Same instructions as Mission 1. Thanks!

Problem 17 Suppose $x_1, \dots, x_n$ are coordinates of a normed linear space V with respect to the basis $\beta = \{v_1, \dots, v_n\}$. Let $F, G : V \to \mathbb{R}$ be differentiable functions on V and $h : \mathbb{R} \to \mathbb{R}$ a differentiable function on $\mathbb{R}$. Show: for $c \in \mathbb{R}$ and for $i = 1, \dots, n$,

$$\frac{\partial}{\partial x_i}\left[cF(x) + G(x)\right] = c\frac{\partial F}{\partial x_i} + \frac{\partial G}{\partial x_i} \qquad \& \qquad \frac{\partial}{\partial x_i}\left[h(F(x))\right] = h'(F(x))\frac{\partial F}{\partial x_i}.$$

Problem 18 Continuing the previous problem, assume $F_i : V \to \mathbb{R}$ is differentiable for any $i \in \mathbb{N}$. Prove the extended product rule:

$$\frac{\partial}{\partial x_i}\left[FG\right] = \frac{\partial F}{\partial x_i}G + F\frac{\partial G}{\partial x_i} \qquad \& \qquad \frac{\partial}{\partial x_k}\left[\prod_{j=1}^{m} F_{i_j}\right] = \sum_{j=1}^{m} \frac{\partial F_{i_j}}{\partial x_k}\prod_{l \neq j} F_{i_l}$$

for $m \in \mathbb{N}$ where $\prod_{l \neq j}$ means l ranges over the list $1, 2, \dots m$ with j deleted.

Problem 19 Let $det : \mathbb{R}^{n \times n} \to \mathbb{R}$ be the mapping defined by

$$\det(A) = \sum_{i_1, \dots, i_n} \epsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \cdots A_{i_n n}$$

where $\epsilon_{i_1 i_2 \dots i_n}$ is the completely antisymmetric symbol for which $\epsilon_{12\dots n} = 1$. The standard coordinates of A are A_{ij} since $A = \sum_{i,j} A_{ij} E_{ij}$ where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Calculate $\frac{\partial}{\partial A_{ij}}\det(A)$ and explain the meaning of this quantity as it relates to the usual formulae for the determinant. You might find it helpful to work out the 2×2 or 3×3 case. Also,

$$A_{i_1 1} A_{i_2 2} \cdots A_{i_n n} = \prod_{k=1}^{n} A_{i_k k}.$$

Problem 20 If $x^2 + y^2 + z^2 + w^2 = 1$ and $xywz = 1$ then calculate $\left.\frac{\partial z}{\partial x}\right|_y$. That is, take z, w to be dependent variables and calculate the derivative of z with respect to x while holding y-fixed.

Problem 21 Let $G(x, y, a, b) = (x^2 - y^2 - ax + by, 2xy - xb - ya)$. Suppose $M = G^{-1}(2, 1)$.

(a.) Solve for a, b as functions of x, y

(b.) use the implicit function theorem to show where it is possible to solve for a, x as functions of b, y (no need to actually solve it, demonstration of existence suffices)

(c.) use the implicit function theorem to show where it is possible to solve for a, y as functions of b, x (no need to actually solve it, demonstration of existence suffices))

(d.) use the implicit function theorem to show where it is possible to solve for x, y as functions of a, b. (no need to actually solve it, demonstration of existence suffices)) *note: I don't expect you to analyze the subtle question of if it is still possible to solve where there implicit function theorem breaks down. I merely wish for you to find the low-hanging fruit which the implicit function theorem provides*

Problem 22 Let $F(x, y, z, w) = (e^x \cosh y, e^x \sinh y, e^z \cos w, e^z \sin w)$ for all $(x, y, z, w) \in \mathbb{R}^4$. Show this mapping is locally invertible. Prove that no global inverse exists.

Problem 23 Define $F(x, y, z) = (x/y, y/z, z)$ for $y, z \neq 0$. Calculate J_F and determine where F can be F is locally invertible. Calculate $F^{-1}(a, b, c)$.

Problem 24 Let $F(x, y) = (x^3 - 3xy^2, 3x^2y - y^3)$ for all $(x, y) \in \mathbb{R}^2$. Show F is locally invertible at all points in the plane except one. Find the inverse for F restricted to the sector $-\pi/3 < \theta < \pi/3$ for $r > 0$ (I use the usual polar coordinates in the plane)

Bonus 3: Let $F(x, y, z) = \dfrac{1}{x^3 + y^3 + z^3 - 3xyz}(x^2 - yz, z^2 - xy, y^2 - xz)$. Find the inverse function of F, or, if not globally possible, find a local inverse for F.

Hint: I used $\mathcal{H}_3 = \mathbb{R}^3$ with typical element $x + jy + j^2z$ and $j^3 = 1$ to construct this example. There is a natural isomorphism given by

$$\mathbf{M}(x + jy + j^2z) = \begin{bmatrix} x & z & y \\ y & x & z \\ z & y & x \end{bmatrix}$$

for which $\mathbf{M}(\zeta\eta) = \mathbf{M}(\zeta)\mathbf{M}(\eta)$ and $\mathbf{M}(1) = I$ the identity matrix. I should mention that if $F : \mathcal{H}_3 \to \mathcal{H}_3$ is real differentiable then F is $\mathcal{H}_3$-differentiable at p if and only if $J_F(p) = \mathbf{M}(dF_p(e_1))$. It turns out that means the formula for F can be written manifestly as a function of $\zeta = x + jy + j^2z$. For example,

$$H(\zeta) = \zeta^2 = (x + jy + j^2z)^2 = x^2 + 2yz + j(z^2 + 2xy) + j^2(y^2 + 2xz)$$

Is the algebra formula for the real mapping $H(x, y, z) = (x^2 + 2yz, z^2 + 2xy, y^2 + 2xz)$. If you look back at Problem 13, you can see that $J_H(\zeta) = 2\mathbf{M}(\zeta)$.