Same instructions as Mission 1. Thanks!

Problem 73 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with non-vanishing ∇f. Let M be the hypersurface which is formed by the solution set of $f(x) = c$; that is $M = f^{-1}\{c\}$. Furthermore, let $n = \omega_{\vec{n}}$ be **unit-normal form** in the sense that $\text{Ker}(n_p) = T_pM$ for each $p \in M$ and $\vec{n} \bullet \vec{n} = 1$. We define the volume form vol_M on the hypersurface by $\text{vol}_M = \star n$ where $\star$ is the eulicidean Hodge dual. Show:

$$df \wedge \text{vol}_M = |\nabla f| dx^1 \wedge \cdots \wedge dx^n$$

Problem 74 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \text{vol}_M = 2\pi R$$

where $S_R = F^{-1}(R)$ for $F(x, y) = x^2 + y^2$. Suggestion,

$$n = \frac{1}{R}(xdy - ydx)$$

has $\vec{n} = \frac{1}{R}\langle -y, x\rangle$ with unit-length on S_R.

Problem 75 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \text{vol}_M = 4\pi R^2$$

where $S_R = F^{-1}(R)$ for $F(x, y, z) = x^2 + y^2 + z^2$.

Problem 76 Consider the 1-form $\alpha = xdz + ydw - (x^2 + y^2 + z^2 + w^2)dt$ on $\mathbb{R}^5$. Calculate $\int_S d\alpha \wedge d\alpha$, where $S \subset \mathbb{R}^5$ is given by $x^2 + y^2 + z^2 + w^2 = 1$ and $0 \leq t \leq 1$. Use the generalized Stokes' Theorem and the identity $d\alpha \wedge d\alpha = d(\alpha \wedge d\alpha)$ to make life easier.

Problem 77 Renteln Exercise 8.60 page 246-247. (volume of n-sphere)

Problem 78 Renteln Exercise 3.28 page 90. (Maxwell's Equations)

Problem 79 Renteln Exercise 3.29 page 93. (conservation of charge from $d^2 = 0$)

Problem 80 Maxwell's equations are written in differential form on $\mathbb{R}^4$ in my notes. Essentially, ignoring a factor of c, the coordinates on spacetime are (t, x, y, z). Pull-back Maxwell's equations to volume of constant time $t = t_o$. What are the new equations which hold on the slice of spacetime where time is constant? Are these equations familar from Physics 232 (if you've had the course, note, set $\mu_o = \epsilon_o = 1$ for our convenience here, I've not been careful about dimensional analysis in my notes in certain places...)

Bonus 10: (Hokage level) Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the projection defined by $\pi(x) = x - (x \bullet e_j)e_j$ for each $x \in \mathbb{R}^n$ for $j = 1, \dots, n$. Suppose $\mathcal{P}$ is an $(n-1)$-dimensional paralell-piped which is formed by $[0,1]$-weighted linear combinations of $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ suspended at base-point $p \in (0, \infty)^n$;

$$\mathcal{P} = \left\{ p + \sum_{j=1}^{n-1} \alpha_j v_j \;\middle|\; \alpha_j \in [0,1] \right\}$$

Let $n \in \mathbb{R}^n$ be a unit-vector in $\{v_1, \cdots, v_{n-1}\}^\perp$. The $(n-1)$-**volume** of $\mathcal{P}$ is given by $\mathbf{vol}(\mathcal{P}) = |\det[v_1|\cdots|v_{n-1}|n]|$. We can study the area of the **shadows** formed by $\mathcal{P}$ on the coordinate hyperplanes. Let $\mathcal{P}_j = \pi_j(\mathcal{P})$ define the shadow of $\mathcal{P}$ on the $x_j = 0$ coordinate plane. Notice,

$$\mathcal{P}_j = \left\{ \pi_j(p) + \sum_{i=1}^{n-1} \alpha_i \pi_j(v_i) \;\middle|\; \alpha_j \in [0,1] \right\}$$

