Same instructions as Mission 1. Thanks!

Remark: I decided against using Renteln Chapter 8 because the calculation is rather involved. However, for a surface the Cartan calculus makes quick work of the problem of curvature. In this Mission I walk you through the high points of the calculation and we get straight to the point of calculating curvature for a few interesting surfaces. The Cartan method of moving frames is far more general than its application here, Renteln gives you some indication of that in Chapter 7, but I may not have time to lecture on that in this course. To be frank, to cover the subject matter in Renteln deeply it takes a couple courses. My apologies for not assigning you dozens of really interesting problems in his text, there is so much more there to learn.

Problem 81 De Rahm, Hodge and others developed a theory to analyze closed vs. exact differential forms. See my notes for an example of how the shape of the domain can come into play. One interesting theorem Hodge proved was that if ω was any p-form on a Riemannian manifold then there exists a $(p-1)$-form α and a $(p+1)$-form β and a *harmonic form* γ such that

$$\omega = d\alpha + \delta\beta + \gamma.$$

In the special case $M = \mathbb{R}^3$ it is the case $\gamma = 0$. **Use the theorem due to Hodge to prove that any vector field can be written in terms of the gradient of a scalar function and the curl of some vector field; that is, for any vector field $\vec{F}$ there exists another vector field $\vec{G}$ and a function g such that $\vec{F} = \nabla g + \nabla \times \vec{G}$.** I think if you examine the case $\omega = \omega_{\vec{F}}$ then it ought to be about a line or two once you unravel the notation. I let Hodge do the really hard part. (you need to use the preceding problem to understand the coderivative part)

Problem 82 Consider $\omega = (x+y)dx + (y+z)dy + (z+x)dz$ on $\mathbb{R}^3$. Verify Hodge's Theorem (see preceding problem) by finding α and β such that $\omega = d\alpha + \delta\beta$. Begin your quest by understanding what the degrees of α and β must be in your context.

Problem 83 Consider the one-form $\omega = xdx + ydy + zdz$ on $\mathbb{R}^3$. Find the foliation of three dimensional space into two-dimensional submanifolds whose tangent spaces are spanned by vector fields which are found in $\ker(\omega)$. Check the condition needed to show ω is dual to a two-plane field distribution on $\mathbb{R}^3$; that is verify $\omega \wedge d\omega = 0$. Incidentally, there is one point left out, perhaps it would be more honest to say find a foliation of $\mathbb{R}^3 - \{(0,0,0)\}$.

Problem 84 Consider $\omega = dy + dz + xydx + xzdx$. Show that $\omega \wedge d\omega = 0$ on all of $\mathbb{R}^3$. What foliation of $\mathbb{R}^3$ does ω describe. Recall, we discussed that $\omega = dz$ corresponds to foliating $\mathbb{R}^3$ into $z = c$ (a family of horizontal planes, each leaf in the foliation labeled by c). Try to find the corresponding family of surfaces for the ω given here.

Problem 85 We spent some time studying *distributions*. I think I could have said more about the dual description. Here is a simple example: in $\mathbb{R}^3$ we can study the distribution generated by ∂_x, ∂_y lets say $E = \langle \partial_x, \partial_y \rangle$ or we can say $E = \langle dz \rangle$. To be more explicit, E is either formed from the span of the given vector fields or as the kernel of the annhilating form dz. In $\mathbb{R}^4$ if we studied $E = \langle \partial_x, \partial_y \rangle$ in (t, x, y, z) space then $E = \langle dt, dz \rangle$ would be the dual description. More explicitly,

$$E = \text{span}\{\partial_x, \partial_y\} = \ker(dt) \cap \ker(dz)$$

The **leaves** of the foliation to E are simply two-dimensional spaces with constant t and z. Each leaf takes x, y as coordinates. In general, in $\mathbb{R}^n$ we can describe a k-dimensional distribution either with k-vector fields, or with $n - k$-annihilating one-forms:

$$E = \langle X_1, \dots, X_k \rangle = \langle \omega_1, \dots, \omega_{n-k} \rangle$$

The Frobenius Theorem explains when such a distribution naturally aligns itself with submanifolds of dimension k. In the case the distribution is **integrable** the **leaves** of the foliation are the submanifolds whose tangent spaces naturally correspond to E pointwise. As we discussed, **involutivity** sufficed to give integrability of the distribution. What I may have failed to emphasize is the corresponding result in terms of differential forms. In particular, E is integrable if $d\omega_1, \dots, d\omega_{n-k}$ annihilate $E = \langle \omega_1, \dots, \omega_{n-k} \rangle$. Or, we can equivalently capture integrabilty of E via the existence of smooth one-forms α_{ij} such that:

$$d\omega_i = \sum_{j=1}^{n-k} \omega_j \wedge \alpha_{ij}.$$

Lectures 26 and 27 from 2015 are somewhat helpful , but, you can answer these without watching.

(a.) Let $V = \partial_x + y\partial_z$ and $W = \partial_y + x\partial_z$. Show $E = \langle V, W \rangle$ is involutive.

(b.) Find a one-form ω for which $E = \langle \omega \rangle$. Hint: calculate $E^\perp$ and use the work-form map to translate to a one-form.

(c.) Find the leaves in the foliation induced from E.

To answer the last part, there are two natural approaches. Pick a point p. First, you could find parametrizations of the leaf through p by flowing along V and W appropriately. Second, you could try to write $\omega = dF$ for some $F = F(x, y, z)$ then $F(x, y, z) = c$ implicitly describes the leaf for c such that $F(p) = c$. Perhaps I should discuss why such an F should exist in this context.

