Same instructions as Mission 1. Thanks!

Problem 9 Let $F(A) = A^3$ for $A \in \mathbb{R}^{n\times n}$. Prove F is differentiable on $\mathbb{R}^{n\times n}$ by proposing a formula for $dF_A(H)$ and showing your proposed differential is linear in H and satisfies the needed limit. *Please do not use partial differentiation (yet).*

Problem 10 Let $F(A) = A^3$. Calculate $\frac{\partial F}{\partial X_{ij}}(A)$ with respect to the standard matrix basis. Explain why $dF_A(H) = A^2H + AHA + HA^2$. *(explain both the existence of dF_A as well as the formula as it can be ciphered from your partial derivatives, cite an appropriate theorem from my notes, do not use the previous problem in your argument)*

Problem 11 Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $F(x, y) = x \bullet y$. Calculate $dF_{(a,b)}(h, k)$.

Problem 12 Use chain-rule for $f(x, y) = \sqrt[y]{x}$ composed with $\gamma(t) = (t, t)$ to calculate $\frac{d}{dt}\left[f(\gamma(t))\right]$. *Thus, in view of the fact $\sqrt[t]{t} = f(\gamma(t))$ you have calculated $\frac{d}{dt}\left[\sqrt[t]{t}\right]$.*

Problem 13 Find the Jacobian matrix for the following maps from $\mathbb{R}^n$ to $\mathbb{R}^n$:

(a.) $F(x, y) = (x^2 - y^2, 2xy)$

(b.) $G(x, y, z) = (x^2 + 2yz, z^2 + 2xy, y^2 + 2xz)$

(c.) $H(x, y) = \dfrac{1}{x^2 - y^2}(x, -y)$

Problem 14 Let V, W be finite-dimensional normed linear spaces. Show that if $F : V \to W$ is Frechet differentiable at $a \in V$ then F is continuous at a.

Problem 15 Let V be a real NLS with basis $\beta = \{v_1, \dots, v_n\}$. Define the **$i$-th coordinate function** $x_i : V \to \mathbb{R}$ by:

$$x_i(a_1v_1 + \cdots + a_nv_n) = a_i$$

for $i = 1, 2, \dots, n$. Prove the following:

(a.) x_i is differentiable and hence continuous on V,

(b.) $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ for $1 \le i, j \le n$. Recall $\delta_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases}$

Problem 16 Let $F(X) = \exp(X)$ for $X \in \mathbb{R}^{n\times n}$ for which $X^3 = 0$. Show F is differentiable on $\mathbb{R}^{n\times n}$. Also, calculate the linearization of F at I.

Bonus 2: Let V be a real vector space with norm $||\cdot||$. The purpose of this problem is to establish the following equivalence: the norm is induced from an inner product $\Leftrightarrow$ the norm satisfies the parallelogram law below:

$$\boxed{||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)} \quad \star$$

for all $x, y \in V$. The proof is somewhat involved:

(a.) Suppose there exists an inner product $\langle , \rangle : V \times V \to \mathbb{R}$ for which $||x|| = \sqrt{\langle x, x\rangle}$ for all $x \in V$. Show $||\cdot||$ so-defined satisfies the parallelogram law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

for all $x, y \in V$.
(this proves the $\Rightarrow$ of the claim, the rest of the problem goes to the other direction)

(b.) Suppose there exists an inner product $\langle , \rangle : V \times V \to \mathbb{R}$ for which $||x|| = \sqrt{\langle x, x\rangle}$ for all $x \in V$. Show $||\cdot||$ so-defined satisfies derive the the *polar form identity*:

$$\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

(c.) Assume V is a given real normed linear space with norm $||\cdot||$ **which satisfies the identity** $\star$. In view of the result of the previous part, it is natural to define $g : V \times V \to \mathbb{R}$ by the following formula:

$$g(x, y) = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

as a potential inner-product induced from the given norm.

- **(i.)** show $g(x, y) = g(y, x)$,
- **(ii.)** show $g(x, x) = ||x||^2$ and hence explain why g is positive definite,
- **(iii.)** show $g(x + y, z) = g(x, z) + g(y, z)$. (be sure to implement $\star$!)
- **(iv.)** show $g(kx, y) = kg(x, y)$ for all $k \in \mathbb{N}$ by induction on k,
- **(v.)** show $g(-x, y) = -g(x, y)$ and show $g(zx, y) = zg(x, y)$ for all $z \in \mathbb{Z}$,
- **(vi.)** show $g\left(\frac{p}{q}x, y\right) = \frac{p}{q}g(x, y)$ for all $p, q \in \mathbb{Z}$ with $q \neq 0$
- **(vii.)** Fix $y \in V$ and define $h(x) = g(x, y)$. Show $h : V \to \mathbb{R}$ is continuous on V,
- **(viii.)** let $r \in \mathbb{R}$ then there exists a sequence of rational numbers p_n/q_n converging to r as $n \to \infty$ by the density of the rational numbers in $\mathbb{R}$. Use the equivalence of sequential limits and topological $(\epsilon - \delta)$ limits paired with the continuity of h (see part vii.) to show $g(rx, y) = rg(x, y)$ for all $r \in \mathbb{R}$.
- **(ix.)** show $g(x, ry + z) = rg(x, y) + g(x, z)$ for all $x, y, z \in V$ and $r \in \mathbb{R}$. *hint: use i., iii. and x. .*

Thus we have shown g so-defined is a symmetric, positive definite, bilinear form on V which means g defines an inner-product. *This completes the $\Leftarrow$ part of the claim.*