Problem 81 Suppose α is a p-form on $\mathbb{R}^3$. Let $*$ denote the standard euclidean Hodge dual. Is it possible that $*d(\alpha) = d(*\alpha)$?

Problem 82 The Hodge dual operation on $\mathbb{R}^3$ allows us to intoduce another way to differentiate a p-form α:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha = (-1)^{3p} * d * \alpha$$

where I have set $n = 3$ since I intend to use this **coderivative** δ on $\mathbb{R}^3$. Let $\alpha = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$ where a, b, c are smooth functions on $\mathbb{R}^3$. Calculate the formula for $\delta\alpha$.

Problem 83 De Rahm, Hodge and others developed a theory to analyze closed vs. exact differential forms. See my notes for an example of how the shape of the domain can come into play. One interesting theorem Hodge proved was that if ω was any p-form on a Riemannian manifold then there exists a $(p-1)$-form α and a $(p+1)$-form β and a *harmonic form* γ such that

$$\omega = d\alpha + \delta\beta + \gamma.$$

In the special case $M = \mathbb{R}^3$ it is the case $\gamma = 0$. **Use the theorem due to Hodge to prove that any vector field can be written in terms of the gradient of a scalar function and the curl of some vector field; that is, for any vector field $\vec{F}$ there exists another vector field $\vec{G}$ and a function g such that $\vec{F} = \nabla g + \nabla \times \vec{G}$.** I think if you examine the case $\omega = \omega_{\vec{F}}$ then it ought to be about a line or two once you unravel the notation. I let Hodge do the really hard part. (you need to use the preceding problem to understand the coderivative part)

Problem 84 Consider $\omega = (x + y)dx + (y + z)dy + (z + x)dz$ on $\mathbb{R}^3$. Verify Hodge's Theorem (see preceding problem) by finding α and β such that $\omega = d\alpha + \delta\beta$. Begin your quest by understanding what the degrees of α and β must be in your context.

Warning: the notations *, $*$ and $_*$ mean different things. I do not use $_*$, but other authors use $df(v) = f_*(v)$; that is f_* is the push-forward by f. I use f^* as the pull-back by f and finally $*\gamma$ is the Hodge dual of γ which is only defined with respect to a metric (and for us, either $\mathbb{R}^3$ euclidean or $\mathbb{R}^4$ Minkowski for the application to Electrodynamics in a later chapter). Hodge duality is far less basic than the other two stars of this discussion. If you wish to read on Hodge duality in some generality then you might look at David Bleecker's text *Gauge Theory and Variational Principles* which is inexpensive in Dover format.

Problem 85 Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(r, \theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$. View r, θ, ϕ as Cartesian coordinates on the domain of F and view x, y, z as coordinates for the codomain of F. In particular, this indicates $F^1 = x$ and $F^2 = y$ and $F^3 = z$. Consider the differential form $\gamma = dx \wedge dy \wedge dz$. Calculate $F^*(\gamma)$.

Problem 86 Continuing the previous problem, find $F^*(\beta)$ where $\beta = \frac{1}{x^2+y^2}[-ydx + xdy]$.

Problem 87 Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space V. Let $\Lambda^k T$ be the function from $\underbrace{V \times V \times \cdots \times V}_{k-\text{copies of } V}$ to $\Lambda^k(V)$ defined as follows:

$$\Lambda^k T(v_1, v_2, \ldots, v_k) = T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k)$$

Briefly explain why $\Lambda^k T$ is a multilinear completely antisymmetric mapping. Then, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then show:

$$\Lambda^n T = M v^1 \wedge v^2 \wedge \cdots \wedge v^n$$

where $\beta^* = \{v^1, v^2, \ldots, v^n\}$ is the dual basis to β. Also, explain what is M as it relates to T.

Problem 88 Prove $\det(AB) = \det(A)\det(B)$ via wedge products. You might want to use the $\epsilon_{i_1,i_2,\ldots,i_n}$ formula for the determinant of $A, B \in \mathbb{R}^{n \times n}$

Problem 89 Suppose $F(r,s,t) = (rst, r^2{+}s^2, s^2{+}t^2, 3)$ defines a mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$. Furthermore, suppose $\mathbb{R}^4$ has Cartesian coordinates (x, y, z, w). Consider the one-forms on $\mathbb{R}^4$ as follows:

$$\alpha = (x^2 + z)dw \qquad \& \qquad \beta = (z^2 + w^2)(dx + dy)$$

Verify the identities:

(**i.**) $d[F^*(\alpha)] = F^*[d\alpha]$
(**ii.**) $d[F^*(\beta)] = F^*[d\beta]$
(**iii.**) $F^*[\alpha \wedge \beta] = F^*[\alpha] \wedge F^*[\beta]$

Problem 90 Maxwell's equations are written in differential form on $\mathbb{R}^4$ in my notes. Essentially, ignoring a factor of c, the coordinates on spacetime are (t, x, y, z). Pull-back Maxwell's equations to volume of constant time $t = t_o$. What are the new equations which hold on the slice of spacetime where time is constant? Are these equations familar from Physics 232 (if you've had or ICED that course)