Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. No fuzzies thanks. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Define $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $s(x, y) = x + y$ and $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $p(x, y) = xy$. Show these maps are continuous.

Problem 2 Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for a normed linear space V such that if $x = \sum_{i=1}^{n} x_i v_i$ then $|x_i| \leq \|x\|$. Suppose $T : V \to V$ is a linear transformation. Show $\lim_{x \to a} T(x) = T(a)$ for $a \in V$ by an $\varepsilon\delta$-argument.

Problem 3 Suppose V, W are real normed linear spaces. We say f is continuous at $a \in V$ if $\lim_{x \to a} f(x) = f(a)$. Furthermore, f is continuous on $U \subseteq V$ if f is continuous at each point in U. Let $f : V \to W$ and $S \subseteq W$ then

$$f^{-1}(S) = \{x \in V \mid f(x) \in S\}$$

is the **inverse image** of S under f. Show: $f : U \to W$ is continuous on U if and only if the inverse image of each open set in W under f is an open set in U. Note, by default, we consider the emptyset $\emptyset$ an open set.

Problem 4 Let $\eta(A, B) = \text{trace}(A^T B)$ for all $A, B \in \mathbb{R}^{n \times n}$. Show that η is an inner product on $\mathbb{R}^{n \times n}$. This shows that $||A|| = \sqrt{\text{trace}(A^T A)}$ is a norm as it is simply the norm induced from η. Also, calculate $||A||$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Problem 5 Let $A, B \in \mathbb{R}^{n \times n}$ and $|| \cdot ||$ denote the Frobenius norm. Show that $||AB|| \leq ||A|| \, ||B||$.

Problem 6 Show that $\text{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} \mid \det(A) = 1\}$ is topologically closed. *It's also closed as a subgroup of the general linear group of invertible matrices in $\mathbb{R}^{n \times n}$. In fact, for reasons we will eventually appreciate, $SL(n, \mathbb{R})$ is a Lie group, that is a group which is also a smooth manifold in the natural sense.*

Problem 7 Show $\mathbb{R}^{m \times n}$ is a complete space. Assume it is already known that $\mathbb{R}^{m \times n}$ is a normed linear space with respect to the Frobenius norm $||A|| = \sqrt{\text{trace}(A^T A)}$.

Problem 8 Define $\sin(A) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$ for $A \in \mathbb{R}^{n \times n}$. Show that $\sin(A)$ exists for any $A \in \mathbb{R}^{n \times n}$. *Hint: in the notes we prove e^A exists for any A.*

Bonus 1: Suppose P is a paralellogram in $\mathbb{R}^3$ in the octant with positive coordinates. Furthermore, define $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2\}$.

(a.) find $Area(P)$.

(b.) define $L_{ij}(\vec{v}) = (\vec{v} \bullet \widehat{x}_i)\,\widehat{x}_i + (\vec{v} \bullet \widehat{x}_j)\,\widehat{x}_j$ and prove using properties of dot-products that L_{ij} is a linear transformation.

(c.) assume that linear transformations map parallelograms to lines, parallelograms or points and use this presupposition to establish the following equation:

$$Area(P)^2 = Area(L_{12}(P))^2 + Area(L_{31}(P))^2 + Area(L_{23}(P))^2$$