Same instructions as Mission 1. Thanks!

**Problem 31** Consider $x, y \in \mathbb{R}^n$. Define $Q(t) = \|x - ty\|^2$ for $t \in \mathbb{R}$. Find the minimum value for $Q$ and show how this can be used to derive the inequality $x \bullet y \leq \|x\|\|y\|$.

**Problem 32** Let $V$ be a real inner product space with inner product $\langle , \rangle$. Suppose $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for $V$ and let $x = \sum_i x_i v_i$ and $y = \sum_j y_j v_j$. Define $\|x\| = \sqrt{\langle x, x \rangle}$. Let $Q(t) = \|x - ty\|^2$ for $t \in \mathbb{R}$. Find the minimum value for $Q$ and show how this can be used to derive the inequality $\langle x, y \rangle \leq \|x\|\|y\|$.

**Problem 33** Let $Q(x, y, z) = 31x^2 + 15y^2 + 15z^2 - 22xy - 22xz + 10yz$. Find the matrix for $Q$ and use technology to calculate the eigenvalues for $Q$. Let $y_1, y_2, y_3$ denote the eigencoordinates, write the formula for $Q$ in terms of the eigencoordinates. For the record, I did not design this problem to have that eigenvalue. It just happened.

**Problem 34** Suppose

$$f(x, y, z) = 9000 + 31(x-1)^2 + 15y^2 + 15(z+2)^2 - 22(x-1)y - 22(x-1)(z+2) + 10y(z+2)$$

Notice $(1, 0, -2)$ is a critical point for $f$. Classify the critical point as min/max or saddle.

**Problem 35** Calculate the multivariate Taylor series centered about $(0, 0, 0)$ for $f(x, y, z) = \dfrac{1 + y^2}{1 - 2xz}$ to order 4. Analyze: is $(0, 0, 0)$ a critical point? If so, analyze if it yields a min/max/saddle or if the second derivative test is not applicable for the given problem.
*Hint: all the cool kids use geometric series*

**Problem 36** Find the geodesics in the tunnel given by $(x, y, z)$ for which $y^2 + z^2 = R^2$.

**Problem 37** Find the geodesics on the cone $\phi = \pi/3$ where $\phi$ denotes the usual spherical angle.

**Problem 38** Let $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$ denote the Lagrangian of a particle with mass $m$ under the force a spring with potential energy $U(x, y) = \frac{k}{2}(x^2 + y^2)$. Notice $L = T - U$ where $T$ is the kinetic energy. **Calculate the Euler-Lagrange equations and show energy $E = T + U$ is conserved along the solution to the Euler-Lagrange equation**

**Problem 39** Let $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + g(r)$ where $g$ is a differentiable function of the polar radius $r$. Find the Euler Lagrange equations. Also, suppose we define angular momentum $J = \dfrac{\partial L}{\partial \dot{\theta}}$, show $J$ is conserved.

**Problem 40** A marble slides without friction on a bowl of radius $R$. If the marble has mass $m$ and the force of gravity is given by $-mg\,\widehat{z}$ then find the equations of motion for the marble (differential equations suffice as an answer here). Also, show momentum in the direction of a rotation about the $z$-axis is conserved.

**Bonus 5:** Imagine a pendulum of length $l_1$ which consists of a very light rod which does not flex and a bob of mass $m_1$. Next, a second pendulum of length $l_2$ which consists of a very light rod which does not flex and a bob of mass $m_2$ is attached so that $l_2$ hangs freely off $m_1$. All of this is attached to point and allowed to swing back and forth under the influence of gravity. Assume this is near the surface of the earth where $F = mg$ applies. Find the equations of motion for this double pendulum. Let $\theta_1$ and $\theta_2$ be the angles which $l_2$ and $l_2$ make with respect to $-\widehat{z}$. Write the equations of motion in terms of these anglular variables.

**Bonus 6:** Let $q_i$ and $\dot{q}_i$ for $1 \leq i \leq n$ denote generalized coordinates of a physical system with Lagrangian $L(q_i, \dot{q}_i)$. Define the **Hamiltonian** by

$$H(q_i, p_i) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i)$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$ defines the $i$-th generalized momenta of the system. Show that the Euler Lagrange equations imply Hamilton's Equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \& \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}.$$

**Bonus 7:** I have in mind a problem where you derive Coriolis effect and such by imposing rotation of the earth on a rotating frame. I lack inspiration to frame the problem properly, but, if you have interest, shoot me an email and I'll put it together as a bonus problem here.