which shows $\mathcal{P}_j$ is formed by $[0,1]$-weighted linear combinations of $\pi_j(v_1), \dots, \pi_j(v_n)$ of attached at basepoint $\pi_j(p)$. It follows that the $(n-1)$-volume of the $\mathcal{P}_j$ can be calculated as follows:

$$\mathbf{vol}(\mathcal{P}_j) = |\det[\pi_j(v_1)|\cdots|\pi_j(v_{n-1})|e_j]|.$$

since e_j is perpendicular to $\mathcal{P}_j$. I choose to refer to the quantity as volume, but to be honest, in the one-dimensional case we usually call it length, in two-dimensions area. Some people call higher dimensional cases hypervolume. Let's examine some simple cases. In the case $n = 2$ the 1-dimensional paralell-piped is just a line-segment. For example, if $v_1 = (1,1)$ then $(1/\sqrt{2}, -1/\sqrt{2})$ is perpendicular to v_1 and

$$\det \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1 & -1/\sqrt{2} \end{bmatrix} = -2/\sqrt{2} = -\sqrt{2} \quad \Rightarrow \quad \mathbf{vol}(\mathcal{P}) = \sqrt{2}.$$

Of course, this is actually the length of the line-segment. Also, notice

$$\mathbf{vol}(\mathcal{P}_1)^2 + \mathbf{vol}(\mathcal{P}_2)^2 = 1^2 + 1^2 = \sqrt{2}^2 = \mathbf{vol}(\mathcal{P})^2.$$

This is not suprising. However, perhaps the fact this generalizes to n-dimensions in the following sense is not already known to you:

$$\mathbf{vol}(\mathcal{P}_1)^2 + \mathbf{vol}(\mathcal{P}_2)^2 + \cdots \mathbf{vol}(\mathcal{P}_n)^2 = \mathbf{vol}(\mathcal{P})^2$$

Prove it. You might call this the generalized Pythagorean identity, I'm not sure its history or formal name. That said, the formula I give for generalized area could just as well be termed generalized volume. Also, you could **define**

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \left[\begin{array}{c|c|c|c|c} & & & & e_1 \\ & & & & e_2 \\ v_1 & v_2 & \cdots & v_{n-1} & \vdots \\ & & & & e_n \end{array} \right] \in \mathbb{R}^n$$

where we insist the determinant is calculated via the Laplace expansion by minors along the last column. You can show $v_1 \times v_2 \times \cdots \times v_{n-1} \in \{v_1, \dots, v_{n-1}\}^\perp$. But, if n is a unit-vector which spans $\{v_1, \dots, v_{n-1}\}^\perp$ then the $(n-1)$-ry cross-product must be a vector parallel to n and thus:

$$v_1 \times v_2 \times \cdots \times v_{n-1} = [(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n]\, n$$

Note, $n \bullet n = 1$ as we assumed n is unit-vector and we can show

$$(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n = \det[v_1|v_2|\dots|v_{n-1}|n]$$

Notice this generalized cross-product is just an extension of the heurstic determinant commonly used in multivariate calculus to define the standard cross-product. In particular, the following is equivalent to the column-based definition

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n \\ & v_1^T & & \\ & v_2^T & & \\ & \vdots & & \\ & v_{n-1}^T & & \end{bmatrix}$$

where we insist the determinant is calculated via the Laplace expansion by minors along the first row. In any event, my point in this discussion is merely that we can calculate higher-dimensional volumes with determinants and these go hand-in-hand with generalized cross-products. In particular,

$$||v_1 \times v_2 \times \cdots \times v_{n-1}|| = \mathbf{vol}(\mathcal{P})$$

where $\mathcal{P}$ is formed by $[0,1]$-weighted linear combinations of $v_1, \dots, v_{n-1}$. When $n = 2$ this gives vector length, when $n = 3$ this is the familar result that the area of the parallelogram with sides $\vec{A}, \vec{B}$ is just $||\vec{A} \times \vec{B}||$.