Problem 86 A frame $\{E_1, E_2, E_3\}$ in $\mathbb{R}^3$ is typically an **orthonormal frame**; $E_i \bullet E_j = \delta_{ij}$. To each frame $\{E_1, E_2, E_3\}$ we find a coframe $\{\theta_1, \theta_2, \theta_3\}$ where $\theta_i(E_j) = \delta_{ij}$ on $\mathbb{R}^3$. Naturally, we may omit some points and speak of frames and coframes on some domain of $\mathbb{R}^3$ in the same sense. For example, the spherical coordinate frame is technically a frame on some subset of $\mathbb{R}^3$ as we face degeneracy of the frame at the origin and pole.

The natural Riemannian connection in $\mathbb{R}^3$ is given by:

$$\nabla_V(W) = \sum_{i=1}^{3} V[W^i]\frac{\partial}{\partial x_i}$$

since $\Gamma^k_{ij} = 0$ in this context. We introduce **connection coefficients** ω_{ij} for the frame $\{E_1, E_2, E_3\}$ by defining:

$$\omega_{ij}(v) = \nabla_v(E_i) \bullet E_j.$$

Then the meaning is clear, $\omega_{ij}(v)$ measures how much E_i is rotating into the E_j-direction as we approach the point the v-direction (the point notation is omitted in the equation above). Notice, we also have:

$$\nabla_v E_i = \sum_{j=1}^{3} \omega_{ij}(v) E_j$$

It can be shown (we will take it on faith here) that $\omega_{ij} = -\omega_{ji}$ and these solve Cartan's Structure Equations:

$$d\theta_i = \sum_{j=1}^{3} \omega_{ij} \wedge \theta_j \qquad \& \qquad d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.$$

If M is a surface in $\mathbb{R}^3$ then an orthonormal frame $\{E_1, E_2, E_3\}$ is said to be **adapted** to M if $\text{span}\{E_1(p), E_2(p)\} = T_pM$ for each $p \in M$. Then $E_3 \in (T_pM)^\perp$ with respect to the standard Euclidean metric of $\mathbb{R}^3$. The coframe $\{\theta_1, \theta_2, \theta_3\}$ of such an adapted frame has $\theta_3 = 0$ on M hence θ_1, θ_2 serve as a basis for one-forms on M.

(a.) Show for a frame field $\{E_1, E_2, E_3\}$ on $\mathbb{R}^3$ if $W = \sum_{i=1}^{3} f_i E_i$ then

$$\nabla_V W = \sum_{j=1}^{3} \left(V[f_j] + \sum_{i=1}^{3} f_i \omega_{ij}(V) \right) E_j$$

(b.) likewise, show if $\alpha = \sum_{i=1}^{3} a_i \theta_i$ then show

$$d\alpha = \sum_{j=1}^{3} \left(da_j + \sum_{i=1}^{3} a_i \omega_{ij} \right) \wedge \theta_j$$

Problem 87 Continuing the previous problem,

(a.) If E_1, E_2, E_3 is adapted to M in the sense that $\theta_3 = 0$ on M then show Cartan's Structure Equations yield:

$$
\begin{aligned}
d\theta_1 &= \omega_{12} \wedge \theta_2 && \text{structure equation for } d\theta_1 \quad (1)\\
d\theta_2 &= \omega_{21} \wedge \theta_1 && \text{structure equation } d\theta_2 \quad (2)\\
\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 &= 0 && \text{symmetry equation} \quad (3)\\
d\omega_{12} &= \omega_{13} \wedge \omega_{32} && \text{Gauss equation} \quad (4)\\
d\omega_{13} &= \omega_{12} \wedge \omega_{23} && \text{Codazzi equation for } d\omega_{13} \quad (5)\\
d\omega_{23} &= \omega_{21} \wedge \omega_{13} && \text{Codazzi equation for } d\omega_{23} \quad (6)
\end{aligned}
$$

(b.) For the adapted frame E_1, E_2, E_3 we have E_3 is everywhere normal to the tangent space of M; that is, E_3 serves as a unit-normal vector field on M. In classical differential geometry we use the normal to define the **shape operator**. In particular, in our current context,

$$S(v) = -\nabla_v E_3 = \omega_{13}(v)E_1 + \omega_{23}(v)E_2.$$

the shape operator measures the change in the normal along the surface. The **mean curvature** H and the **Gaussian curvature** K are defined by $2H = \text{trace}(S)$ and $K = \det(S)$. Your mission, should you accept it, is to show that:

$$\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2 \qquad \& \qquad \omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$$

(c.) given your work above, explain why $d\omega_{12} = -K\theta_1 \wedge \theta_2$.

Problem 88 There is a choice of spherical coordinates where spherical angles α, β naturally provide $\theta_1 = R\cos\alpha\, d\beta$ and $\theta_2 = Rd\alpha$ with $\omega_{12} = \sin\alpha\, d\beta$. Given these toys, calculate the Gaussian curvature of the sphere of radius R.

Problem 89 Calculate E_1, E_2, E_3 and θ_1, θ_2 and ω_{12} as well as K for the cone parametrized via:

$$X(u, v) = (v\cos u, v\sin u, v)$$

Problem 90 Calculate E_1, E_2, E_3 and θ_1, θ_2 and ω_{12} as well as K for the Helicoid parametrized via ($b \neq 0$ is a constant)

$$X(u, v) = (u\cos v, u\sin v, bv)$$