

here I omitted the point-dependence for brevity.

Problem 74 Let $F(x, y, z) = (\cos(x), \sin(xy))$. Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$.

Problem 75 Let $\mathrm{SL}(2) = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1\}$. Find a coordinate chart which includes the identity matrix.

Problem 76 Suppose $B^T = -B \in \mathbb{R}^{2\times 2}$ let $\gamma(t) = e^{tB}$ define a curve in $\mathbb{R}^{2\times 2}$. Show that γ gives a path in $\mathrm{SL}(2)$ and express the tangent vector to the path at $t = 0$ in terms of coordinate vector fields derived from the coordinate system you derived in the previous problem. To find the tangent vector to the curve you should push-forward the derivation $\frac{d}{dt}\big|_0$ in the domain of the path to $d\gamma_o(\frac{d}{dt}\big|_0) \in T_I\mathrm{SL}(2)$.

Problem 77 Observe $\chi = (\theta, \phi)$ gives a coordinate chart on $S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. The inverse of this chart is given by the patch $\chi^{-1}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Calculate the push-forward under χ^{-1} of ∂_θ and ∂_ϕ in terms of the standard coordinate derivations $\partial_x, \partial_y, \partial_z$ the Cartesian coordinate system (x, y, z) for $\mathbb{R}^3$.

Problem 78 Let $\alpha = ydx + zdy$ and $\beta = xydz \wedge dt + dx \wedge dy$. Assume that x, y, z, t are independent Cartesian coordinates on $\mathbb{R}^4$.

(a.) calculate $d\alpha$

(b.) calculate $d\beta$

(c.) calculate $\alpha \wedge \beta$

(d.) calculate $d(\alpha \wedge \beta)$ in two ways.

Problem 79 Use the table of basic Hodge duals in my notes extended linearly to prove $*\omega_{\vec{v}} = \Phi_{\vec{v}}$. Also, show that $*\Phi_{\vec{v}} = \omega_{\vec{v}}$.

Problem 80 Explain what vector-calculus identities follow from $d(d\alpha) = 0$ and the correspondances $df = \omega_f$, $d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}}$ and $d\Phi_{\vec{G}} = (\nabla \cdot \vec{G})dx \wedge dy \wedge dz$. I did one of these in lecture.